Analytical Estimates for Optimum Transfer Paths

H. MUNICK[1] and R. McGILL[2]

Grumman Aircraft Engineering Corp., Bethpage, N. Y.

THIS paper essentially extends the work of Lawden (1).[3] The problem considered is the minimization of characteristic velocity, or the amount of fuel consumed, using two impulses.

Lawden studied the case of transfer from an elliptic orbit to a higher energy circular orbit. Consider a space vehicle traveling in an elliptic orbit with velocity components (u, v) resolved perpendicular to and along the radius vector. Suppose at some point on this elliptic orbit the space vehicle is at a distance α from the focus, with velocity components (u_0, v_0).

Suddenly an impulse is applied, giving rise to new components of velocity (u_1, v_1). The vehicle then goes into a new elliptical orbit, the transfer ellipse. Upon arrival at the higher energy circular orbit, a second impulse is applied, correcting the space vehicle's velocity components to those of a circular orbit at a distance β from the focus. The characteristic velocity is given by

$$\sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2} + \sqrt{[u_2 - (\mu/\beta)^{1/2}]^2 + v_2^2} \quad [1]$$

Lawden then defines dimensionless parameters x, y by

$$u_1 = x\sqrt{\mu/\alpha} \qquad v_1 = y\sqrt{\mu/\alpha} \quad [2]$$

$$u_0 = x_0\sqrt{\mu/\alpha} \qquad v_0 = y_0\sqrt{\mu/\alpha} \quad [3]$$

This transformation reduces the problem to minimizing

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{x^2 - 2r^{3/2}x + y^2 + 3r - 2} \quad [4]$$

subject to

$$(1 - r^2)x^2 + y^2 \geq 2(1 - r) \quad [5]$$

where $r = \alpha/\beta$.

For the case where r is approximately unity, an analytic estimate can be given. An example of this is transfer from anywhere on a nearly circular orbit to a higher energy neighboring circular orbit (see Fig. 1). A second example is transfer from the near apogee region of an elliptic orbit to a higher energy circular orbit (see Fig. 2).

Case 1

Consider a small quantity $\epsilon > 0$. Let

$$r = 1 - \epsilon \quad [6]$$

Rewrite Equation [4] as

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(x - r^{3/2})^2 + y^2 + (3r - 2 - r^3)} \quad [7]$$

Using Equation [6], and omitting terms in ϵ of order higher than one, Equations [7 and 5] become

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{[x - 1 + (3/2)\epsilon]^2 + y^2} \quad [8]$$

$$x^2 + (y^2/2\epsilon) \geq 1 \quad [9]$$

Equations [8 and 9] can be given a geometric interpretation. The first member of [8] represents the distance from (x, y) to (x_0, y_0), the second the distance from (x, y) to $[1 - (3/2)\epsilon, 0]$. The constraint given by Equation [9] says (x, y) must lie on or outside the ellipse given by the equality in Equation [9]. As is presented in Fig. 3, any point on the dashed line connecting (x_0, y_0) to $[1 - (3/2)\epsilon, 0]$ gives the minimum characteristic velocity. Once (x_{opt}, y_{opt}) is found $(u_{1,opt}, v_{1,opt})$ is given by Equation [2]. From Lawden's results the second impulse is determined.

A second case of interest occurs for r slightly greater than unity (see Fig. 4).

