Review

Studies in Mathematics and Mechanics, *presented to Richard von Mises by Friends, Colleagues and Pupils, and edited by Garrett Birkhoff, Gustav Kuerti and Gabor Szegö. New York Academic Press,* 1954. 353 *pp.* $9.00.

This tribute to the eminent work of Richard von Mises in applied mathematics and mechanics takes the admirable form of a series of 42 articles, on branches of these subjects in which he had worked, by "friends, colleagues and pupils." It is preceded by a valuable bibliography of von Mises's writings, and a brief but penetrating study by Philipp Frank of the views and aims which governed the selection by von Mises of a wide variety of scientific, mathematical, philosophical and literary subject-matter for investigation. Here are noticed briefly only four articles in the volume, which seem to be those of most interest to aeronautical scientists.

Professor Temple contributes an article on his very important concept of "weak functions" or "generalised functions," which is treated further in *J. London Math. Soc.* 28, 134 (1953) and *Proc. Roy. Soc.* 228, 175 (1955). In these investigations he gives a simple and rigorous foundation to the theory of delta functions, principal values and finite parts of integrals, and the Fourier analysis of singular functions of all kinds. Much of this work greatly simplifies the analysis in many well-known kinds of aerodynamical calculation.

A simple model of shear flow turbulence based on considering the motion from a Lagrangian point of view and postulating certain resistances and rotations for fluid elements depending on their difference of velocity from the mean and on the local mean shear is put forward by Dr. J. M. Burgers.

The oscillating vortex wake behind an obstacle at Reynolds numbers of order 10^2 is studied by means of the Oseen approximation in a paper by Professor C. C. Lin.

Professor Howard Emmons derives a general correlation for experimental results on natural convection, film condensation, film boiling and film melting, by re-analysing the significance of the relevant Rayleigh and Reynolds numbers.

M. J. LIGHTHILL.

Errata

It is regretted that in the paper by L. E. Fraenkel "On the Unsteady Motion of a Slender Body through a Compressible Fluid" published in the February 1955 issue of The Aeronautical Quarterly (Vol. VI, p. 59), the printed impression of some of the pages was very poor.

The most serious result of this was that the identity between equations (16) and (17), page 65, *which should read*

$$K_o(z)=\tfrac{1}{2}\int_0^\infty \exp\left(-u-\frac{z^2}{4u}\right)\frac{du}{u},$$

apparently contained "z" and not "z^2" in the expression on the right hand side.

In addition the letter "d" should have appeared before the "θ" at the very end of equations (76) and (77) on p. 80.

Dynamic Problems of Interplanetary Flight

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SUMMARY: The solution to the general problem of transferring a rocket between two terminals in space with minimum fuel expenditure is explained and the results obtained when application is made to a number of particular problems of space navigation are described. The mathematical techniques which may usefully be employed in the calculation of optimum rocket trajectories are exemplified by a method of solving the problem of obtaining maximum range from a rocket missile over the Earth's surface.

1. Introduction

In the seventeenth and eighteenth centuries the problems of sea navigation acted as a stimulus and a challenge to the scientists of those days and led to many enquiries of a mathematical character being undertaken. The results of these so fertilised the soil of the whole terrain of mathematics that rich harvests were soon gathered from all its fields. The problems of the space navigator are of a far greater complexity and, although the mathematical techniques available for solving them are more powerful than the primitive methods used by these earlier investigators, it is to be expected that the conquest of space and the resulting large demand for methods of computing rocket trajectories will have an equally beneficial effect and will be productive of many new mathematical ideas.

There are three main problems of space navigation. Firstly, there is the problem of computing the most convenient trajectory to be followed by a space ship undertaking a particular interplanetary journey. Among the various possible trajectories satisfying the conditions imposed by such factors as the maximum motor thrust available, the maximum allowable time for the journey, the minimum distance of approach to the sun, and so on, the most satisfactory trajectory will clearly be that requiring the least expenditure of fuel. In the early stages, it will be necessary to relax all conditions to the greatest degree possible in order to achieve fuel economy, while looking forward hopefully to the day when atomic drives become a reality and the necessity for fuel economy becomes less pressing. In such circumstances, the calculation of trajectories of least fuel expenditure will have to be carried out under a variety of conditions imposed by such considerations as time schedules and the safety of passengers. For example, it may be a requirement that certain orbits be avoided because of their passing through regions of high meteor or cosmic ray intensity.

Based on a Section Lecture read before the Society on 27th January 1955.
Received February 1955.

The second main problem is that of computing the position and velocity of a rocket in space from observations of celestial bodies and the subsequent derivation of the elements of its orbit. This is a survey resection problem in three dimensions and, although it is more complex than the corresponding two-dimensional one, the existing apparatus of spherical trigonometry will be quite capable of providing a solution. The directions of the various bodies of the solar system can be found by observing them against the background of the fixed stars and then, knowing the co-ordinates of these bodies, those of the rocket are easily derived. Thirdly, there is the problem of providing the future navigator with sets of tables, by the aid of which he will be able to compute the appropriate manoeuvre necessary to bring his ship back on to a pre-calculated track when a divergence has been observed.

Of these three problems, investigation of the first seems most likely to require the development of new mathematical techniques and it is this problem which is considered further in the next Section.

