A Gravity Assist Primer

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From time to time I am asked by people in the public sector how they can calculate the "energy gain" that occurs as a result of a gravity assist flyby of a massive body. Typically this type of request originates with students doing science projects or adults looking for some spare-time fun. These individuals are of course ardent supporters of the space program, and that being the case, it is unfortunate and unsatisfactory to have to respond with answers such as "It's too complicated," or "You can't just use two-body formulae." Furthermore, the reference material on this subject is quite technical, and therefore not appropriate for an initial computational treatment. Thus, for a long time, I have wanted to derive a procedure to perform the gravity assist calculation that could be used by such individuals. This procedure would begin with a minimum set of input parameters and, presuming only a high school or college freshman mathematics background, would allow computation of the desired results to a reasonably accurate level of approximation. My attempt at such a procedure follows.

We will begin by solving the two-body portion of the problem, i.e., the flyby body (e.g., a planet) and the spacecraft. The results of this portion can then be used to compute changes with respect to the other massive body (e.g., the sun).

When a spacecraft has a flyby of another planet, it is on a type of trajectory or orbit known as a hyperbola with respect to that planet. Such a trajectory is shown in Fig. 1. A hyperbola is a type of conic section, i.e., if one took a cone and passed a plane through it in a certain manner, the points of intersection would define a hyperbola. Planes can also be passed through cones in such ways as to generate intersections that are circles, ellipses, and parabolas, though we are not concerned with them here. The important point in all this is that we will have to use some terminology that applies to conic sections in general and to the hyperbola in particular. Also we will make use of the equation that describes such a curve in polar coordinates, i.e., an angle and a distance. This relation is stated as

$$r = \frac{p}{1 + e\cos f} \qquad (1)$$

where, at any point on the hyperbola, r is the distance from planet to spacecraft and f specifies the angle of this point on the trajectory with respect to a particular reference direction. The angle f is called the true anomaly and the reference direction is the point on the hyperbola that is closest to the planet. This point of closest approach is called the periapse. Figure 1 depicts r and f.

The variables e and p represent other properties of a hyperbola known respectively as the eccentricity and the parameter. The parameter, p, can itself be expressed as

$$p = a\,(1 - e^2) \qquad (2)$$

where a is known as the semi-major axis of the hyperbola and is in fact defined to be a negative number.

From Eq. (1) we can obtain the spacecraft's distance at any point.