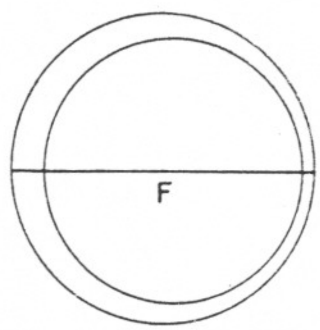

Fig. 1 Transfer from nearly circular orbit to a circular orbit

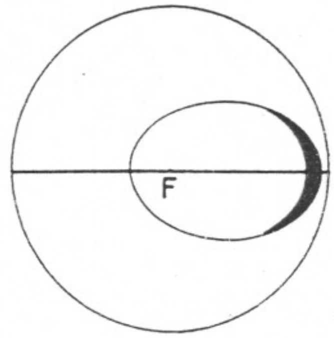

Fig. 2 Transfer from near apogee region to a circular orbit

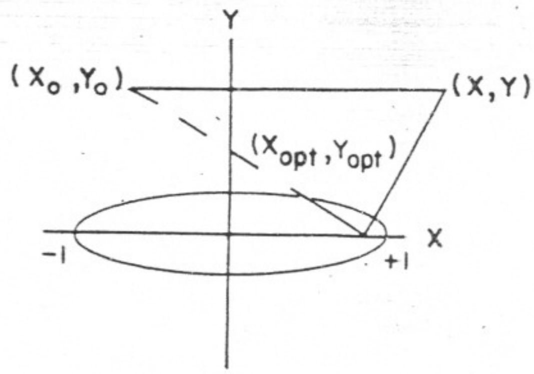

Fig. 3 Optimum solution for r slightly less than unity

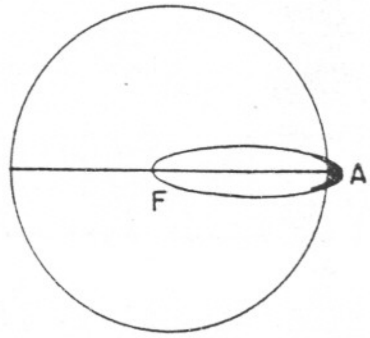

Fig. 4 Transfer from outside of circular orbit to circular orbit

Received June 4, 1959.

[1] Dynamics Group, Research Mathematician. Member ARS.

[2] Systems Section, Research Dynamicist.

[3] Numbers in parentheses indicate References at end of paper.

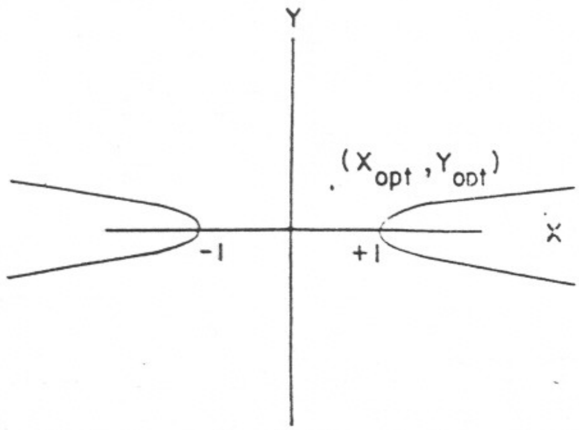

Fig. 5 Transfer for r slightly greater than unity

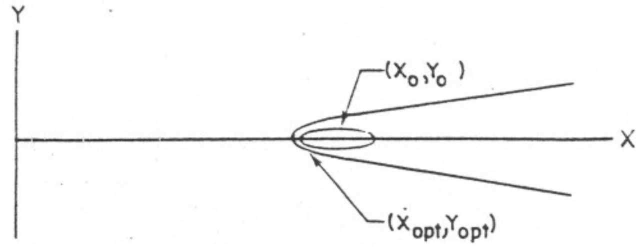

Fig. 6 Optimum transfer for (x_0, y_0) lying inside branch of hyperbola

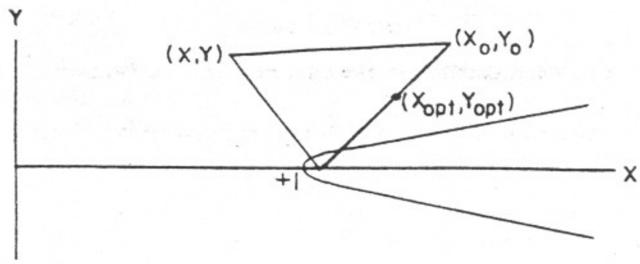

Fig. 7 Optimum solution for (x_0, y_0) lying outside branch of hyperbola

Case 2

Consider a small quantity $\epsilon > 0$

$$r = 1 + \epsilon \qquad [10]$$

To a first-order approximation Equations [8 and 9] become

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{\{x - [1 + (3/2)\epsilon]\}^2 + y^2} \qquad [11]$$

$$x^2 - \frac{y^2}{2\epsilon} \leqq 1 \qquad [12]$$

The solution to this minimum problem, owing to constraint [12], is given by some (x, y) in the region (see Fig. 5) between the two branches of a hyperbola. There are two subcases here of interest:

1 Coordinate (x_0, y_0) lies inside the branch containing $[1 + (3/2)\ \epsilon,\ 0]$. The minimum is given by the point of nearest tangency of an ellipse with foci at (x_0, y_0), $(1 + (3/2)\ \epsilon,\ 0)$, and the branch of the hyperbola (see Fig. 6). Courant and Robbins (2) have treated this type of minimum problem.