NOTATION

$a = P/\mu$
A *see* equation (10)
b *see* equation (33)
c rocket exhaust velocity
d *see* equation (34)
$f(t)$ acceleration due to motor thrust at time t
F_1, F_2 components of retardation due to air resistance and gravity
g acceleration due to gravity
k *see* equation (22)
l_1, l_2 direction cosines of direction of motor thrust
M rocket mass
M_0 value of M on launching
$p_i = \dot{x}_i, (i=1, 2)$
P motor thrust
q_1, q_2 rocket's co-ordinates at "all-burnt"
$\dot{q}_1, \dot{q}_2$ rocket's velocity components at "all-burnt"
R range along the Equator
t time after launching
T value of t at "all-burnt"
x_1, x_2 co-ordinates of the rocket at time t
$X(t)$ *see* equation (37)
y_i, z_i *see* equations (9)

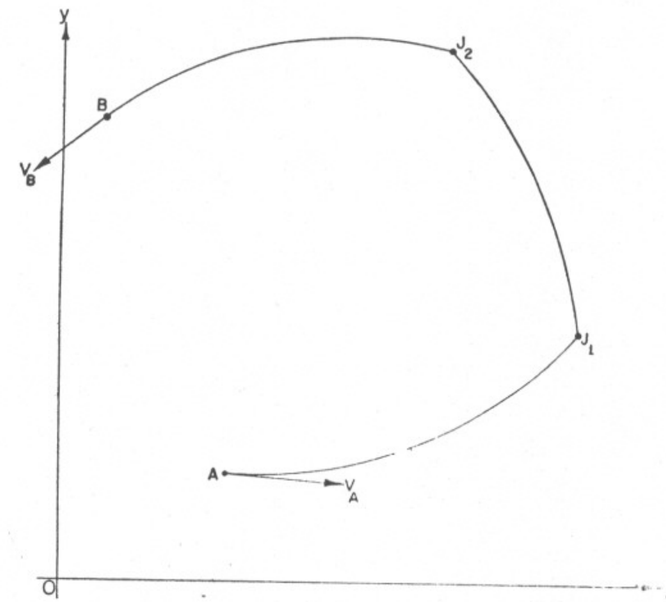

Fig. 1.

α_{ri}, β_{ri} defined after equation (11)
δ_{ik} Kronecker deltas
θ inclination of the rocket's velocity to the horizontal
ϕ inclination of the rocket thrust to the horizontal
μ rate of consumption of propellant
$\tau = M_0/\mu$

2. Rocket Manoeuvres of Least Fuel Expenditure

Suppose a rocket is to be navigated between two terminals A and B, its velocity at these points being specified. If the motion takes place *in vacuo* and the gravitational field is of a general nature, it may be shown[1] that the trajectory of minimum fuel expenditure comprises a number of null-thrust arcs along which the rocket falls freely under gravity, impulsive thrusts from the motor being applied at the junctions between these arcs to effect transfer from one to the next (Fig. 1).

If the motion takes place in the plane of rectangular axes Ox, Oy and the x and y components of the gravitational attraction are $-f$, $-g$, respectively, two components u, v, of a vector quantity called *the primer* are determined by means of the equations

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + u\frac{\partial f}{\partial x} + v\frac{\partial g}{\partial x} &= 0, \\ \frac{d^2v}{dt^2} + u\frac{\partial f}{\partial y} + v\frac{\partial g}{\partial y} &= 0. \end{aligned}\right\} \quad \ldots \quad (1)$$

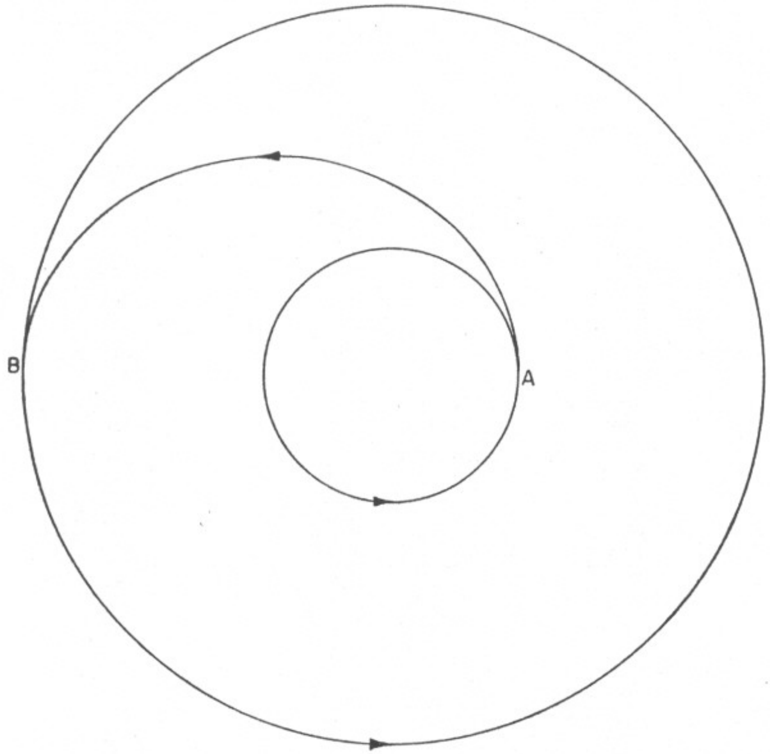

Fig. 2.

