Optimization problems in which the mass and exhaust velocity programs and the burning time are varied, as well as the steering program, can also be handled by classical variational methods, insofar as we only seek stationary solutions. To find solutions that actually provide maxima or minima requires, of course, a special investigation for each problem, since there may well be a stationary solution which is not optimum or an optimum solution which is not stationary.

When the exhaust velocity is a prescribed function of time e.g. constant as is, approximately, the case for chemical propulsion systems) the sophisticated analysis for finding the optimum $M(t)$ is of little help, since it generally says only that impulsive burning is optimal. A more detailed analysis, in which the characteristics of the rocket motors, propellant tanks, et al. are taken into account simultaneously with the trajectory optimization, is then necessary. A practical method of carrying this out involves the logical interconnection of a number of detailed engineering analyses, of which one is the set of differential equations of motion for the trajectory. With the aid of a high-speed digital computer, a self-consistent solution of this scheme of equations can be obtained, each such solution representing a possible and *realistic* missile. By variation of the input parameters, a number of optimization problems may be studied.

In the field of missile analysis, there is a need for both very complicated models, like this design optimization analysis, and for very simple models, like the point particle. In fact, these tend to complement each other. The latter provide analytical solutions in closed form, from which valuable insights regarding trajectories can be obtained. They also furnish simple trajectory patterns which are of great value for the problems involving the simultaneous optimization of design and trajectory. The more complicated models, in turn, make it possible to take into account many effects which must of necessity be omitted from the simple models and also provide a good illustration of the proper use of high-speed computers in missile design.

CHAPTER 5

EARTH SATELLITES AND RELATED ORBIT AND PERTURBATION THEORY

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5–1 ASTRODYNAMICS

The theory that underlies the determination of orbits and ephemerides (tables of position) for space vehicles has deep roots in the scientific culture of our age. Not only are the contributions of such men as Newton, Laplace, Lagrange, and Gauss still basic to practical work in celestial mechanics, but many of their equations and series survive without change. From far earlier the deferents and epicycles of Apollonius, Hipparchus, and Ptolemy, temporarily discredited along with the Ptolemaic system by the successes of Copernicus and Kepler, have reappeared in modern perturbation theory in what we now refer to as Fourier series and harmonic analysis.

Space navigation requires a discipline, however, that is at once broader and narrower than celestial mechanics—broader in the inclusion of new forces, new tools, and new data, and narrower in the exclusion of certain portions that are useful in stellar and galactic dynamics. Accordingly we define astrodynamics as including the parts of celestial mechanics, geophysics, aerodynamics, exterior propulsion theory, electromagnetic theory, and observation theory that bear upon the trajectories of astronomical objects and space vehicles. In the general field of space navigation, astrodynamics is closely linked with communication and control.

In astrodynamics it is necessary at the start to distinguish clearly the qualitative orbits that are useful in design or feasibility studies, for general discussions and estimates, and for the calculation of rough ephemerides from the precision orbits that are necessary for space navigation and for obtaining improved geophysical data, etc. Much excellent work has been done on "feasibility" orbits, even to the inclusion of one or more disturbing objects or the earth's equatorial bulge. Such trajectory work, however, introduces many simplifying assumptions: that the moon is moving around the earth in a circle, or is stationary in a rotating system, that the earth may be represented by a simplified gravitational model, that the disturbing objects are in the vehicle's orbit plane, etc. Precision trajectory work cannot accept these simplifications. The earth departs in countless ways, great and small, from a sphere or ellipsoid; the coefficients of the terms that express these departures must be evaluated, along with the basic gravitational constant, with the highest accuracy attainable. The moon's orbit differs from a circle in such a complicated way that no simple expressions can define its motion with sufficient approximation for more than a very short interval. The positions of launch points, targets, and observers, and even the lunar ephemerides are

referred to a framework that is affected by precession and nutation; they are therefore only briefly an approximation to the inertial framework that is necessary as a reference in orbit work. In the handling of perturbations many alternatives present themselves for consideration before any one of them can be declared preferable or even adequate; and the relation between the dynamical trajectory and accurate observations, hardly even considered in connection with design orbits, becomes essential to the correction of orbits and to accurate prediction.

Even in "feasibility" orbit studies there are important alternatives to be considered. The possibilities of "inferential" methods, based on the known integrals of the two-, three-, and n-body problems, should be especially explored before "shotgun" calculations with high-speed computers are embarked upon. The latter technique has led to some useful and ingenious discoveries, but there have also been extensive calculation programs undertaken that could have been shown to be valueless by advance use of "inferential" methods.

5–2 KEPLER'S LAWS AND NEWTON'S MODIFICATIONS OF THEM

The three laws of Kepler are fundamental to a preliminary understanding of the orbit problem. These are as follows (Fig. 5–1):

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The straight line joining a planet to the sun sweeps over equal areas in equal intervals of time (the "law of areas").
3. The square of the period of a planet is proportional to the cube of its mean distance (the "harmonic law"). It is convenient to represent this law by the following formula,

$$P^2 = (2\pi/k)^2 a^3$$

where P is the period, a is the mean distance between sun and planet, and $2\pi/k$ is the factor of proportionality, written in this form to make a comparison with later expressions more convenient.

While Kepler and his laws were establishing a basic part of the foundation of what was to become celestial mechanics, Galileo, his contemporary, was discovering equally important terrestrial foundations of mechanics. Galileo and Kepler respected one another and even corresponded, but apparently their mutual respect did not go far enough for either to become really familiar with what the other was doing. Accordingly, it was left for Newton to combine the fundamental work of these two men in such a way as to discover the law

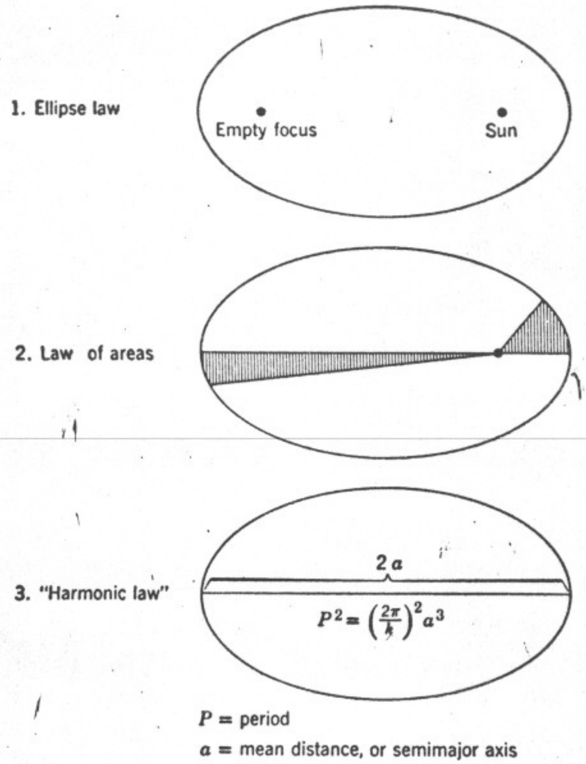

Figure 5–1 Kepler's laws.

of universal gravitation, the cornerstone of celestial mechanics. We may state this law as follows:

Every particle of matter in the universe attracts every other particle with a force that varies directly as the product of their masses and inversely as the square of the distance between them.