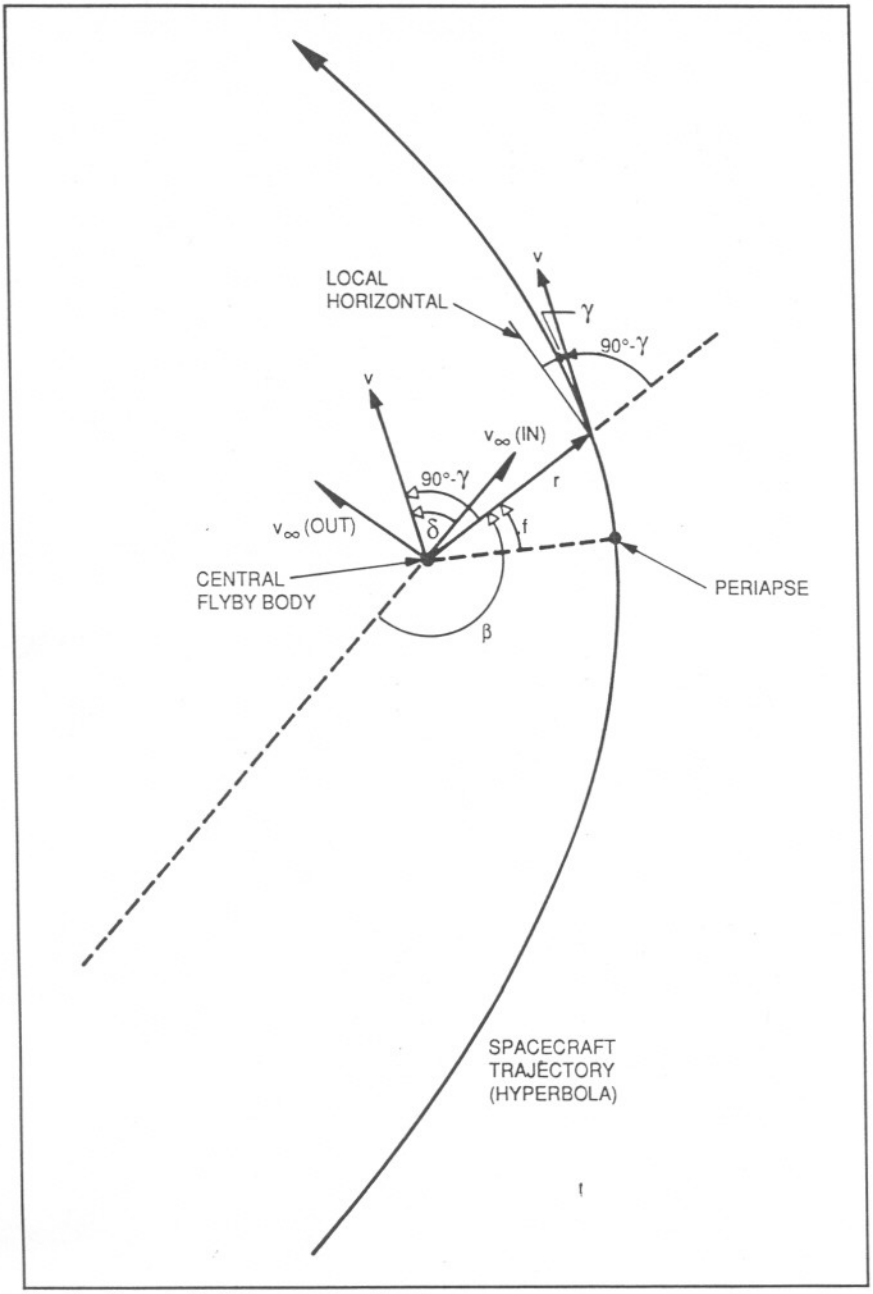

Fig. 1 Hyperbolic flyby around a planet.

We also need to be able to determine its speed. To obtain this we have another relation

$$v = \sqrt{GM\,(2/r - 1/a)} \qquad (3)$$

where GM is a measure of the flyby planet's gravitational field, and is actually the product of the gravitational constant G and the planet's mass M.

With the preceding relations and two quantities that characterize the spacecraft's trajectory, we can solve the two-body flyby problem. The first input quantity is r_p, the spacecraft's distance at periapse, i.e., its closest approach distance. The other is known as v_∞ and is the speed with which the spacecraft approaches the planet when it is still a long way out, i.e., at "infinity" for all practical purposes.

With what we have now, we can compute the quantities e, a, and p that were previously defined. To compute a, we use Eq. (3) at the point where r is infinity for all practical purposes. At this point, the term $2/r$ is equal to zero. Furthermore the spacecraft's speed, v_∞, is given. Equation (3) becomes

$$v_\infty = \sqrt{-GM/a}$$

which after rearrangement is

$$a = -GM/v_\infty^2 \qquad (4)$$

To compute e we use Eq. (1) at the point where $r = r_p$, i.e., the point of closest approach. At this point, from Fig. 1, we see that $f = 0$, and since $\cos(0) = 1$, Eq. (1) can be written at this point as

$$r_p = p/(1+e)$$

Then we use Eq. (2) to substitute for p. Thus

$$r_p = a(1-e)$$

after cancelling some common factors and solving for e, the equation becomes

$$e = 1-(r_p/a)$$

and by using Eq. (4) for a, it becomes

$$e = 1 + (r_p v_\infty^2/GM) \qquad (5)$$

When we have solved for a and e we can find p directly from Eq. (2).

We still need to compute a few other quantities before we can step our way through the flyby. One of these is known as f_∞, the true anomaly the spacecraft has when it has attained the point a long way out that is approximated by infinity. To obtain this quantity we refer back to Eq. (1), but restate it as

$$1 + e\cos f = p/r$$

Then, as the true anomaly goes to f_∞, the distance r also goes to infinity and the right-hand side of this equation goes to zero. With this in mind we can, after a little algebra, state it as

$$f_\infty = \cos^{-1}(-1/e) \qquad (6)$$

To obtain the next quantity of interest, we can use Eq. (4) expressed as

$$v_\infty^2 = -GM/a$$

and substitute this into Eq. (3) expressed as

$$v = \sqrt{(2GM/r) + v_\infty^2} \qquad (7)$$

What we specifically seek is the spacecraft's speed at periapse, v_p, which is obtained from Eq. (7) where $r = r_p$.