2 Coordinate (x_0, y_0) lies anywhere in the plane outside of the region given by subcase 1. For this situation the answer is found by joining (x_0, y_0) to $[1 + (3/2)\ \epsilon,\ 0]$ and choosing any (x, y) on this line lying in the region (see Fig. 7).

A different case of interest is for r close to zero. An example of this is transfer of a space vehicle near the Earth's surface to a higher energy circular orbit at a considerable distance from the Earth's surface (see Fig. 8).

Case 3

Consider a small quantity $\epsilon > 0$. Let

$$r = \epsilon \qquad [13]$$

Neglecting terms in ϵ higher than first order, Equations [4 and 5] become

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{x^2 + y^2 - 2 + 3\epsilon} \qquad [14]$$

$$x^2 + y^2 - 2 \geqq -2\epsilon \qquad [15]$$

Lawden concludes in his analysis that the answer to the minimum problem occurs on the boundary given by the equality in Equation [15]. This reduces Equation [14] to

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{\epsilon} \qquad [16]$$

From Equation [16] we observe the answer is given as the intersection point of the normal from (x_0, y_0) to the circle given by the equality in Equation [15].

Lawden's results are confirmed if a geometric interpretation is given to Equations [14 and 15] (see Fig. 9). The first member of Equation [14] gives the distance from (x, y) to

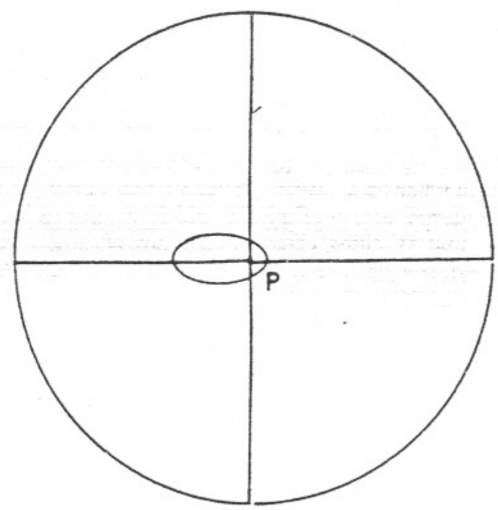

Fig. 8 Transfer from near Earth's surface to a distant circular orbit

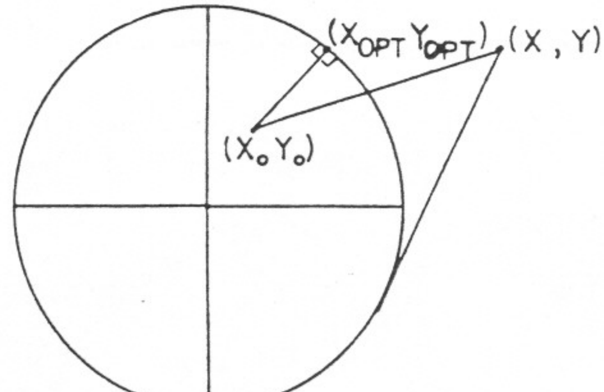

Fig. 9 Optimum solution for transfer from near Earth's surface to a distant circular orbit

(x_0, y_0), the second gives the length of the tangent from (x, y) to the boundary curve given by the equality in Equation [15]. The normal distance is less than the sum of any other two distances.

Acknowledgment

The authors would like to thank G. E. Taylor of the Dynamics Group for his efforts in helping complete this problem.

Nomenclature

u = velocity component perpendicular to the radius vector
v = velocity component along the radius vector
α = distance from focus of point on elliptic orbit
β = radius of higher energy circular orbit
μ = gravitational constant
x = dimensionless parameter related to u
y = dimensionless parameter related to v
r = ratio of α to β
ϵ = small positive quantity

References

1 Lawden, D. F., "Entry into Circular Orbits—I," *J. Brit. Interplanet. Soc.*, vol. 10, no. 1, pp. 5–17, Jan. 1951.
2 Courant, R. and Robbins, H., "What Is Mathematics?," Oxford University Press, New York, pp. 338–341.

Projected Orbits of 24-hr Earth Satellites

LOUIS B. WADEL[1]

Chance Vought Aircraft, Inc., Dallas, Texas

The projected orbit of a 24-hr Earth satellite is a "figure 8," oriented north-south and centered at the Equator. This path is calculated in terms of latitude and longitude as a function of the angle made by the satellite's orbital plane with the Earth's equatorial plane. If this angle is zero, the path reduces to a single point on the Equator.