The quantities u, v, vary with t along any track of the type specified. The track is optimal with respect to fuel expenditure if the quantities u, v are such that $u^2+v^2 \leqslant 1$ at all points on the track and satisfy the following conditions at each junction:—

(*i*) u, v are continuous and represent the direction cosines of the direction of thrust,

(*ii*) $\dot{u}$, $\dot{v}$ are continuous,

(*iii*) $u\dot{u}+v\dot{v}=0$.

Corresponding results which take into account aerodynamic forces will be found in Ref. 2.

Particular problems to which this general result may be applied are:—

(*a*) The basic manoeuvre of transferring a rocket from one orbit about a central attracting body into another not necessarily coplanar with the first. The time of transit, the point of departure and the point of arrival, may or may not be specified.

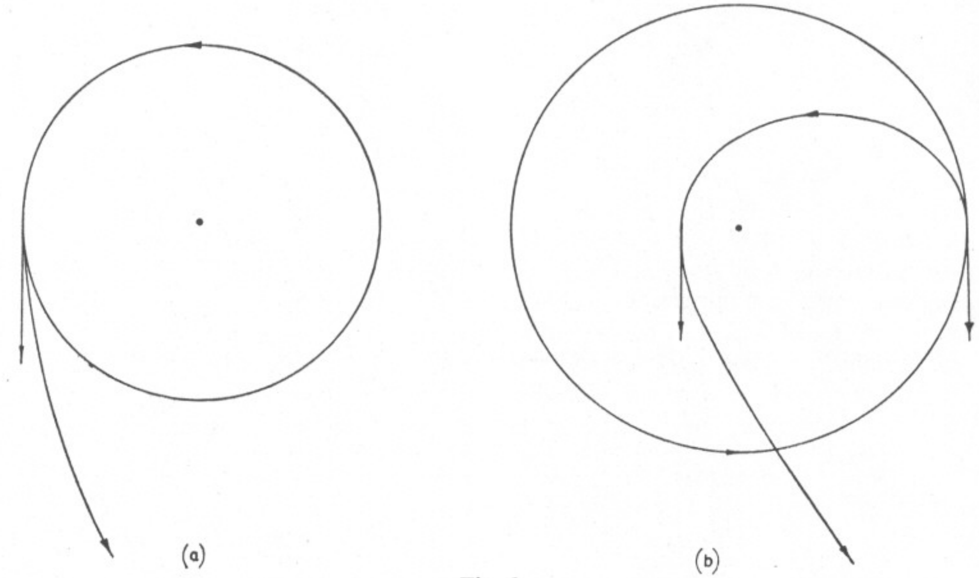

Fig. 3.

The solution to the simplest case of this problem, namely, transfer between coplanar circular orbits, was discovered by Hohmann in 1925[3]. Hohmann found that minimum fuel expenditure is involved if transfer takes place along an elliptical orbit tangential to both circular orbits (Fig. 2), impulsive thrusts being applied by the motors at each point of contact, A and B, to effect transference into and out of the connecting orbit. Although the requirement of impulsive thrusts cannot be achieved in practice, there will usually be no difficulty in attaining a very close approximation. If the two orbits are those of the Earth and Mars for example, it may be calculated that the time which would be spent in the orbit of transfer would be about 260 days, as compared with two periods of acceleration of a few minutes each. However, account might have to be taken of the fact that the motor thrust is limited if transfer between two orbits about the Earth is being considered, for then the time of transfer will be comparable with the time of thrust, or if it is proposed to use a micro-thrust motor employing a jet of ionised particles.

The present position in relation to the general problem is that the mode of optimal transfer between any pair of coplanar elliptical orbits is known[4,5] but the general three-dimensional manoeuvre remains to be treated.

(*b*) It seems probable that a space ship embarking upon an interplanetary journey will depart from a circular orbit about the Earth and terminate its passage by entering into another circular orbit about the target planet. The problem is therefore presented of optimising these manoeuvres for a given pair of planets and circular orbits about them. Again, the solution is available[6] in the two-dimensional, but not in the three-dimensional, case.

(*c*) The solution to the previous problem being known, there exists the subsidiary problem of determining the most economical manoeuvre whereby a rocket may escape from a given circular orbit, arriving at infinity with the velocity necessary to set it into the orbit of transfer. Assuming the thrust unlimited in magnitude, there are two possibilities. If the velocity required at infinity is less than a certain critical value, escape should be achieved by a single impulsive thrust tangential to the circular orbit (Fig. 3*a*) causing the rocket to move into a hyperbolic orbit. If, however, the velocity at infinity is to exceed this critical value, a more complex manoeuvre saves fuel. An impulsive thrust opposing the motion is first applied to set the rocket into an elliptical orbit approaching the centre of attraction (Fig. 3*b*). At perigee a second impulsive thrust transfers the ship into a hyperbolic orbit along which it proceeds to infinity. This manoeuvre, which was first proposed by Oberth[7], becomes more economical the closer the approach that can be made to the centre of attraction. By reversing the order of events in these manoeuvres, economical entries into circular orbits may be made with any velocity of approach. Full details will be found in Refs. 8 and 9.

If, however, the thrust is limited in magnitude, so that impulsive thrusts cannot be approximated, this problem does not admit of so simple a solution. Equations are available from which the optimal trajectory may be deduced[10], but they cannot be integrated analytically. Their numerical integration has not yet been performed, but it appears that they require the thrust to be offset from the direction of motion towards the centre of attraction.

This section is concluded by describing two other problems relating to optimal rocket trajectories.