We will use this speed to calculate the spacecraft's orbital angular momentum per unit mass, a quantity that characterizes the rotation of the vehicle's trajectory, and that always has the same value throughout the flyby. This quantity, designated by h, is defined as

$$h = rv \sin(90 \text{ deg} - \gamma)$$

or

$$h = rv \cos \gamma \qquad (8)$$

since $\sin(90 \text{ deg} - \gamma) = \cos \gamma$. Here γ is known as the flight-path angle and is shown in Fig. 1. This angle just specifies the orientation of the spacecraft's speed with respect to a direction that would be horizontal at that point on the trajectory. Thus $\gamma = 0$ deg means the craft is in horizontal flight; $\gamma = +90$ deg means the vehicle is climbing vertically; values of γ between 0 and +90 deg refer to a climb that is in between horizontal flight and vertical ascent. Likewise $\gamma = -90$ deg means the vehicle is diving vertically; values between −90 and 0 deg refer to a descent that is in between vertical descent and horizontal flight. The approach, which characterizes the first half of the flyby, is all descent, i.e., the craft is getting closer to the planet. Thus flight-path angles are all negative here. The departure, which characterizes the second half of the flyby, is all ascent, i.e., the craft is getting farther from the planet. Thus flight-path angles are all positive here. The change from negative to positive value of γ must then be at the periapse point, and so here we know that $\gamma = 0$ deg.

Because we know the value of γ at this one point, we can use it to compute the quantity h. We use Eq. (8) with the value of r_p as given, and v_p as obtained from Eq. (7). Thus

$$h = r_p v_p \cos \gamma$$

or

$$h = r_p v_p = \text{const} \qquad (9)$$

since $\cos(0) = 1$. Now we have values for all the basic quantities that we need. We have v_∞, r_p, and GM as inputs and the quantities a, e, p, f_∞, v_p, and h are obtained from the relevant equations just provided. We can use these in the computations that we need to step our way through the flyby.

We will take our steps in increments of true anomaly, f, beginning near $f = -f_\infty$, increasing to $f = 0$ at periapse and ending near $f = f_\infty$. Then, for each value of f, and using e and p from above, we can compute r from Eq. (1). With this we can use Eq. (7), and values of GM and v_∞ from above, to compute v at each point.

Now we need to define a new angle β, sometimes called the range angle. This angle is shown in Fig. 1 and is given by the expression

$$\beta = f_\infty + f \qquad (10)$$

Since the flyby begins near $f = -f_\infty$, we see that at the start β is near zero; the flyby ends near $f = +f_\infty$ and so at the end $\beta \approx 2f_\infty$. So this angle is just an indicator of how far we are through the flyby, ranging from 0 to $2f_\infty$.

Next we will need to compute the flight-path angle at each point. We use Eq. (8) and rearrange it to solve it for γ

$$\cos \gamma = h/(rv)$$

where we have previously computed h from Eq. (9). We must also utilize the fact that points that have negative values of f will also have negative values of γ. Thus

$$\gamma = \mp \cos^{-1}(h/rv) \qquad (11)$$

where we use the minus or plus sign, as appropriate.

Finally we compute at each point the angle δ, which is also shown in Fig. 1. This angle is a measure of how much the spacecraft's velocity orientation has been rotated from its starting direction toward its final direction. With the quantities we have, we can compute δ at each point from the relation

$$\delta = \beta - \gamma - 90 \text{ deg} \qquad (12)$$

As an example of how the preceding computations are performed, the Appendix illustrates the Voyager 1 flyby of the planet Jupiter, which occurred on March 5, 1979. Note that as the true anomaly is varied from about $-f_\infty$ (−139 deg) to $+f_\infty$ (139 deg), the distance to Jupiter decreases to its minimum value, then increases symmetrically; likewise the speed begins approximately at v_∞, reaches its maximum value at closest approach, then drops back, again symmetrically, to v_∞. Thus Voyager 1 left Jupiter with the same speed, relative to the planet, that it approached Jupiter. The range angle starts near zero and ends with a value of about $2f_\infty$, as expected. The flight-path angle begins around −90 deg, implying a dive toward Jupiter, reaches horizontal flight at closest approach ($\gamma = 0$ deg), and, on departure, climbs symmetrically to around +90 deg. Finally, the turn angle, δ, is seen to start at zero and end with a value of 98.6 deg.