This law we may express by the following formula:

$$\mathbf{f}_{12} = -\mathbf{f}_{21} = k^2 m_1 m_2 \mathbf{r}_{12}/r_{12}{}^3$$

where m_1 and m_2 are the masses of the two particles, $\mathbf{r}_{12}$ is the vector distance measured from m_1 toward m_2, r_{12} is the magnitude thereof, $\mathbf{f}_{12}$ is the force acting on m_1 in the direction of m_2, and $\mathbf{f}_{21}$ is the equal and opposite force acting on m_2 in the direction of m_1, and finally k^2 is the factor of proportionality, the constant of gravitation that is sometimes represented by G.

With the law of universal gravitation and the three laws of motion that Newton had developed and expanded from a Galilean foundation, Newton was able to redevelop Kepler's laws. When he did so, he found five important modifications in the Keplerian concepts.

1. The motion of each planet will be disturbed ("perturbed") by the attraction of each of the others, so that it will depart from the orbit it would follow under the attraction of the sun alone. Figure 5–2 illustrates this kind of "perturbation." A comet, let us say, is traveling in orbit A, approximately a Keplerian orbit with the sun at one focus. On one of the occasions when it crosses the orbit of Jupiter it finds Jupiter nearby, and in accordance with the law of universal gravitation the attraction of Jupiter becomes momentarily very large, pulling the comet out of orbit A and hurling it off toward the sun in a new direction. Shortly thereafter the attraction of the sun again becomes predominant and the comet takes up orbit B, which is substantially another Keplerian ellipse with the sun at the focus. It is rare that the attraction of Jupiter has as great an effect as that illustrated, but much smaller effects are never absent. The attractions of Jupiter and the other planets are continually altering the orbits of the comets, of the minor planets, and of the major planets themselves. The Keplerian orbits that these objects would follow, if each were alone with the sun in space, are at best only approximations to the actual paths. Thus a distinction appears between the "two-body problem," including Keplerian orbits, and "perturbation theory,"

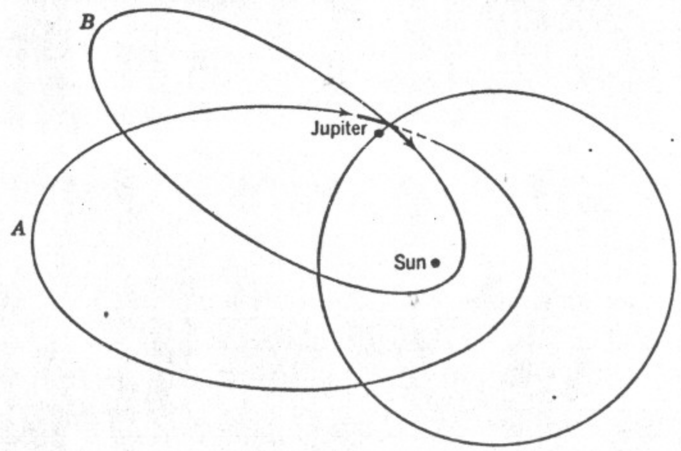

Figure 5–2 The two-body and three-body problems—perturbation.

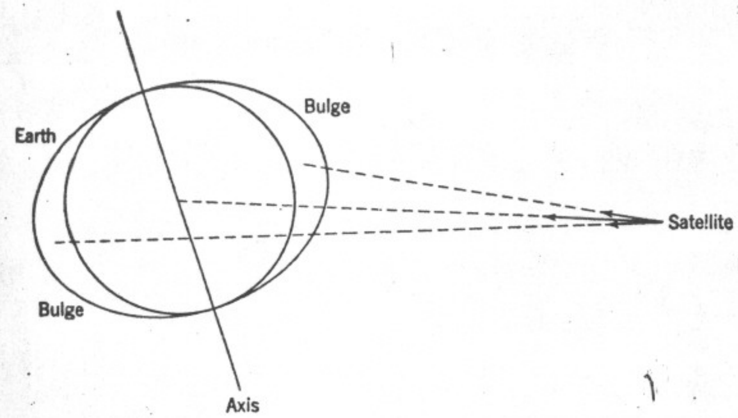

Figure 5-3 The perturbation due to an "equatorial bulge."

including the three-body and many-body problems. The two-body problem is usually, but not always, a good first approximation to the many-body problem.

2. The law of universal gravitation, and consequently Kepler's laws, insofar as Newton had verified them up to this point, applies only to particles. Newton was able to show, however, that the attraction for an exterior particle exerted by a spherical mass homogeneous in spherical concentric layers will be directly proportional to the total mass of the sphere and inversely proportional to the square of the distance of the particle from the sphere's center. The establishment of this principle was evidently important to Newton's verification of the law of universal gravitation, which he found in comparing the motions of the moon and the famous apple.

Now it happens that the sun, the earth, and the other planets are very nearly homogeneous in spherical concentric layers, so that they may be treated as point masses when we are considering their attractions on relatively distant objects. Because of its rotation, however, each of these objects has an equatorial bulge whose attraction on a nearby object, such as a satellite, must be taken into account as a perturbation. Thus the attraction of the earth's equatorial bulge produces an important set of perturbations in the motion of the moon; the reaction on the earth is one of the causes of precession and nutation.* For the moon the perturbations due to the earth's bulge are

* Precession and nutation are conical and sinusoidal angular motions of the earth's axis, respectively.

less than those due to the field of the sun, but for a nearby artificial satellite, the bulge influence is far more important (Fig. 5-3).

3. Any of the conic sections (Fig. 5-4) is a possible orbit for an object moving under the attraction of the sun alone. Thus comets, many of which move in orbits that are indistinguishable from parabolas, were shown by Newton to obey the same laws as the planets. Heliocentric hyperbolic orbits are rare, but when we deal with the motions of the meteorites that plow into the earth's atmosphere, or of rocket space ships that are attempting to leave the earth for voyages into space, we find that the portions of the orbits close to

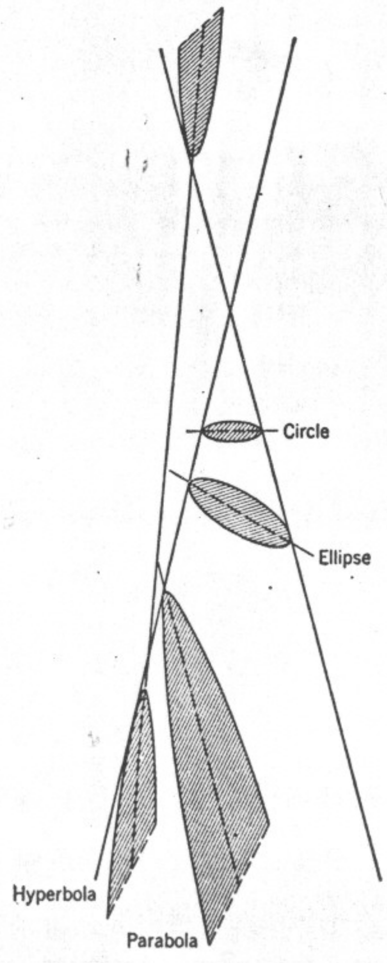

Figure 5-4 The conic sections.

the earth are very nearly representable by geocentric hyperbolas. The hyperbolic orbit, accordingly, will become increasingly important in the future.

