A TREATISE

ON THE

ANALYTICAL DYNAMICS
OF PARTICLES AND RIGID BODIES

WITH AN INTRODUCTION TO THE
PROBLEM OF THREE BODIES

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are orbits, the differential equations of motion of the particle possess an integral linear and homogeneous in the velocities $(\dot{x}, \dot{y})$.

7. The equations of motion of a free system of m particles are

$$\frac{d^2 x_s}{dt^2} = X_s \qquad (s = 1, 2, \ldots, 3m).$$

If an integral exists of the form

$$\sum_{s=1}^{3m} f_s \dot{x}_s - Ct = \text{Constant},$$

where $f_1, f_2, \ldots, f_{3m}$ are functions of $x_1, x_2, \ldots, x_{3m}$, and C is a constant, shew that this integral can be written

$$\sum_{s=1}^{3m} k_s \dot{x}_s + \sum_{r,s=1}^{3m} a_{rs} (x_s \dot{x}_r - x_r \dot{x}_s) - Ct = \text{Constant},$$

where the quantities k_s and a_{rs} are constants. (Pennacchietti.)

8. Two particles move on a surface under the action of different forces depending only on their respective positions: if their differential equations of motion have in common an integral independent of the time, shew that the surface is applicable on a surface of revolution. (Bertrand.)

CHAPTER XIII

THE REDUCTION OF THE PROBLEM OF THREE BODIES

154. *Introduction.*

The most celebrated of all dynamical problems is known as the *Problem of Three Bodies*, and may be enunciated as follows:

Three particles attract each other according to the Newtonian law, so that between each pair of particles there is an attractive force which is proportional to the product of the masses of the particles and the inverse square of their distance apart: they are free to move in space, and are initially supposed to be moving in any given manner; to determine their subsequent motion.

The practical importance of this problem arises from its applications to Celestial Mechanics: the bodies which constitute the solar system attract each other according to the Newtonian law, and (as they have approximately the form of spheres, whose dimensions are very small compared with the distances which separate them) it is usual to consider the problem of determining their motion in an ideal form, in which the bodies are replaced by particles of masses equal to the masses of the respective bodies and occupying the positions of their centres of gravity*.

The problem of three bodies cannot be solved in finite terms by means of any of the functions at present known to analysis. This difficulty has stimulated research to such an extent, that since the year 1750 over 800 memoirs, many of them bearing the names of the greatest mathematicians, have been published on the subject†. In the present chapter, we shall discuss the known integrals of the system and their application to the reduction of the problem to a dynamical problem with a lesser number of degrees of freedom.

* The motions of the bodies relative to their centres of gravity (in the consideration of which their sizes and shapes of course cannot be neglected) are discussed separately, e.g. in the Theory of Precession and Nutation. In some cases however (e.g. in the Theory of the Satellites of the Major Planets) the oblateness of one of the bodies exercises so great an effect, that the problem cannot be divided in this way.

† For the history of the Problem of Three Bodies, cf. A. Gautier, *Essai historique sur le problème des trois corps* (Paris, 1817): R. Grant, *History of Physical Astronomy from the earliest ages to the middle of the nineteenth century* (London, 1852): E. T. Whittaker, *Report on the progress of the solution of the Problem of Three Bodies* (Brit. Ass. Rep. 1899, p. 121): and E. O. Lovett, *Quart. Journ. Math.* XLII. (1911), p. 252, who discusses the memoirs of the period 1898–1908.

155. *The differential equations of the problem.*

Let P, Q, R denote the three particles, (m_1, m_2, m_3) their masses, and (r_{23}, r_{31}, r_{12}) their mutual distances. Take any fixed rectangular axes $Oxyz$, and let (q_1, q_2, q_3), (q_4, q_5, q_6), (q_7, q_8, q_9), be the coordinates of P, Q, R, respectively. The kinetic energy of the system is

$$T = \tfrac{1}{2}m_1(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \tfrac{1}{2}m_2(\dot{q}_4^2 + \dot{q}_5^2 + \dot{q}_6^2) + \tfrac{1}{2}m_3(\dot{q}_7^2 + \dot{q}_8^2 + \dot{q}_9^2);$$

the force of attraction between m_1 and m_2 is $k^2 m_1 m_2 r_{12}^{-2}$, where k^2 is the constant of attraction: we shall suppose the units so chosen that k^2 is unity, so that this attraction becomes $m_1 m_2 r_{12}^{-2}$, and the corresponding term in the potential energy is $-m_1 m_2 r_{12}^{-1}$. The potential energy of the system is therefore

$$\begin{aligned} V &= -\frac{m_2 m_3}{r_{23}} - \frac{m_3 m_1}{r_{31}} - \frac{m_1 m_2}{r_{12}} \\ &= -m_2 m_3\{(q_4-q_7)^2 + (q_5-q_8)^2 + (q_6-q_9)^2\}^{-\frac{1}{2}} \\ &\quad - m_3 m_1\{(q_7-q_1)^2 + (q_8-q_2)^2 + (q_9-q_3)^2\}^{-\frac{1}{2}} \\ &\quad - m_1 m_2\{(q_1-q_4)^2 + (q_2-q_5)^2 + (q_3-q_6)^2\}^{-\frac{1}{2}}. \end{aligned}$$