We can check our calculations by using a formula that will allow us to directly compute the total turn angle that is produced by a flyby. The equation is

$$\delta_{TOTAL} = 2 \sin^{-1}\{1 / [1 + (r_p v_\infty^2/GM)]\}$$

$$\delta_{TOTAL} = 2 \sin^{-1}(1/e)$$

For the Voyager 1 Jupiter flyby, this becomes

$$\delta_{TOTAL} = 2 \sin^{-1}(1/1.318978) = 98.6 \text{ deg}$$

which checks with our previous results.

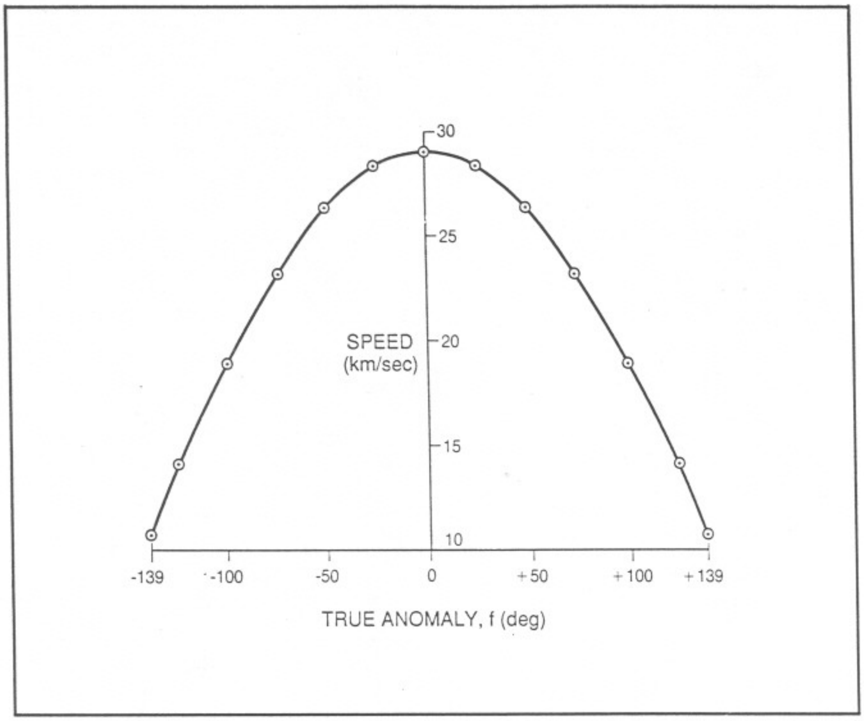

Fig. 2 **Speed of Voyager 1 with respect to Jupiter during Jupiter flyby.**

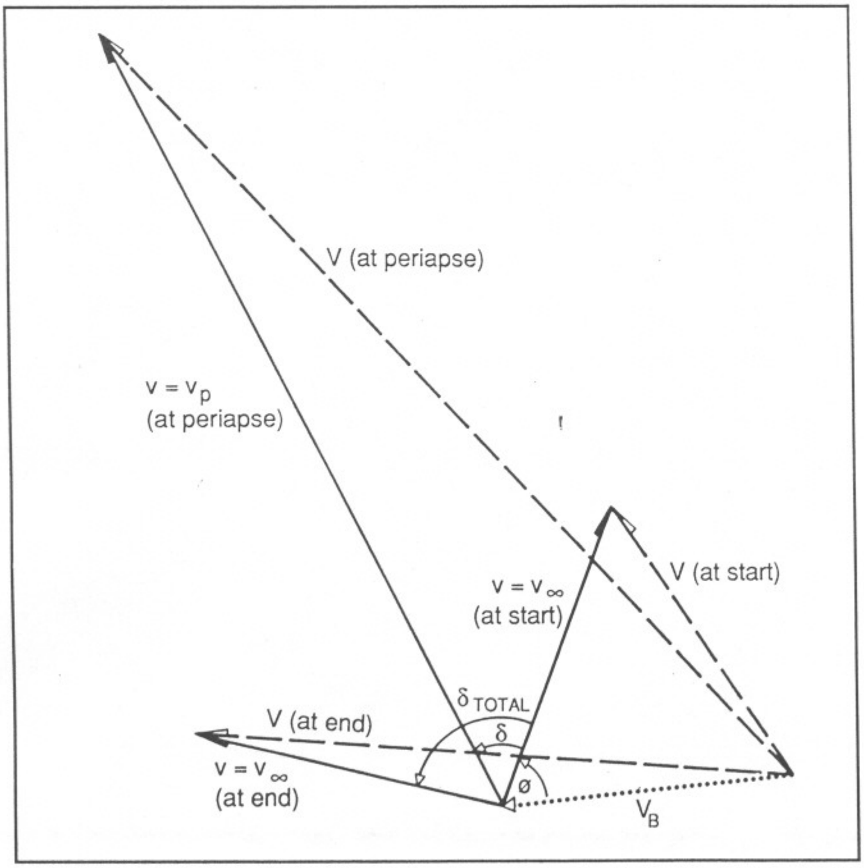

Fig. 3 **Gravity assist vector diagram.**

Figure 2 depicts Voyager 1's speed with respect to Jupiter. In this figure it is evident that the speed reaches a maximum at periapse, but thereafter drops back to its original value by the end of the flyby. Thus Jupiter produced no net change in Voyager 1's speed (or energy) with respect to the planet itself. However, Jupiter did cause Voyager's velocity orientation to be rotated by 98.6 deg. It is this fact that produces the well-known gravity assist effect. To see how this comes about we must now look at the velocity of the central body of the flyby, in this case Jupiter, with respect to the third body, in this case the sun.

Figure 3 shows the classic "vector diagram" that is often used to graphically display the gravity assist effect. The solid vectors represent the magnitude and orientation of the spacecraft's speed with respect to the planet at the beginning and end of the flyby, as well as at periapse. We have just seen that the magnitudes at the beginning and end are the same, i.e., equal to v_∞, but that the flyby has rotated the orientation. We only need to know two other quantities. The first is the speed of the flyby body with respect to the third body, known here as V_B, and shown in Fig. 3 as a dotted vector. The other is the orientation angle ϕ that this vector makes with the spacecraft's v_∞ at the beginning of the flyby, also shown in the figure.