4. Newton found that Kepler's second law was accurate without change, but in Kepler's third law he found that the constant of proportionality between the square of the period and the cube of the mean distance involved the sum of the masses of the two objects, m_1 and m_2, in a way not suspected by Kepler:

$$P^2 = (2\pi)^2 a^3 / k^2 (m_1 + m_2)$$

For the planets m_1 represents the mass of the sun and m_2 the mass of the individual planet, which is usually negligible in comparison. Accordingly, the modification that Newton found did not seriously affect Kepler's third law as it applied to the planets. Far more important was the fact that it could now be applied to the motions of satellites about planets. In these circumstances m_1 represents the mass of the planet and m_2 the mass of the satellite, which is usually as negligible in comparison with the mass of the planet as is the mass of the planet in comparison with that of the sun. The factor $m_1 + m_2$ now appears in many of the equations of the two-body problem in such a way as to make them all applicable to planetocentric orbits when the perturbations are not too great.

5. Kepler's laws appear as integrals of the two-body problem. There are, of course, many other such integrals. All of them provide us with equations that are useful in precision orbit work. Some of them are of such simple character as to be about as useful as Kepler's laws in "feasibility" studies. Of these the most conspicuous is the *vis viva* integral or energy integral,

$$V^2 = k^2 (m_1 + m_2) \left(\frac{2}{r} - \frac{1}{a} \right)$$

where V is the velocity of m_2, r is its distance from m_1 (previously designated by r_{12}), and k, m_1, m_2, and a have the significances previously assigned to them. This formula expresses the fact that the sum of the kinetic and potential energies is constant; V^2 is proportional to the kinetic energy and the $2/r$ term represents the potential energy.

An illustration of the usefulness of the *vis viva* integral is found in Fig. 5–5, which shows the orbits of a set of particles projected from the same point with the same velocity. Since V and r are the same for each of these orbits, it follows that the semimajor axis, a, will also

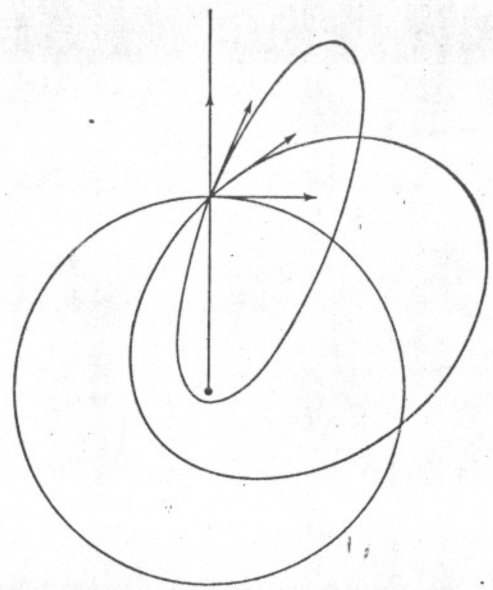

Figure 5–5 Covelocity orbits.

be the same for each. Let us suppose that the circular orbit is the orbit of a satellite of the earth projected horizontally and just above its surface. The velocity will then be slightly under 5 miles/sec. If the angle of projection is somewhat above the horizontal, the projectile will rise to a height and then fall back to the surface, with a hypothetical portion of the orbit buried inside the earth. If the angle of projection is higher, the particle will rise to a greater height but come back nearer to its starting point, and if the angle of projection is 90°, the particle will rise straight up and fall straight back. For each of these orbits, of course, we are neglecting the rotation of the earth as well as the effect of drag. It follows that the height to which the vertically projected object will rise is equal to the radius of the earth.

The conic sections are illustrated from another point of view in Fig. 5–6, in which we have a family of orbits with the same tangent but different velocities at their common point. Let us suppose for illustration that the circle represents the orbit of the earth, which is very nearly circular. On this assumption the earth's velocity of 18.5 miles/sec is directed at right angles to the sun. The effect of

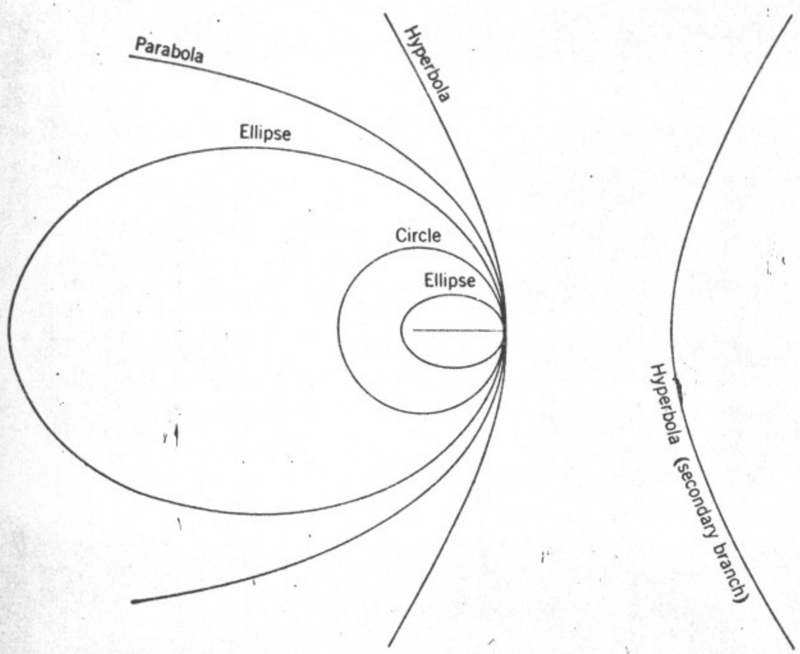

Figure 5-6 Cotangential conic orbits.

the sun's attraction is to cause the earth to fall $\frac{1}{9}$ in. toward the sun in the same second that it travels 18.5 miles along the tangent. The combination of the two is just right to bring the earth back to the same distance that it had before. And so, second by second, the earth falls toward the sun but never gets any closer.

Let us suppose that the earth's velocity is slowed down to about 11 miles/sec. The sun's effect, of $\frac{1}{9}$ in., will be unchanged, and so the earth will fall closer to the sun, taking up as a path the smaller ellipse shown in Fig. 5-6. As it falls toward the sun it will gain in velocity, until at the point opposite the start it will have gained so much that the centrifugal force will exceed the gravitational pull of the sun, even though the latter has increased also. At this point, then, the earth will begin to climb away from the sun, slowing down as it goes until it arrives back at the same point with the same velocity in the same direction.

If we could slow the earth down to zero, it would fall $\frac{1}{9}$ in. toward the sun in the first second and considerably more in ensuing seconds, speeding up until it reached the sun in about 2 months. This, happily, is one of the less likely ways in which the earth will come to an end. Other causes will more strongly influence the earth's longevity.

If we should speed the earth up to about 24 miles/sec, the $\frac{1}{9}$ in. by which it would fall to the sun would be insufficient to bring it back to the same distance, and it would find itself climbing away from the sun in the larger of the two ellipses shown in Fig. 5-6. Slowing down to the point opposite the start, it would find there that its centrifugal force was insufficient to overcome the gravitational attraction, though the latter would be less also. Accordingly, the earth would begin to fall back toward the sun, regaining speed as it went until it reached the same point with the same velocity as before.