When a rocket is perturbed by a planetary body, a transfer of energy takes place between it and the body which may be to the advantage of the rocket. Thus, by passing close to the Moon, a space ship outward bound from an orbit about the Earth to Mars can acquire a considerable amount of energy without any expenditure of fuel. The best way of utilising such perturbation effects is not known, although there exist a few purely numerical studies, some of which will be found in Ref. 11.

Finally, there is the most urgent of all astronautical problems, that of calculating the optimal trajectory from a ground launching station into a satellite orbit about the Earth. This problem is complicated by the existence of aerodynamic forces and by the necessity for making allowance for the limited thrust which can be provided by the rocket motors. The work of Tsien and Evans[12] represents an advance along the path leading towards a complete solution to this problem. They have calculated the mode of programming the fuel expenditure of a vertically ascending rocket necessary to achieve maximum height. The motor thrust was assumed unrestricted in magnitude. Clearly a compromise has to be effected between a rapid rate of fuel expenditure, leading to high velocities in the lower levels of the atmosphere and hence to a fuel wastage due to the excessive work which must be done against air resistance, and a slow rate of fuel expenditure, which is uneconomical for other reasons. Tsien and Evans's solution requires that an initial impulsive thrust is to be followed by a period of variable finite thrust, at the end of which the motor is to be shut down and the vehicle is then to coast to a standstill.

They proposed that the initial impulse should be provided by a booster which would fall away shortly after the rocket left its launching tower. If c is the jet velocity, assuming only that the air resistance is proportional to the square of the velocity (*i.e.* both this force and gravity may vary in any manner with the height), optimal programming has been achieved if the speed of the rocket during the phases when the motor is operating is λc, where

$$\lambda^3 + \lambda = \frac{\text{rocket weight}}{\text{air resistance at jet speed}} . \qquad (2)$$

The booster is supposed to accelerate the rocket initially to a speed in conformity with this equation. A small rocket can be programmed to move in the manner required by this equation, but since the rocket weight increases as the cube of the linear dimensions, whereas the air resistance only increases as the square, the value of λ determined by equation (2) increases with the rocket size and there proves to be no possibility of a really large rocket following this optimal schedule. It is suggested that to reach maximum height with such a rocket, as close an approximation as possible should be made to the optimal solution, by first boosting to the maximum degree possible and thereafter operating at maximum thrust. For the similar three-dimensional problem of placing a large rocket in a satellite orbit, it is to be expected that the optimal solution will require a thrust beyond that attainable. Taking into account the limited thrust available, optimal conditions will be best approached by first boosting to the maximum degree possible and thereafter operating at maximum thrust. When just sufficient momentum has been acquired to carry the vehicle to the level of the orbit, the motors should be shut down. The rocket then coasts to the orbit and a final short burst from the motors will be necessary to effect transfer from the elliptical coasting orbit into the satellite orbit. The problem then remains of programming the direction of thrust during the first phase to achieve maximum fuel economy. Equations are available, from which, by numerical integration the solution to this problem may be found[13].

3. Programming for Maximum Range of a Rocket Missile

To illustrate the kind of mathematical techniques which are usefully employed when considering the type of problem described at the end of Section 2, a method is described of calculating the trajectory of a rocket missile so that it will achieve its maximum range over the Earth's surface.

Theoretically, optimisation should be carried out with respect to both the magnitude of the motor thrust and its direction. However, for a large long-range rocket weapon, the optimal magnitude of thrust proves to be impracticably large for reasons which have already been mentioned and it will therefore be assumed that when the motor is operating, it is always developing its maximum thrust. It remains to achieve optimal conditions with respect to the direction of thrust. It is also assumed that the period of motion divides into two phases. During the first phase, the motor operates at full power and drives the missile towards the vertex of its trajectory. The motion during the second phase is one of free fall under gravity

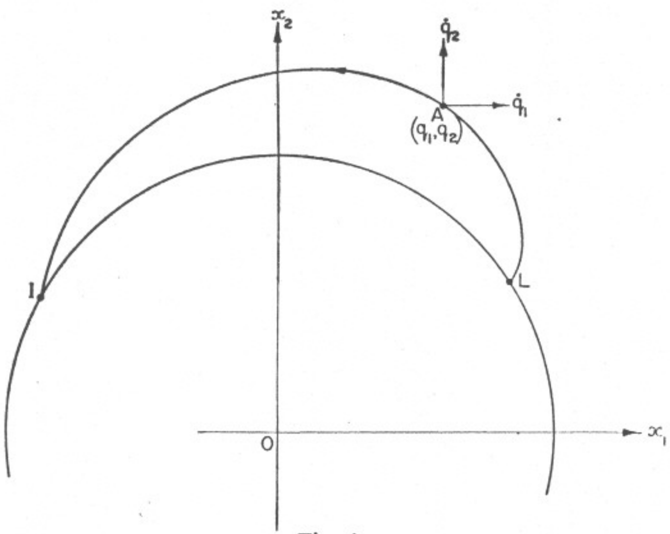

Fig. 4.

alone until the atmosphere is re-entered, when the missile may follow a ballistic trajectory under the additional force of air resistance or further extend its range by gliding towards its point of impact.