We wish to compute, at each point in the flyby, the speed of the spacecraft with respect to the sun, known simply as V, from the information we now have. Mathematically this is just the vector addition of the spacecraft's speed with respect to the flyby body at that point, and the speed of the flyby body with respect to the sun. This vector sum is represented by a dashed vector in Fig. 3, which forms one side of a triangle that can be completed by using trigonometry. At any point in the flyby we have the solid vector, which has been previously computed as the quantity v. We can also assume that the quantity V_B is constant during the flyby, so that we also have this value as an input. From Fig. 3, it is evident that at any point in the flyby the angle between the two triangle sides that we know is just the sum $\phi + \delta$. Since we also have previously computed the value of δ at each point in the flyby, we also know this sum. Thus, by knowing two sides of a triangle and the angle between them, we can solve for the third side, using the well-known law of cosines, which can be stated in terms of our variables as

$$V = \sqrt{v^2 + V_B^2 - 2v\, V_B \cos(\phi + \delta)} \quad (14)$$

The computation of V is added as the final column in the Appendix. We can also plot the resulting profile of V as a function of true anomaly f. This is done on Fig. 4, where it is clear that the speed does not return to its original value. In the case of Voyager 1 at Jupiter, the initial speed is 12.6 km/sec and the final speed is 23.4 km/sec. This amounts to a speed increase of 10.8 km/sec relative to the sun, and this represents the gravity assist effect provided by the planet Jupiter to the Voyager 1 spacecraft.