An increase in the earth's velocity to 26 miles/sec would carry it off on the parabola shown in Fig. 5-6. Still greater velocities would carry it away from the sun along a hyperbolic orbit such as the one shown. In either case the attraction of the sun would be insufficient to slow the earth down enough for it to be caused to return.

5-3 TWO-BODY FORMULAS

The "elements" of the two-body orbit of an object are a set of six independent constants that specify the orbit's orientation in space, its size and shape, and the position occupied by the object at a specified time or the time at which it is at a specified point. For geocentric orbits such a set (Fig. 5-7) consists in part of the three "orientation" elements:

- Ω, the longitude of the node, measured in the plane of the equator from the direction of the vernal equinox to the direction of the ascending node, or intersection of the orbit with the equator—for heliocentric orbits substitute "ecliptic" for "equator,"
- i, the inclination, or angle between the plane of the orbit and the plane of the equator,
- ω, the argument of perigee, or angle between the direction of the ascending node and the direction of the perigee—"perihelion" for heliocentric orbits.

The remaining three elements are the "dimensional" elements:

- a, the semimajor axis or mean distance,
- e, the eccentricity, or the ratio of the distance from the center of the orbit to the focus (the center of the earth) to the mean distance, and
- T, the time of perigee passage.

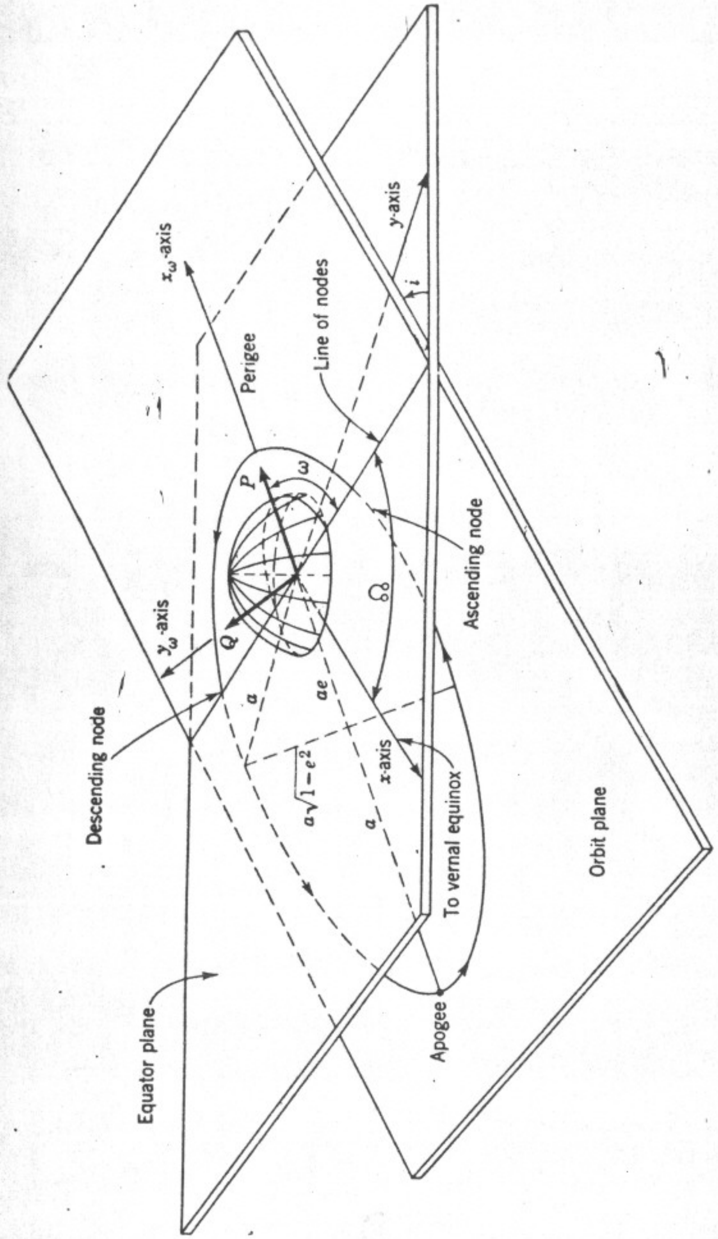

Figure 5–7 The elements of a geocentric orbit.

These six constants are often replaced by others, in part or altogether. For example, the orientation unit vectors,

P, directed to perigee,
Q, parallel to the velocity vector at perigee, and
W, perpendicular to the orbit plane and making up a right-handed system with **P** and **Q**, are often used as orientation elements in place of Ω, i, and ω.

Sometimes used in place of a or e are (Fig. 5–8):

$q = a(1 - e)$, the perigee distance, or
$p = a(1 - e^2)$, the "parameter" or semi-latus rectum, or
$n = k_e \sqrt{m_1 + m_2}/a^{3/2}$, the mean angular velocity, or "mean motion," where k_e is the gravitational constant used with geocentric orbits and m_1 and m_2 are the masses of earth and vehicle, respectively.

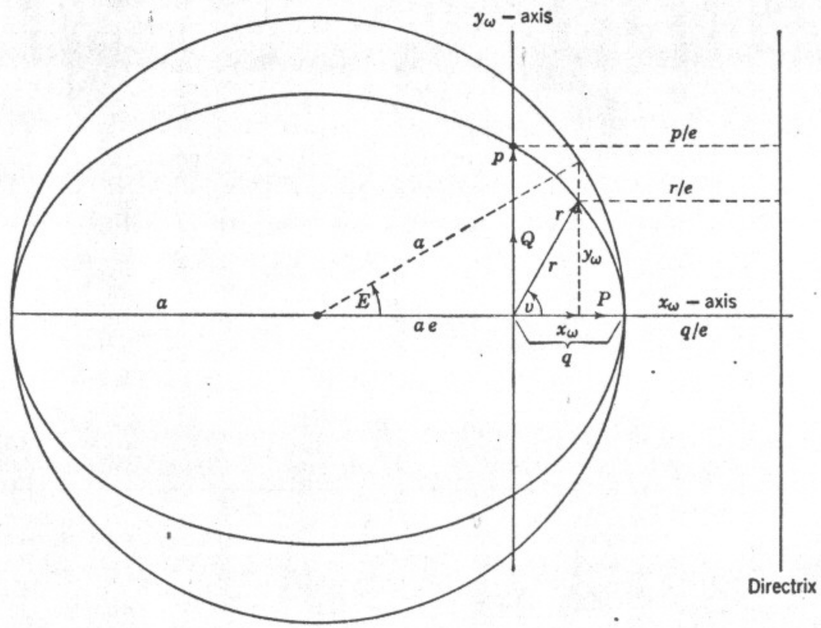

Figure 5–8 Coordinates in the plane of a two-body elliptical orbit

The elements of the orbit serve to tie together variables that include (Figs. 5–7, 5–8):

x, y, z, rectangular coordinates referred to the equator and equinox,
$x_\omega, y_\omega, z_\omega$, rectangular coordinates referred to the orbit plane and perigee,
r, the radius vector,

and the three angles or "anomalies,"

v, the "true anomaly,"
E, the "eccentric anomaly," and
M, the "mean anomaly."