For simplicity, consider the two-dimensional problem of a missile launched from a point on the Equator and moving in the plane of this great circle. Let Ox_1, Ox_2 (Fig. 4) be rectangular axes through the Earth's centre, rotating with it and lying in the Equatorial plane and let (x_1, x_2) be the co-ordinates of the rocket at time t. If c is the jet velocity and M is the rocket's mass at any instant, the acceleration due to the motor thrust is $-c\,dM/(M\,dt)$. Since the fuel consumption programme is to be assumed fixed, this acceleration will be a known function of t, say $f(t)$. If (l_1, l_2) are the direction cosines of the direction of thrust during the first phase, the equations of motion of the rocket are

$$\ddot{x} + F_i = f l_i, \quad (i=1, 2) \qquad (3)$$

where $(-F_1, -F_2)$ are the components of the missile's acceleration due to the air resistance, the gravitational attraction and the centrifugal and Coriolis forces associated with the rotating frame of reference. If it is assumed that there is no alteration in the settings of any aerodynamic control surfaces and that the orientation of the rocket relative to the direction of motion remains invariant during the time that air resistance is operative, it is permissible to write

$$F_i = F_i(x_1, x_2, \dot{x}_1, \dot{x}_2, M, t), \qquad (4)$$

explicitly.

Suppose that the functions $l_i(t)$ determine the thrust direction programme resulting in the achievement of maximum range. Then equations (3) can be solved for $x_i = x_i(t)$, specifying the optimal trajectory. At $t=0$ the rocket is launched from

L (Fig. 4) and x_i, $\dot{x}_i$ take given values. Suppose that the motor operates until $t=T$, "all-burnt" occurring at A. Let A be the point (q_1, q_2) and let $(\dot{q}_1, \dot{q}_2)$ be the velocity components at this point. From A to the point of impact I, the rocket falls under gravity. The position of I is completely determined by the design of the missile, the position of A and the missile's velocity at this point. Thus the range R along the Equator from the position of L to the point I is an explicit function of the variables q_i, $\dot{q}_i$,

i.e.

$$R = R(q_1, q_2, \dot{q}_1, \dot{q}_2). \qquad (5)$$

This will be a known function of q_i, $\dot{q}_i$ for any particular missile. The thrust direction programme must now be selected so that q_i, $\dot{q}_i$ take values which maximise R.

Consider a small variation in the thrust direction programme in which the functions l_i are replaced by $l_i + \delta l_i$. Since $l_1^2 + l_2^2 = 1$, the δl_i are related by the equation

$$l_i \delta l_i = 0, \qquad (6)$$

but are otherwise arbitrary (the repeated index summation convention is being used in equation (6) as elsewhere in the argument). Taking the first variation of the equations of motion (3), we obtain

$$\delta \ddot{x}_i + \frac{\partial F_i}{\partial \dot{x}_j}\delta \dot{x}_j + \frac{\partial F_i}{\partial x_j}\delta x_j = f \delta l_i. \qquad (7)$$

Since the launching point and the missile's initial velocity are not subject to variation, $\delta x_i = \delta \dot{x}_i = 0$ at $t=0$. Equations (7) accordingly fix the functions δx_i in terms of the δl_i and hence the varied trajectory.

Putting $\delta \dot{x}_i = \delta p_i$, equations (7) are replaced by the first order system

$$\left.\begin{aligned} \delta \dot{p}_i + \frac{\partial F_i}{\partial \dot{x}_j}\delta p_j + \frac{\partial F_i}{\partial x_j}\delta x_j &= f\delta l_i, \\ \delta \dot{x}_i - \delta p_i &= 0. \end{aligned}\right\} \qquad (8)$$

Solving these equations by the method of the variation of parameters[14], leads to consideration of the system of equations

$$\left.\begin{aligned} \dot{z}_i + \frac{\partial F_i}{\partial \dot{x}_j} z_j + \frac{\partial F_i}{\partial x_j} y_j &= 0, \\ \dot{y}_i - z_i &= 0, \end{aligned}\right\} \qquad (9)$$

for the four functions y_i, z_i $(i=1, 2)$. Let

$$y_i = y_{ik}(t), \quad z_i = z_{ik}(t) \quad (i=1, 2;\ k=1, 2, 3, 4)$$

represent a linearly independent set of solutions of the equations (9).

Also, let A be the determinant of the fourth order given by

$$A = \begin{vmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \end{vmatrix} . \qquad (10)$$

Then equations (8) can be integrated over the time interval $(0, T)$ into the forms

$$\delta x_r(T) = \delta q_r = \int_0^T \frac{f}{A} \alpha_{ri} \delta l_i \, dt,$$

$$\delta p_r(T) = \delta \dot{q}_r = \int_0^T \frac{f}{A} \beta_{ri} \delta l_i \, dt, \qquad (11)$$

where α_{ri}, β_{ri} are the determinants formed from A by replacement of its i^{th} row by the elements

$$y_{r1}(T),\ y_{r2}(T),\ y_{r3}(T),\ y_{r4}(T),$$

and

$$z_{r1}(T),\ z_{r2}(T),\ z_{r3}(T),\ z_{r4}(T),$$

respectively.

The variation in the thrust direction programme results in a variation in R (equation (5)) of

$$\delta R = \frac{\partial R}{\partial q_r} \delta q_r + \frac{\partial R}{\partial \dot{q}_r} \delta \dot{q}_r$$

$$= \int_0^T \frac{f}{A} \left(\frac{\partial R}{\partial q_r} \alpha_{ri} + \frac{\partial R}{\partial \dot{q}_r} \beta_{ri} \right) \delta l_i \, dt, \qquad (12)$$

after using equations (11).