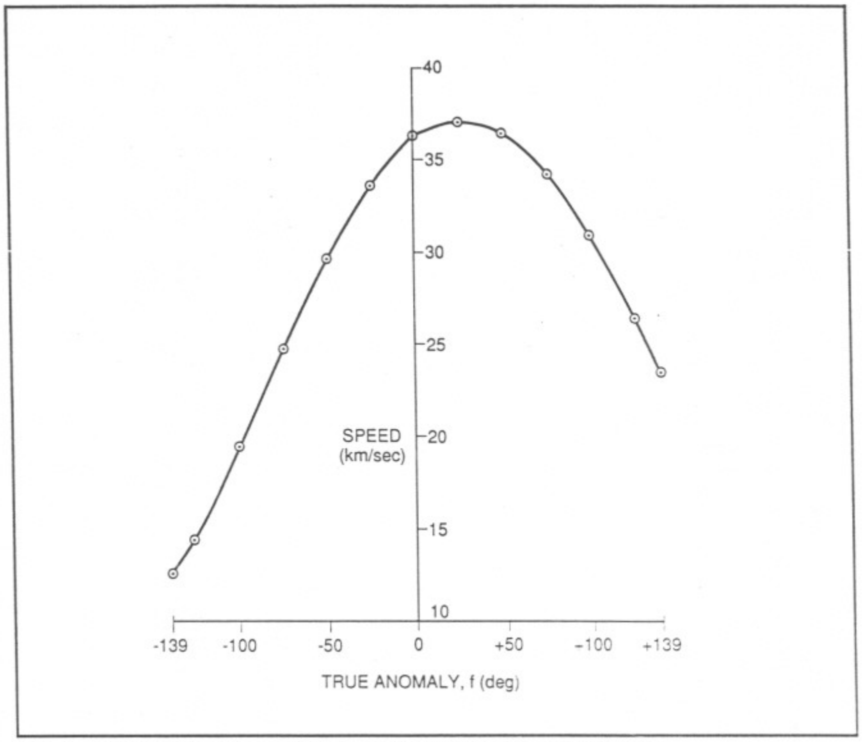

Fig. 4 Speed of Voyager 1 with respect to sun during Jupiter flyby.

Similar computations can be done for the Voyager 2 flybys of Jupiter, Saturn, and Uranus. In order to accomplish this the following data are needed:

Voyager 2 Jupiter Flyby

r_p = 721376 km $\quad V_B$ = 12.69 km/sec
v_∞ = 7.6159 km/sec
GM = 126685919 km³/sec² $\quad \phi$ = 48.3 deg

Voyager 2 Saturn Flyby

r_p = 160689 km $\quad V_B$ = 9.59 km/sec
v_∞ = 10.6731 km/sec
GM = 37929891 km³/sec² $\quad \phi$ = 98.2 deg

Voyager 2 Uranus Flyby

r_p = 107061 km $\quad V_B$ = 6.71 km/sec
v_∞ = 14.7321 km/sec
GM = 5793947 km³/sec² $\quad \phi$ = 106.0 deg

In the process of doing these calculations, there exists one potential numerical problem. At $f=0$, we expect that $\cos\gamma = 1.0$, so that $\gamma = 0$. However, depending on the number of digits used, the expression $h/(rv) = \cos\gamma$ may slightly exceed 1.0, which would not make sense. If this occurs, fix its value at 1.0, so that $\cos\gamma$ is realistic and $\gamma = 0$.

When the computations are done you should find that Voyager 2's speed was increased by 10.1 km/sec by the planet Jupiter, increased by 4.9 km/sec by the planet Saturn, and increased by 1.9 km/sec by the planet Uranus.

Figs. 5–8 depict the Voyager 1 Jupiter flyby and the Voyager 2 flybys of Jupiter, Saturn, and Uranus. Also shown superimposed on each is the relevant gravity assist vector diagram.

The procedure that has been delineated herein is reasonably accurate for flybys in which the plane of the spacecraft's trajectory is approximately the same as that of the planets' orbits around the sun, which is known as the ecliptic plane. This co-planar condition essentially reduces the problem to one in only two dimensions, thus making it amenable to the solution procedure. Flybys that do not meet this condition have to be solved with a full three-dimensional vector analysis, which is beyond the scope of the current procedure. These three-dimensional gravity assists may or may not change a spacecraft's speed (energy), but they will tend to alter the inclination or orientation of the spacecraft's orbit. Planetary encounters such as Voyager 1 at Saturn and Voyager 2 at Neptune fall into this more complex category and so are not treated here.

Gravity assist is a fascinating subject with a long history of theoretical and now practical development. The concept was documented as early as the 1920's by the Russian scientist and engineer Fridrikh Tsander. The specific application of this technique to a grand tour of the outer solar system, i.e., the Voyager Mission, was conceived contemporaneously by Gary Flandro, Michael Minovitch, and Brent Silver in the 1960's.