To illustrate some of the varied formulas that are useful in orbit work, we collect those that are used to calculate x, y, and z, from the elements a, e, and T, and the components of **P** and **Q**, given also t and n:

$$M = n(t - T)$$

$$E - e \sin E = M \qquad \text{This is "Kepler's equation," which must be solved here by successive approximations.}$$

$$\left.\begin{aligned} x_\omega &= a(\cos E - e) \\ y_\omega &= a\sqrt{1 - e^2} \sin E \end{aligned}\right\} \text{Fig. 5–8}$$

$$\left.\begin{aligned} x &= x_\omega P_x + y_\omega Q_x \\ y &= x_\omega P_y + y_\omega Q_y \\ z &= x_\omega P_z + y_\omega Q_z \end{aligned}\right\} \text{Fig. 5–7}$$

There is a similar set of formulas for the hyperbola. There are allied formulas to obtain x, y, and z, or the total velocity, angle of elevation, etc. There are alternative formulas and check formulas. Choice between them is often dictated by simplicity, by special circumstances such as small inclination, small eccentricity, or regression of the nodes due to perturbations, or by the nature of the initial conditions. The preceding definitions will help the engineer or physicist translate the language of classical astronomy. For detailed techniques of orbit calculation, the reader is referred to the references at the end of this chapter.

5–4 PERTURBATION THEORY

Perturbations are usually thought of as being those parts of the accelerations of an object that cannot be accounted for by a simple inverse-square central force field. If the perturbations were eliminated, the object would move in a simple conic orbit defined by the

usual two-body elements. Such an orbit may be taken as a reference for the integration of the perturbations in determining the actual path. Two-body reference orbits may be replaced on occasion by reference orbits based on other integrals than those of the two-body problem. These integrals may be based on the three-body problem, on special integrals that take into account some of the effects of the earth's equatorial bulge, or on gravity-free drag orbits. The purpose of the reference orbit, whether it is used for a long interval of time or a short one, is the reduction in size of the accelerations that have to be integrated.

Often the perturbing forces may be reduced greatly by relatively simple devices. For example, the sun's direct attraction at the position of the moon is about twice that of the earth, and so, if the earth and the moon were not in motion, the sun would pull the moon away from the earth. The sun's attraction for both the earth and the moon, however, acts primarily to keep them moving along curvilinear orbits instead of along tangents (Fig. 5–9). Only the difference between the sun's gravitational field at the position of the moon and the sun's field at the position of the earth is left over to serve as a perturbing force on the moon, provided the moon's motion is regarded as geocentric. It is by the device of shifting from the sun's central force field to the earth's central force field that this reduction of the perturbations is accomplished. The perturbing force is still about 1/100 of the earth's attraction, however; it is small by comparison but still a very large perturbation as perturbations go. The result is the complicated motion of the moon that was referred to in Section 5–1. For satellites close to the earth the attraction of the sun approximates its attraction for the earth, so that the perturbing effect will be far less than it is on the moon. The effect of the earth's equatorial bulge, on the other hand, will be the greater because of that same proximity. Other perturbing effects may be supplied by thrust, drag, other aero-

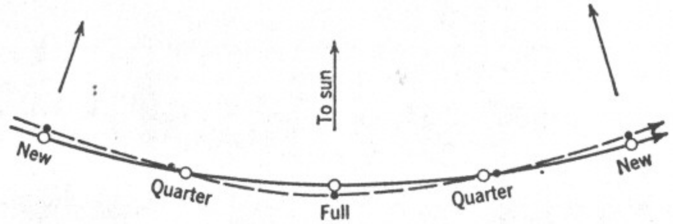

Figure 5–9 The moon's heliocentric motion.

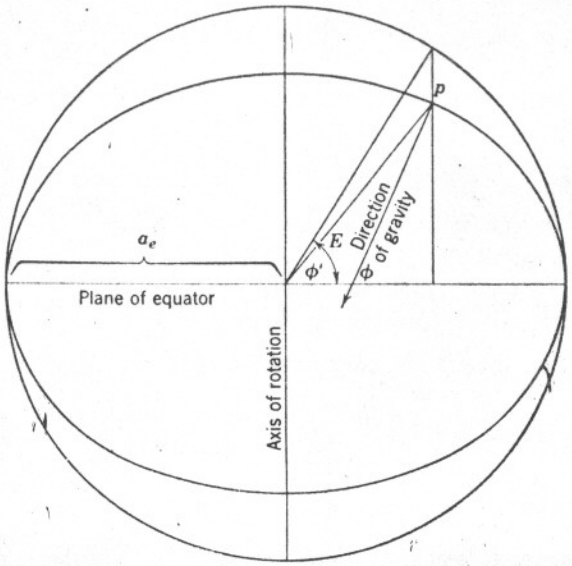

Figure 5–10 The earth's equatorial bulge and the direction of gravity.

dynamic forces, electromagnetic forces, radiation pressure, the difference between Einstein and Newtonian gravitation, etc.

The primary gravitational effects of the earth, the moon, the sun, and the planets are such that we think of the masses of the attracting objects as concentrated at their centers, so that the resulting acceleration terms are directly obtainable from the inverse-square law of universal gravitation. But for each of these terms we must determine the best available value of a constant coefficient that is essentially the constant of gravitation multiplied by the mass of the object. If the attracting object is fairly near, as is the earth for a satellite, it must be represented by an ellipsoid of revolution, with its mass distributed homogeneously in ellipsoidal concentric layers, instead of the spherical ones that enable us to consider that all the mass is at the center. The attraction of the earth, assumed to be an ellipsoid of revolution, may be developed in series, evaluating the coefficient of the second harmonic along with the primary, or zero-order, attraction that is obtained when the earth is assumed to be spherical. This coefficient is closely related to the "flattening" or "ellipticity" or "oblateness" of the earth, and to the coefficient of the principal latitude

term in the expression for the acceleration of gravity at sea level. If the satellite is very near the earth, and if the highest precision is needed, we take into account the coefficient of the fourth harmonic in the attraction of the ellipsoid of revolution, and perhaps also corrections to this figure ranging down to "local anomalies" due to inhomogeneities in the earth's crust. Each correction must be expressed by a mathematical term of selected form and will contribute a coefficient to be evaluated. So also must thrust, drag, etc., be approximated by mathematical expressions for the actual accelerations encountered, and each will require the determination of numerical coefficients.

Another device that will reduce the perturbation terms is found in shifting the center of attraction that is to be used in connection with the two-body reference orbit away from the center of figure of the attracting object. For example, the path of an object in free fall in the vicinity of the earth will be along the direction of gravity (Fig. 5–10), rather than toward the center of the earth. We find, accordingly, that the perturbation terms will be reduced if we remove the reference center of attraction to some point on the line defined by the direction of gravity. The same device will be useful in connection with ICBM trajectories.

One of the well-known effects produced on a satellite orbit by the perturbations of the bulge at the equator of the earth is the regression of the nodes. If a satellite is moving in a predominately eastward direction (Fig. 5–11), it will cross the equator at each passage a little

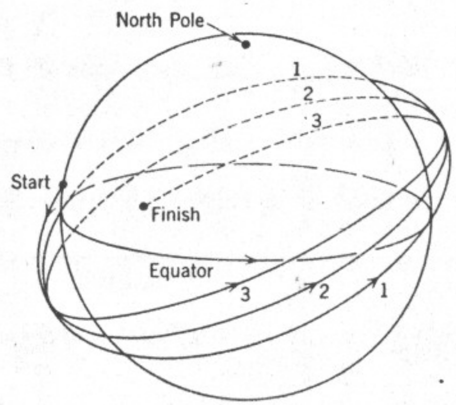

Figure 5–11 The regression of the nodes.