If the original value of R was a maximum, $\delta R = 0$ for all functions δl_i satisfying equation (6). This will be so if

$$l_1 : l_2 = \left(\frac{\partial R}{\partial q_r} \alpha_{r1} + \frac{\partial R}{\partial \dot{q}_r} \beta_{r1} \right) : \left(\frac{\partial R}{\partial q_r} \alpha_{r2} + \frac{\partial R}{\partial \dot{q}_r} \beta_{r2} \right). \qquad (13)$$

This equation determines the optimum thrust direction programme.

Integration of equations (3) and (13), determining the optimal trajectory, will have to be performed numerically. The presence in equation (13) of quantities which have to be evaluated at $t = T$ necessitates performing this integration in a sense opposite to that in which the rocket describes its trajectory. The conditions existing at "all-burnt" will have to be chosen more or less arbitrarily in the first

instance and, after integration back to the launching point L, adjusted until the conditions at L are in agreement with those specified. The computing problem is clearly one in which it will be necessary to acquire the services of an electronic computor. If, however, the problem is simplified by neglecting air resistance, the curvature and rotation of the Earth and variations in gravity with height, an analytical solution is easy to obtain. Such a solution can be used to assess the increase in range to be expected when any thrust direction programme is modified to follow an optimum schedule. This case is considered in the next section.

4. Maximum Range over an Airless Flat Earth

In this case it is convenient to take the origin O at the point of projection and the axes Ox_1, Ox_2 horizontally and vertically respectively. The missile follows a parabolic track from A to the point of impact I where its trajectory intersects Ox_1. It will be found that equation (5) now takes the form

$$R = q_1 + \frac{\dot{q}_1}{g} \{ \dot{q}_2 + \sqrt{(\dot{q}_2^2 + 2 g q_2)} \}, \qquad (14)$$

g being the constant acceleration due to gravity.

The equations of motion (3) can be written

$$\ddot{x}_1 = l_1 f, \quad \ddot{x}_2 + g = l_2 f, \qquad (15)$$

from which it follows that the equations (9) are in this case

$$\dot{z}_i = 0, \quad \dot{y}_i = z_i. \qquad (16)$$

Let $y_{ik}(t)$, $z_{ik}(t)$ be a set of solutions of these equations satisfying the initial conditions

$$z_{ik}(0) = \delta_{ik}, \quad y_{ik}(0) = \delta_{i+2,k}, \qquad (17)$$

δ_{ik} being zero unless the two subscripts are identical, when its value is unity. This set of solutions is fundamental and from it the determinant A can be constructed in the form

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{vmatrix} . \qquad (18)$$

It now follows that

$$\left.\begin{aligned} \alpha_{11} &= T - t, & \alpha_{12} &= 0, \\ \alpha_{21} &= 0, & \alpha_{22} &= T - t, \\ \beta_{11} &= 1, & \beta_{12} &= 0, \\ \beta_{21} &= 0, & \beta_{22} &= 1. \end{aligned}\right\} \qquad (19)$$

Substituting from equations (14) and (19) into equation (13), it is found that the optimum thrust direction programme is determined by the equation

$$\frac{l_1}{l_2} = \frac{1}{\dot{q}_1}\sqrt{(\dot{q}_2{}^2+2gq_2)},$$

i.e.

$$\left.\begin{aligned} l_1 &= \sqrt{\left[\frac{\dot{q}_2{}^2+2gq_2}{\dot{q}_1{}^2+\dot{q}_2{}^2+2gq_2}\right]}, \\ l_2 &= \frac{\dot{q}_1}{\sqrt{(\dot{q}_1{}^2+\dot{q}_2{}^2+2gq_2)}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

It will be observed that the thrust direction is independent of the time t.

First consider the case when the acceleration f due to the motor thrust is constant during the first phase of the motion. Assuming that initially $x_1=x_2=\dot{x}_1=\dot{x}_2=0$, equations (15) may be integrated to yield

$$\dot{x}_1=l_1 ft, \quad \dot{x}_2=(l_2 f-g)t,$$

$$x_1=\tfrac{1}{2}l_1 ft^2, \quad x_2=\tfrac{1}{2}(l_2 f-g)t^2, \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

l_1 and l_2 being given by the equations (20). When $t=T$, $x_1=q_1$, $x_2=q_2$, $\dot{x}_1=\dot{q}_1$, $\dot{x}_2=\dot{q}_2$, thus providing four equations for the determination of the values of q_1, q_2, $\dot{q}_1$, $\dot{q}_2$. It will be found that, if k is the positive root of the equation

$$(k+1)^{3/2}[(k+1)^{1/2}+k^{1/2}]=(f/g)^2, \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

then

$$\left.\begin{aligned} q_1 &= \tfrac{1}{2}gT^2k^{1/2}(k+1)^{1/2}, \\ q_2 &= \tfrac{1}{2}gT^2k, \\ \dot{q}_1 &= gTk^{1/2}(k+1)^{1/2}, \\ \dot{q}_2 &= gTk. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Equations (21), determining the trajectory while the motors are operating, may now be shown to take the form

$$x_1=\tfrac{1}{2}gt^2k^{1/2}(k+1)^{1/2}, \quad x_2=\tfrac{1}{2}gt^2k. \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

These are parametric equations of a straight line through O and at an angle θ to the horizontal given by the equation

$$\tan\theta=\left(1+\frac{1}{k}\right)^{-1/2}. \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