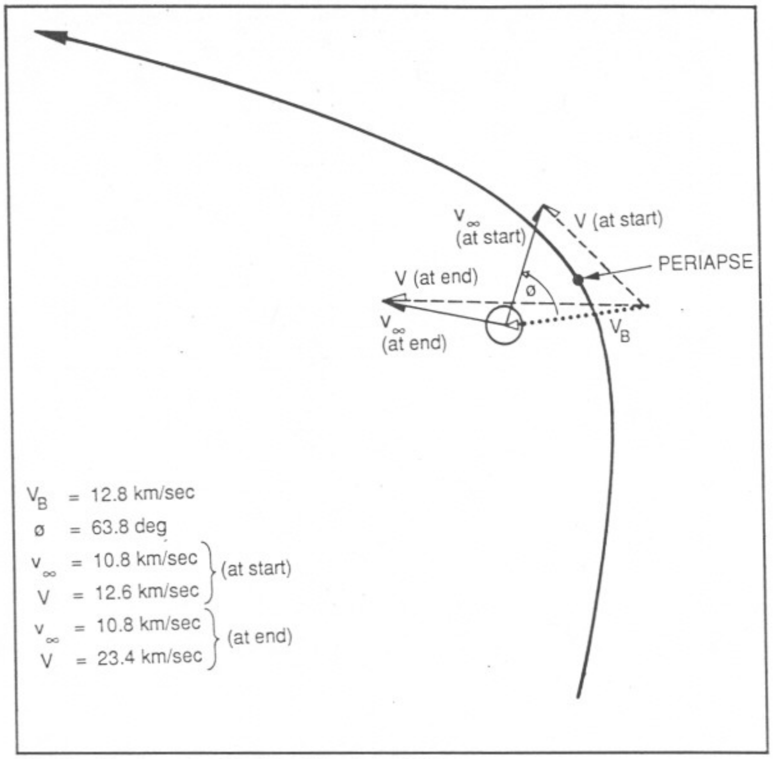

Fig. 5 Voyager 1 at Jupiter, March 5, 1979.

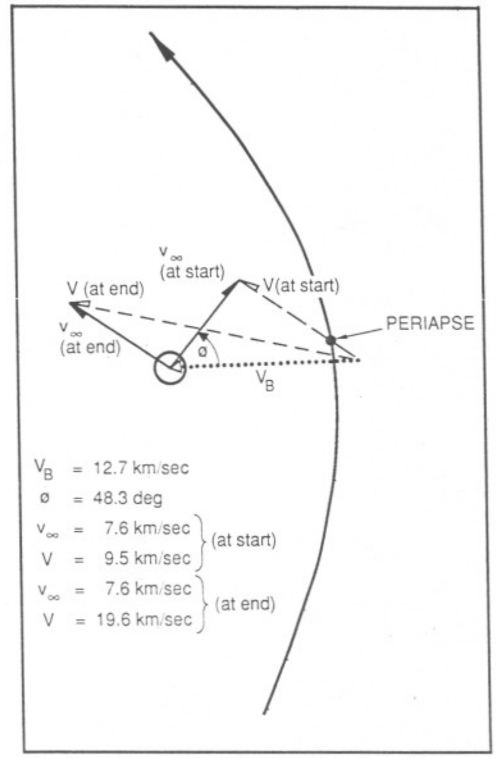

Fig. 6 Voyager 2 at Jupiter, July 9, 1979.

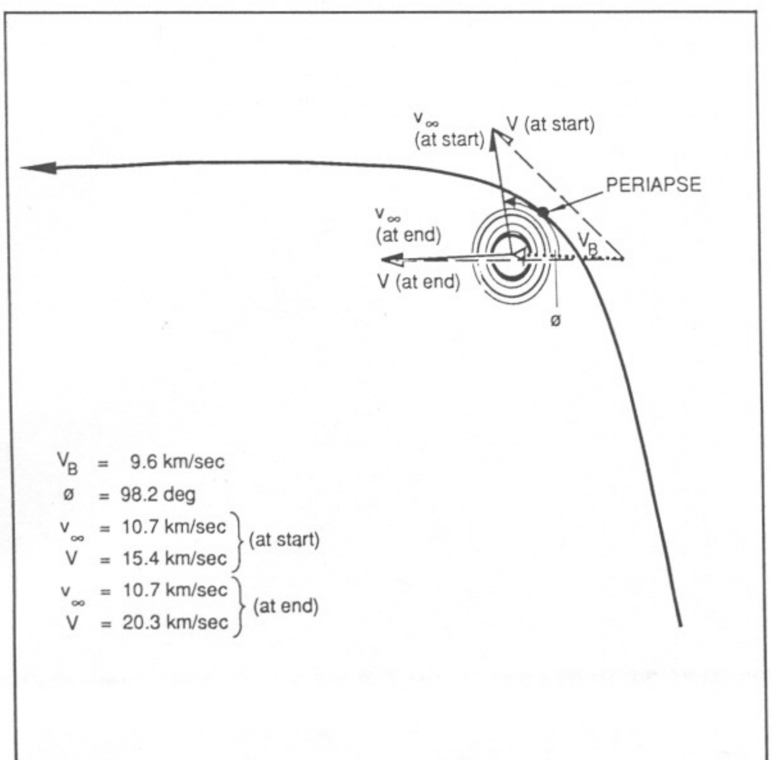

Fig. 7 Voyager 2 at Saturn, August 26, 1981.