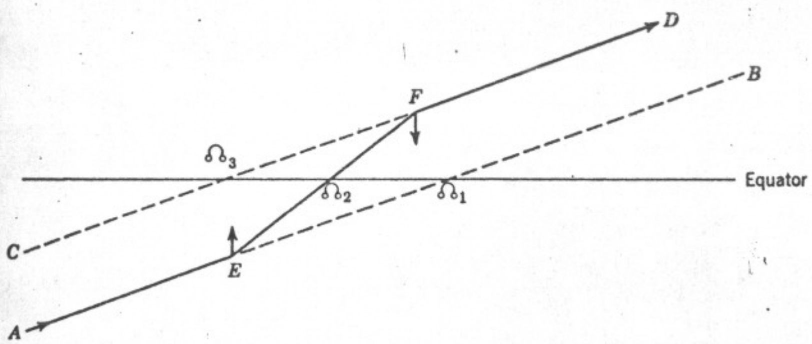

Figure 5–12 Exploration of the regression of the nodes.

bit farther west than the time before. This regression of the nodes is in addition to any apparent westward motion due to the eastward rotation of the earth beneath the orbit. Figure 5–12 supplies us with an explanation of this phenomenon. Let us suppose that a satellite is proceeding along a path from A to B, so that it would cross the equator at $☊_1$ if there were no perturbations. Let us now suppose the effect of the attraction of the bulge to be simplified into an instantaneous acceleration that acts on the satellite when it reaches point E. As a result, it crosses the equator at $☊_2$ and proceeds to F, where a compensating simplified acceleration from the bulge diverts the satellite into the path CD, which crosses the equator at $☊_3$. The effect of the perturbations of the bulge, accordingly, is to make the node regress from $☊_1$ to $☊_3$.

The effect of drag on a satellite orbit may be illustrated by Fig. 5–13. Let us suppose that AA_2 is the satellite orbit and that drag is concentrated in a small region near the perigee A. The effect of the drag is, of course, to diminish the velocity at A, with the result that the satellite does not succeed in rising up to A_2, but only to A'_2 (compare the *vis viva* integral of Section 5–2). Since the semimajor axis is diminished, the period will also be diminished, in accordance with Kepler's third law, and the satellite will return to A sooner than it would otherwise. Whereas the velocity at A is diminished, the average velocity is actually increased.

Figure 5–13 may also be used to illustrate the basic problem that will be encountered when establishing a satellite in a 24-hour equatorial orbit for communication purposes. Let AA_2 represent the orbit actually achieved. Because of the inexactness of the burnout velocity in both magnitude and direction, it will depart from the desired orbit

in two principal ways: (*a*) It will not be perfectly circular, and (*b*) it will not have a period of exactly 24 hours. The effect of a small eccentricity will not be disastrous but will merely cause the satellite to oscillate back and forth with reference to a fixed point on the equator on the surface of the earth. Additional oscillations will be introduced by perturbations. But an error in the period will cause the satellite to depart indefinitely from the desired equatorial reference point, so that it will eventually become useless for a communications base, or for whatever similar purpose the 24-hour orbit is desired.

To adjust the period, corrective thrusts will be necessary. At the same time we might well seek to reduce the eccentricity of the orbit in order to reduce the oscillations of the satellite. If the period is too great, we will accordingly make our corrective thrust at A, reduc-

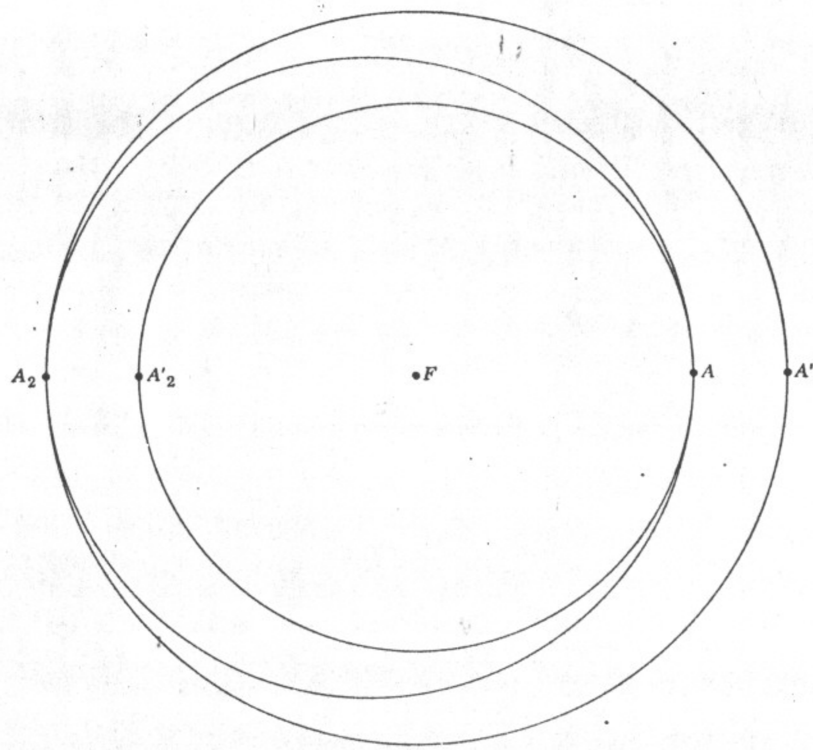

Figure 5–13 Perturbative effects of thrust or drag on eccentricity e, mean distance a, and period P.

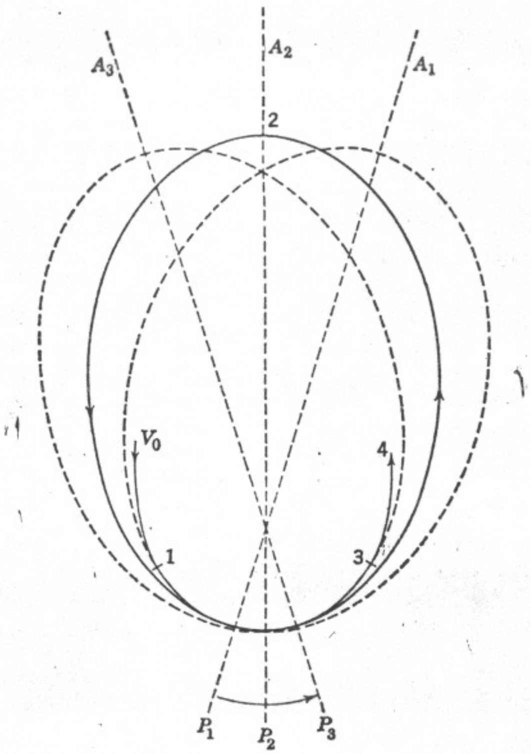

Figure 5–14 Advance of perigee or perihelion (rotation of the line of apsides).

ing both velocity and eccentricity. If the period is too short, however, we will make the corrective thrust at A_2 in a way that increases the velocity, launching the satellite into the larger circular orbit with resulting increase of the period.