If ϕ is the angle made by the direction of the thrust with the horizontal, it may also be proved that

$$\tan\phi=\frac{l_2}{l_1}=\left(1+\frac{1}{k}\right)^{1/2}. \quad . \quad . \quad . \quad . \qquad (26)$$

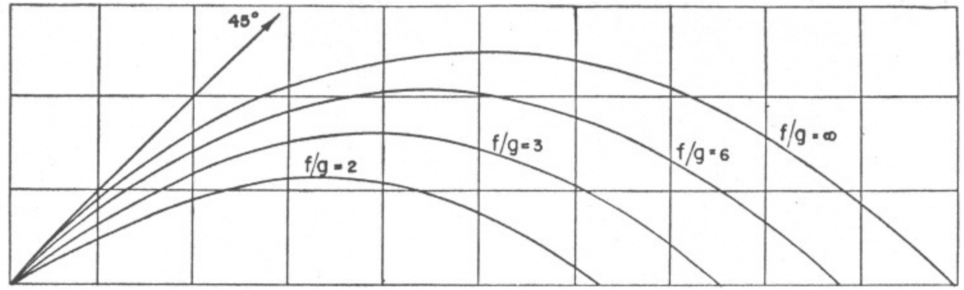

Fig. 5.

It will be observed that as $f \longrightarrow \infty$, *i.e.* as the thrust becomes more powerful and gunnery conditions are approached, $k \longrightarrow \infty$ and hence θ and ϕ both tend to a limiting value of 45°, as might have been expected.

From equation (14), it is found that

$$R_{\max}=\tfrac{1}{2}gT^2k^{1/2}(k+1)^{1/2}[(k+1)^{1/2}+k^{1/2}]^2. \quad . \quad . \quad . \quad . \qquad (27)$$

Complete trajectories for the cases $f/g=2, 3, 6, \infty$, have been drawn in Fig. 5. For each trajectory T was selected so that the fuel consumption was the same in all cases.

Equations (25) and (26) show that optimum conditions are achieved by deflecting the direction of thrust from the direction of motion by an angle

$$\cot^{-1}[k^{1/2}(k+1)^{1/2}+k^{1/2}(k+1)^{1/2}] \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

in the upwards direction. It is of interest to discover the effectiveness of this device. If the thrust magnitude programme is maintained unaltered, but the thrust direction is supposed to be always in the direction of motion, the equations of the trajectory will be those given in Ref. 15. Assuming that the missile is initially stationary it must first ascend vertically. However, there are a single infinity of possible trajectories issuing from the launching point and, for comparison with the optimum trajectory, we must select that which results in the greatest range being achieved. In a particular case it was assumed that $f=3g$, $g=32{\cdot}2$ ft./sec.^{2}, $T=50$ sec. The resulting maximum range, when the thrust was not deflected from the direction of motion, proved to be 97 miles. The corresponding solution of equation (22) was found to be $k=1{\cdot}271$. Equation (27) then yielded $R_{\max}=104$ miles, an increment of about 7 per cent. Smaller values of f/g result in a larger percentage improvement in range, and larger values in a smaller improvement.

Attention is now directed to the problem of attaining maximum range with a rocket whose thrust remains constant. Since the mass will diminish as fuel is consumed, it follows that the acceleration f due to the motor will steadily increase.

If P denotes the thrust, M_0 the rocket's initial mass and μ the rate at which propellant is consumed, μ must be constant if P and the jet velocity do not vary, and hence the component of the rocket's acceleration due to P at time t will be given by the equation

$$f(t)=\frac{P}{M_0-\mu t}=\frac{a}{\tau-t}, \qquad (29)$$

where $a=P/\mu$, $\tau=M_0/\mu$. The equations of motion of the missile are accordingly.

$$\ddot{x}_1=\frac{al_1}{\tau-t}, \quad \ddot{x}_2=\frac{al_2}{\tau-t}-g, \qquad (30)$$

where, for maximum range, l_1, l_2 are specified by equations (20).

We now proceed as before and calculate that

$$\left.\begin{aligned} q_1&=\tfrac{1}{2}gT^2(2-b)p^{\frac{1}{2}}(p-b)^{\frac{1}{2}}, \\ q_2&=\tfrac{1}{2}gT^2[(2-b)p-1], \\ \dot{q}_1&=gTp^{\frac{1}{2}}(p-b)^{\frac{1}{2}}, \\ \dot{q}_2&=gT(p-1), \end{aligned}\right\} \qquad (31)$$

p being the positive root of the equation

$$p^{\frac{3}{2}}[p^{\frac{1}{2}}+(p-b)^{\frac{1}{2}}]=d^2, \qquad (32)$$

where

$$b=\frac{2\tau}{T}-\frac{2}{\log\dfrac{\tau}{\tau-T}}, \qquad (33)$$

$$d=\frac{a}{gT}\log\frac{\tau}{\tau-T}. \qquad (34)$$

It now follows from equation (14) that

$$R_{\max}=\tfrac{1}{2}gT^2d^4\left(1-\frac{b}{p}\right)^{\frac{1}{2}}p^{-2}. \qquad (35)$$