A SATELLITE in a circular orbit about the Earth (assumed spherically symmetrical) will have a period of rotation equal to the Earth's rotational period if its altitude above the surface of the Earth equals approximately 22,000 miles (1).[2] If the orbit lies in the plane of the Equator and the satellite moves eastward, the projection of the orbit upon the Earth's surface consists of a single stationary point on the Equator.

If the orbit makes an angle α with the equatorial plane, the orbit's projection is a closed "figure 8" path, oriented north-south, centered at the Equator which it crosses at 12-hr intervals, and reaching extreme latitudes of $\pm\alpha$. Detailed properties of this path are calculated below as a function of α.

The projected path will be obtained by first calculating the projected orbit's great circle path in terms of latitude and longitude for a fictitious nonrotating Earth, and then modifying the results to account for the Earth's rotation. The general equation for a great circle is (2)

$$\cos\phi = c_1 \sin\phi \cos\theta + c_2 \sin\phi \sin\theta \qquad [1]$$

where

θ = east longitude
ϕ = 90 deg $-\ \lambda$
λ = north latitude

Crossing of the Equator may be assumed to occur at $\theta = 0$, and the resultant path shifted later to any other crossing point desired. The Equation [1] must be satisfied by the point $\theta = 0$, $\phi = 90$ deg, requiring $c_1 = 0$, and consequently

$$c_2 \tan\phi \sin\theta = 1 \qquad [2]$$

Differentiating Equation [2] and setting $d\phi/d\theta = 0$ yields

$$c_2 = \cot\phi_m = \tan\lambda_m = \tan\alpha \qquad [3]$$

where ϕ_m is minimum ϕ obtained, corresponding to λ_m, maximum latitude reached. The great circle Equation [2] can now be expressed as

$$\tan\alpha \tan\phi \sin\theta = 1$$

or

$$\tan\alpha \cot\lambda \sin\theta = 1 \qquad [4]$$

On the surface of a sphere the equation for differential element of arc length ds is given (2) by

$$(ds)^2/R_e^2 = [1 + \sin^2\phi (d\theta/d\phi)^2](d\phi)^2 \qquad [5]$$

where R_e = radius of the sphere (Earth).

Equation [5] may be divided by $(dt)^2$ and, for the present case of constant speed travel, ds/dt set equal to a constant V. Then, utilizing the relation [4] between ϕ and θ, Equation [5] may be integrated to yield

$$\sin\lambda = \sin\alpha \sin (V/R_e)t = \sin\alpha \sin\omega_e t = \sin\alpha \sin 2\pi t \qquad [6]$$

where

ω_e = angular velocity of Earth in radians per day $= 2\pi$
t = time in days measured from time of Equator crossing

Equation [4] may be rearranged as

$$\sin\theta = \cot\alpha \tan\lambda \qquad [7]$$

Once α is specified, latitude can be calculated as a function of time from Equation [6], and then longitude (for a nonrotating Earth) calculated from Equation [7]. The actual geographi-

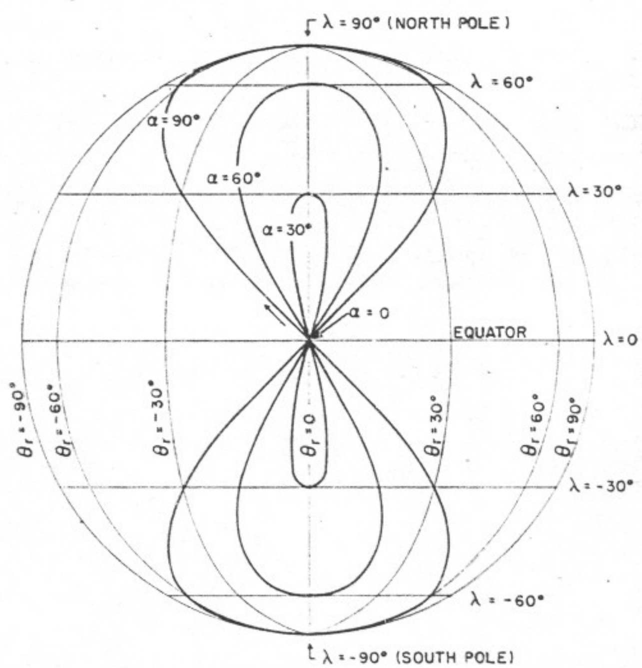

Fig. 1 Projected orbits

Received July 1, 1959.
[1] Supervisor, Missile Systems Evaluation.
[2] Numbers in parentheses indicate References at end of paper.