Still another perturbation, the advance of perigee, is illustrated by Fig. 5–14. (In heliocentric orbits this is the well-known advance of perihelion that is partly due to Einstein effects.) The actual path of the object is indicated by positions 0, 1, 2, 3, 4. At position 1 let us calculate an "osculating" elliptic reference orbit, i.e., an elliptic orbit that has the same position and velocity as the actual orbit. This is the orbit the object would then follow if all perturbations ceased. Its perigee and apogee are on the line P_1A_1. By the time the object reaches position 2, at its apogee, the reference orbit will have the perigee and apogee on the line P_2A_2. Finally at position 3

let us determine a new osculating elliptic reference orbit, which in this instance will have its perigee and apogee along the line P_3A_3. The perigee, accordingly, will have advanced through the arc P_1P_3.

5–5 PERTURBATION CALCULATIONS

The perturbative accelerations may be integrated numerically, or they may be expanded into series and integrated term by term. The numerical integration process is known as "special perturbations," and the series integration method is known as "general perturbations." The latter procedure can be handled only by successive approximations, since the series expressions initially integrated must be based on some kind of zero-order approximation. The result will be "first-order perturbations," which are in effect series that can be resubstituted into the basic expressions to obtain "second-order perturbations." Numerical integration will achieve similar results if the accelerations initially integrated are obtained, approximately, from a zero-order approximation. But in numerical integration we may evaluate the integrals at each step and use these, instead of the zero-order approximation, for the calculation of the accelerations. The step-by-step process, then, yields accurate results except for the steady accumulation of error in the summation processes involved in integration. For satellite integrations, with so many revolutions achieved in such a short space of time, the accumulation of error is disastrous. General perturbations, however, have the disadvantage of requiring a great many terms to achieve the same accuracy. It is probable that the best means of handling satellite orbits will be based on some combination of the two techniques, with the aid of improved approximations beyond the two-body reference orbit.

Whether we seek to integrate special perturbations or general perturbations, we must first make a decision about what we shall integrate. Three principal alternatives present themselves.

1. We may integrate the sum total of the accelerations, with no reference to an osculating or other reference orbit. Such an integration is not strictly a perturbation method, since the term perturbation implies that a distinction is made between the principal terms and the perturbation terms in the accelerations. If we integrate the total accelerations, such a distinction would be valueless. By convention the integration of the total accelerations is nevertheless referred to as a perturbation method and is known as Cowell's method. It was first used by Cowell and Crommelin in their prediction of the return

of Halley's comet in 1909. Most lunar trajectories have been integrated by this method, although it is easy to show that other methods are preferable. The method is applicable only to special perturbations, since general perturbations require the use of a reference orbit. In Fig. 5-14 Cowell's method would be represented by the heavy line or actual path only and would make no use of the osculating elliptic orbits there represented.

2. We may integrate the departures from the osculating reference orbit. In special perturbations the principal representative of this process is called Encke's method. Starting at position 1 in Fig. 5-14 we would calculate successive positions, probably equally spaced in time, in the reference ellipse whose major axis is indicated by P_1A_1. We would then integrate the perturbative accelerations into perturbative displacements, with which to correct the two-body positions in the reference orbit to the actual positions in the actual path. By the time we reached position 3 we would probably find that the actual position was so far from the reference two-body positions that the perturbative accelerations, which would include terms resulting from the displacement as well as terms introduced by the non-two-body forces, would be as large as the total accelerations. Then Encke's method would no longer be advantageous as compared with Cowell's method. We would then calculate a new osculating reference orbit, with major axis P_3A_3, and start over with our perturbations greatly reduced in size.

3. The method of variation of parameters avoids the gradual increase in the perturbation terms that are due to the increasing displacement of the actual position from the reference position, thus avoiding the periodic large corrections in the reference orbit. It accomplishes these ends by causing the reference orbit to vary gradually in such a way that it always yields exactly the same position and velocity as those associated with the actual path. That is, the varying reference orbit is always "osculating." The constant elements of the two-body problem become varying parameters defining the varying orbit. The variations of the parameters are determined directly from the perturbative accelerations.

When the perturbations are very large, neither Encke's method nor the method of variation of parameters offers any advantage over Cowell's method, and the last should be used because it requires less calculation. When the perturbations are small, however, and especially when the two-body motion is very rapid, Cowell's method is disadvantageous and may even be incapable of handling the problem.

This circumstance we encounter with the orbit of the minor planet Icarus. This little planet comes closer to the sun than any other, passing about halfway between the orbit of Mercury and the sun. When it does so, it moves very rapidly and its two-body accelerations become large and unmanageable. Table 5-1 indicates what one of its integration tables would look like if we attempted to use the same 10-day interval that can be used at aphelion. The table is clearly quite unusable, and the interval would have to be cut to a small

TABLE 5-1

Cowell's Method versus the Method of Variation of Parameters

Cowell's method

	$\ddot{x}$	$\delta\ddot{x}$	$\delta^2\ddot{x}$	$\delta^3\ddot{x}$
June 13	− 1.00539		− 0.27914	
		− 0.56978		− 0.54620
June 23	− 1.57517		− 0.82534	
		− 1.39512		− 2.21389
July 3	− 2.97029		− 3.03923	
		− 4.43435		+34.74936
July 13	− 7.40464		+31.71013	
		+27.27578		−75.06246
July 23	+19.87114		−43.35233	

Variation of parameters

	$\dot{a}_y$	$\delta\dot{a}_y$	$\delta^2\dot{a}_y$	$\delta^3\dot{a}_y$
June 13	−0.000,0023		− 7	
		+ 1		+ 1
June 23	−0.000,0022		− 6	
		− 5		+16
July 3	−0.000,0027		+10	
		+ 5		+15
July 13	−0.000,0022		+25	
		+30		−58
July 23	+0.000,0008		−33	

fraction of a day before the integration could be carried through. At such an interval the accumulation of error in the integration would be prohibitive. By contrast the variations of the parameters, of which one is shown, are very small, even though two more decimal places are carried than in the Cowell table. It will be observed, however, that the same oscillation of sign begins to appear in the variation of parameters table, requiring careful calculation at a reduced interval to hold the error down to an acceptable standard for precision work.

The same disadvantage of Cowell's method appears in lunar flight trajectories. At the start the perturbations of the moon are actually quite negligible, so that by Encke's method or the variation of parameters a simple two-body orbit could be used in the early stages. By Cowell's method the whole integration must be performed at extremely short steps because of rapid changes which would be taken fully into account by a two-body orbit if it were used.

5–6 OBSERVATIONS AND CORRECTION

Since purely ballistic trajectories are subject to rather large uncertainties because of burnout errors in position and velocity, accurate observations become an essential part of the programming of subsequent corrective thrusts.

In this connection we must think of astronomical rather than navigational accuracy, i.e., of observations that are good to $0''.1$ rather than to $1'.0$. Such accuracy will require differential measures of the position of the vehicle against a stellar background. Thus far attempts to get anything approaching this accuracy in satellite observation work have been near failures, if not complete failures. The ballistic cameras along the Cape Canaveral chain have been more successful, and claims of observational accuracy down to about $2''$ have been made. Observations of such an order of accuracy must obviously be corrected painstakingly for refraction, or at least for differential refraction, aberration, and various other physical or instrumental errors.