The equations of the trajectory during the period of operation of the motors are

$$\left.\begin{aligned} x_1&=p^{\frac{1}{2}}(p-b)^{\frac{1}{2}}X(t), \\ x_2&=pX(t)-\tfrac{1}{2}gt^2, \end{aligned}\right\} \qquad (36)$$

where

$$X(t)=\frac{gT}{\log\dfrac{\tau}{\tau-T}}\left[t-(\tau-t)\log\frac{\tau}{\tau-t}\right]. \qquad (37)$$

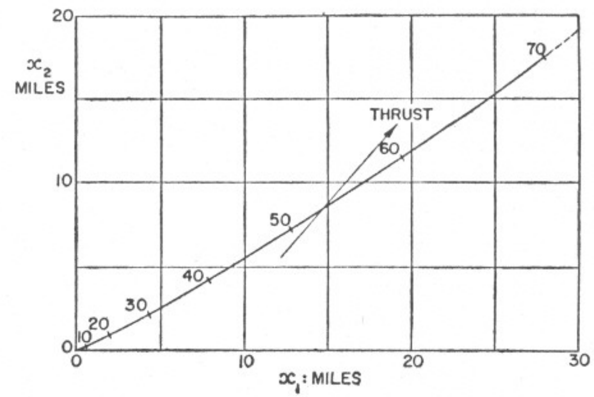

Fig. 6.

The angle of inclination of the thrust to the horizontal is given by the equation

$$\tan\phi=\left(1-\frac{b}{p}\right)^{-\frac{1}{2}}. \qquad (38)$$

The angle made by the direction of motion with the horizontal at time t is given by the equation

$$\tan\theta=\left(1-\frac{b}{p}\right)^{-\frac{1}{2}}\left[1-\frac{\tau}{pT}\,\frac{\log\dfrac{\tau}{\tau-T}}{\log\dfrac{\tau}{\tau-t}}\right]. \qquad (39)$$

This equation shows that the rocket must be launched at an angle θ_0 to the horizontal, where

$$\tan\theta_0=\left(1-\frac{b}{p}\right)^{-\frac{1}{2}}\left[1-\frac{\tau}{pT}\log\frac{\tau}{\tau-T}\right]. \qquad (40)$$

In the particular case of the V.2 rocket missile, $P=60{,}000$ lb. wt., $M_0=28{,}500$ lb., $\mu=275$ lb./sec. Hence $a=7{,}028$ ft./sec., $\tau=103{\cdot}6$ sec. Taking the duration of the thrust to be $T=70$ sec., it is found that $b=1{\cdot}184$, $d=3{\cdot}510$. The positive root of equation (32) is then $p=2{\cdot}658$, resulting in a value for $R_{\max}$ from equation (35) of 277 miles.

Substituting appropriate values for the parameters appearing in equations (38) and (40), it is calculated that the angles made by the thrust and the initial direction of motion with the horizontal are

$$\phi=49^\circ\ 12', \quad \theta_0=23^\circ\ 23', \qquad (41)$$

respectively. The portion of the trajectory from the launching point to "all-burnt" has been drawn to scale in Fig. 6. Time, in 10 sec. intervals, has been marked off along the curve.

The maximum range of this weapon is usually stated to be about 220 miles. The difference of 57 miles between this and the calculated figure will be accounted for to some extent by the effect of atmospheric resistance which has been neglected here. Against this must be set the effects of the Earth's curvature and the reduction in gravity with height. Both these factors will tend to increase the range. It is therefore difficult to assess with precision the improvement in range which would result if the thrust direction programme of this rocket were converted to optimum. Kooy and Uytenbogaart[16] estimate that the effect of air resistance on the range does not exceed 6 per cent. It accordingly seems reasonable to expect an improvement of about 10 per cent., *i.e.* of the same order as that found in the previous numerical case.

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The Measurement of High Speed Air Velocity and Temperature using Sound Wave Photography

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SUMMARY: A transient sound wave is generated by discharging a spark in an air stream, the velocity and temperature of which are required. After a time interval of about 25 microseconds a shadowgraph photograph of the sound wave is taken; a second photograph is obtained after a further known time interval of the same order. The two exposures, both on the same negative, show the propagation of the sound wave in the air stream and from this the Mach number, the true velocity, the local velocity of sound, and hence the temperature, can be calculated.

Using this method, measurements of Mach numbers of the order of 0·5 gave values between 98 per cent. and 100 per cent. of those calculated from pressure measurements. Typical examples of the sound wave photographs are shown. With further development along the lines indicated in the paper greater accuracy should be possible. The local speed of sound was measured to an estimated accuracy of $\pm 1{\cdot}5$ per cent.

Since only a very short time interval is needed to obtain the photographs, the method appears promising for investigating explosions of brief duration, or dealing with flows of pulsating character.

1. Introduction

In high-speed gas-flow analysis the use of pitot and static tubes together with shadowgraph, schlieren and interference photographs is well known. Of these methods the shadowgraph and schlieren techniques are generally used to give qualitative results, although some quantitative work is done. Unfortunately such quantitative work not only involves a good deal of mathematical analysis, but seldom gives accurate results. The interferometer is capable of determining the change of density of the flow, but extremely sensitive and expensive apparatus, used with great care, is needed and this technique is not generally used, except for dealing with special problems. By far the greatest proportion of the quantitative work that has been done has been based on pitot and static pressure readings.

From a knowledge of the pitot and static pressures at any point the Mach number of the flow can be determined, provided that the ratio of the specific heats γ is known. A further knowledge of the temperature at the point enables the true speed to be calculated. In wind tunnel work, it is usual to assume that isentropic

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