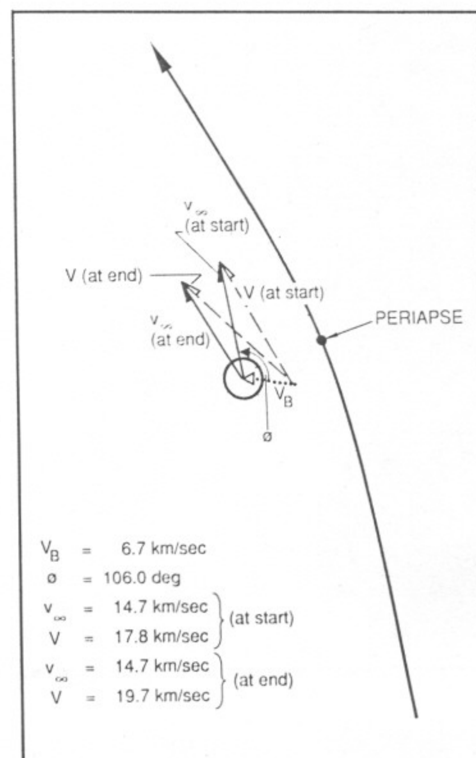

Fig. 8 Voyager 2 at Uranus, January 24, 1986.

Appendix: Voyager 1 Jupiter Flyby

$r_p = 348435$ km $V_B = 12.83$ km/sec
$v_\infty = 10.7692$ km/sec $\phi = 63.8$ deg
$GM = 126685919$ km^3/sec^2

$$a = -\frac{126685919 \text{ km}^3/\text{sec}^2}{(10.7692 \text{ km/sec})^2} = -1092349 \text{ km}$$

$$e = 1 + \frac{(348435 \text{ km})(10.7692 \text{ km/sec})^2}{126685919 \text{ km}^3/\text{sec}^2}$$
$$= 1.318978$$

$$p = (-1092349 \text{ km})\,[1 - (1.318978)^2]$$
$$= 808014 \text{ km}$$

$$f_\infty = \cos^{-1}(-1/1.318978) = 139.302 \text{ deg}$$

$$v_p = \sqrt{\frac{2(126685919 \text{ km}^3/\text{sec}^2)}{348435 \text{ km}} + (10.7692 \text{ km/sec})^2}$$
$$= 29.03699$$

$$h = (348435 \text{ km})(29.03699 \text{ km/sec}) = 10117504 \text{ km}^2/\text{sec}$$

f (deg)	$r = \frac{p}{1 + e\cos f}$ (km)	$v = \sqrt{\frac{2GM}{r} + v_\infty^2}$ (km/sec)	$\beta = f_\infty + f$ (deg)	$\gamma = \cos^{-1}\left(\frac{h}{rv}\right)$ (deg)	$\delta = \beta - \gamma - 90°$ (deg)	$v = \sqrt{v^2 + v_B^2 - 2vV_B\cos(\phi + \delta)}$ (km/sec)
−139	177394255	10.8353	0.3	−89.7	0	12.62
−125	3318806	13.8679	14.3	−77.3	1.6	14.45
−100	1048060	18.9137	39.3	−59.3	8.6	19.38
−75	602377	23.1645	64.3	−43.5	17.8	24.79
−50	437279	26.3705	89.3	−28.7	28.0	29.68
−25	368049	28.3618	114.3	−14.2	38.5	33.54
0	348435	29.0370	139.3	0	49.3	36.06
+25	368049	28.3618	164.3	+14.2	60.1	37.07
+50	437279	26.3705	189.3	+28.7	70.6	36.52
+75	602377	23.1645	214.3	+43.5	80.8	34.43
+100	1048060	18.9137	239.3	+59.3	90.0	30.95
+125	3318806	13.8679	264.3	+77.3	97.0	26.32
+139	177394255	10.8353	278.3	+89.7	98.6	23.39