Since the accuracy of $2''$ corresponds to 1 part in 10^5, these optical observations offer a stirring challenge to modern electronic means of observation. But it is much to be hoped that electronic observations of radial distance and velocity, that is, "slant range" and "slant range rate," may eventually achieve the accuracy of astronomical observations. They offer much to the simplification of preliminary orbit determinations and to the accuracy of precision orbits. Rocket-

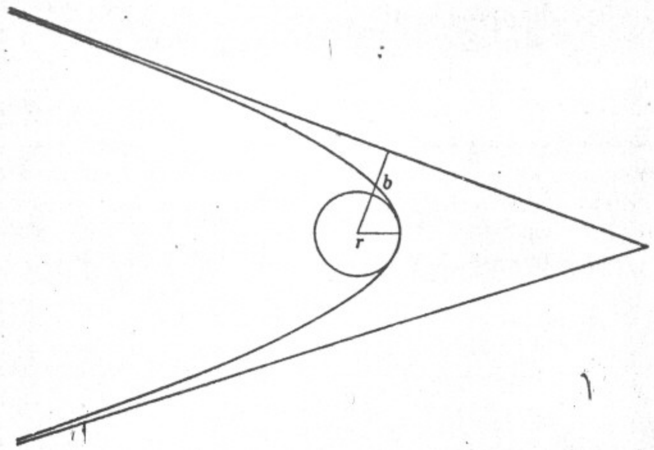

Figure 5-15 Effective collision radius.

unit when we shift to geocentric orbits. The distinction between the equatorial radius and a laboratory unit related to the centimeter, however, is not so serious as in heliocentric orbit problems. The geocentric gravitational constant is determined from the acceleration of gravity, as a more accurate source of information than even the motion of the moon, and must take into account very accurately the effect of the earth's rotation and of the resulting equatorial bulge.*

5-8 INFERENTIAL METHODS VERSUS EXTENSIVE CALCULATION

In preliminary studies we may make effective use of two-body or three-body integrals and infer results that might at first seem to require extensive calculation. One of the useful tools in inferential work is the "effective radius" (Fig. 5-15). If a two-body trajectory, ignoring the effect of the moon's attraction, indicates that a rocket will pass by the moon in a straight line at the distance b, the effect of the moon's attraction may often be approximated by the hyperbola to which the straight line is an asymptote. If the minimum distance in the hyperbola, r, is the actual radius of the moon, the moon's

* S. Herrick, R. M. L. Baker, Jr., and C. G. Hilton, "Gravitational and Related Constants for Accurate Space Navigation," *Proceedings of the Eighth International Astronautical Congress, Barcelona, 1957*, Vienna, Springer-Verlag, pp. 197-235, 1958; American Rocket Society Preprint, No. 497-57; *U. C. L. A. Astron. Papers*, **1**, No. 24, 297-338 (1958).

effective radius is b. The moon as a target is made larger, with an effective radius from one to three times its actual radius, depending on the velocity with which the rocket approaches the moon. If our calculations take into account the moon's attraction, of course, the closest approach should be compared with the moon's actual radius rather than its "effective radius."

The calculational technique, as opposed to the inferential one, has been very effective in disclosing previously undisclosed facts. One of these discoveries is illustrated by Fig. 5-16. Let us suppose that the design orbit for a trip to the moon is the intermediate one of the three shown, with a planned encounter with the moon at T_3. If the actual velocity is greater than the design velocity, the rocket will travel in the outer one of the three orbits, arriving at the moon's distance earlier and encountering the moon at T_2. If, however, the

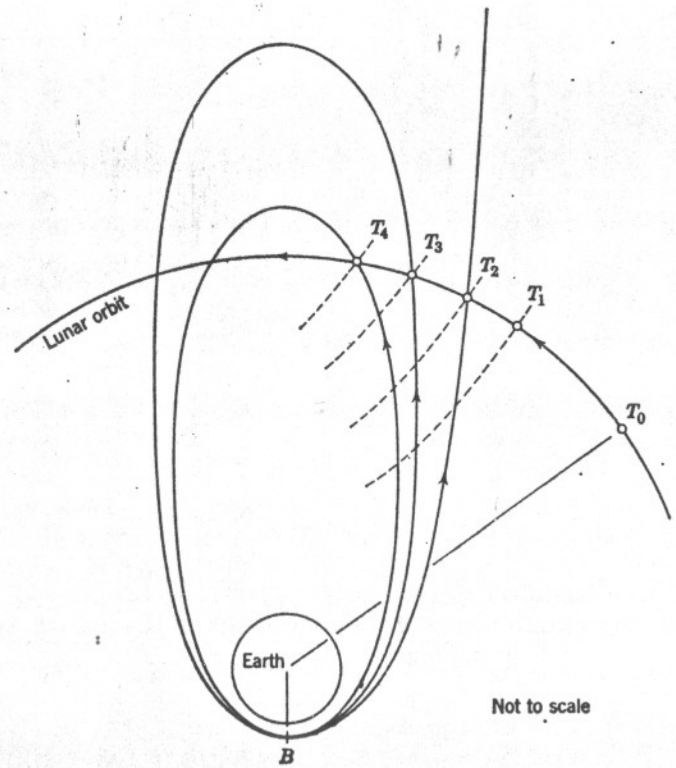

Figure 5-16 Lunar impact geometry in lunar orbit plane.

Figure 5-17 Paths followed during extraction of orbital energy from the moon by an object passing nearby.

actual velocity is less than the ign velocity, the rocket will travel on the smaller of the elliptic os, arriving at the moon's distance later and encountering the moot T_4. Thus the range of velocities that permits impact on the mois larger than would otherwise be the case, because the moon's mn compensates for the spacing of the trajectories. This effect wascovered by a number of men, but probably first at the RAND Coration by H. Lieske; the actual drawing in Fig. 5-16 was dony L. G. Walters of Aeronutronics Systems, Inc. The region in vh impact could be made is exaggerated by the drawing.

Sometimes inferential methocre more effective than calculation, however, and give warning thaxtensive calculation would be unproductive. Figure 5-17 illusies such a circumstance. It was proposed that a rocket sent up e by the moon could be swung by the moon into an interplanet orbit, with augmentation of its energy. In fact, the most that be gained from the moon is twice its velocity of about 1.2 miles/. Unfortunately, the problem requires very careful maneuverin that the moon will be passed at which the earth's attraction is balanced by the moon, with a zero velocity, it would simply be unaffected by the moon's sphere of primary attraction, which would pass on, leaving the vehicle to fall back toward the earth. The only place to enter the region of the moon's primary attraction with a zero geocentric velocity would be directly ahead of the moon, in a sort of circular window whose radius would be equal to the moon's effective radius. Then the hyperbolic orbit would encounter the surface of the moon.

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CHAPTER 6

THE VANGUARD IGY EARTH SATELLITE LAUNCHING TRAJECTORIES AND ORBITS

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6–1 THE IGY SATELLITE LAUNCHING PROBLEM

The earth satellites established during the International Geophysical Year will increase our knowledge of the world around us. The selection of the orbits of these satellites is governed by various factors. Some of these have to do with the satellite launching vehicles. Others are related to the needs of the satellite tracking programs. Still others are functions of the needs of the scientific experiments to be conducted in the satellites.

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