The equations of motion of the system are

$$m_k \ddot{q}_r = -\partial V/\partial q_r \qquad (r = 1, 2, \ldots, 9),$$

where k denotes the integer part of $\frac{1}{3}(r+2)$. This system consists of 9 differential equations, each of the 2nd order, and the system is therefore of order 18.

Writing

$$m_k \dot{q}_r = p_r \qquad (r = 1, 2, \ldots, 9),$$

and

$$H = \sum_{r=1}^{9} \frac{p_r^2}{2m_k} + V,$$

the equations take the Hamiltonian form

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, 9),$$

and these are a set of 18 differential equations, each of the 1st order, for the determination of the variables $(q_1, q_2, \ldots, q_9, p_1, p_2, \ldots, p_9)$.

It was shewn by Lagrange* that this system can be reduced to a system which is only of the 6th order. That a reduction of this kind must be possible may be seen from the following considerations.

In the first place, since no forces act except the mutual attractions of the

* *Recueil des pieces qui ont remporté les prix de l'Acad. de Paris*, IX. (1772). Lagrange of course did not reduce the system to the Hamiltonian form. Cf. Bohlin, *Kongl. Sv. Vet.-Handl.* XLII. (1907), No. 9, for an improved Lagrangian reduction.

particles, the centre of gravity of the system moves in a straight line with uniform velocity. This fact is expressed by the 6 integrals

$$\begin{cases} p_1 + p_4 + p_7 = a_1, \\ p_2 + p_5 + p_8 = a_3, \\ p_3 + p_6 + p_9 = a_5, \end{cases}$$

$$\begin{cases} m_1 q_1 + m_2 q_4 + m_3 q_7 - (p_1 + p_4 + p_7)\,t = a_2, \\ m_1 q_2 + m_2 q_5 + m_3 q_8 - (p_2 + p_5 + p_8)\,t = a_4, \\ m_1 q_3 + m_2 q_6 + m_3 q_9 - (p_3 + p_6 + p_9)\,t = a_6, \end{cases}$$

where $a_1, a_2, \ldots, a_6$ are constants. It may be expected that the use of these integrals will enable us to depress the equations of motion from the 18th to the 12th order.

In the second place, the angular momentum of the three bodies round each of the coordinate axes is constant throughout the motion. This fact is analytically expressed by the equations

$$\begin{cases} q_1 p_2 - q_2 p_1 + q_4 p_5 - q_5 p_4 + q_7 p_8 - q_8 p_7 = a_7, \\ q_2 p_3 - q_3 p_2 + q_5 p_6 - q_6 p_5 + q_8 p_9 - q_9 p_8 = a_8, \\ q_3 p_1 - q_1 p_3 + q_6 p_4 - q_4 p_6 + q_9 p_7 - q_7 p_9 = a_9, \end{cases}$$

where a_7, a_8, a_9 are constants. By use of these three integrals we may expect to be able to depress further the equations of motion from the 12th to the 9th order. But when one of the coordinates which define the position of the system is taken to be the azimuth ϕ of one of the bodies with respect to some fixed axis (say the axis of z), and the other coordinates define the position of the system relative to the plane having this azimuth, the coordinate ϕ is an ignorable coordinate, and consequently the corresponding integral (which is one of the integrals of angular momentum above-mentioned) can be used to depress the order of the system by *two* units; the equations of motion can therefore, as a matter of fact, be reduced in this way to the 8th order. This fact (though contained implicity in Lagrange's memoir already cited) was first explicitly noticed by Jacobi* in 1843, and is generally referred to as the *elimination of the nodes*.

Lastly, it is possible again to depress the order of the equations by two units as in § 42, by using the integral of energy and eliminating the time. So finally *the equations of motion may be reduced to a system of the 6th order.*

* *Journ. für Math.* XXVI. p. 115. From the point of view of the theory of Partial Differential Equations, we may express the matter by saying that the integrals of angular momentum give rise to an involution-system, consisting of two functions which are in involution with each other and with H: and hence the Hamilton-Jacobi partial differential equation with 6 independent variables can be reduced to a partial differential equation with 6 − 2 or 4 independent variables: this will be the Hamilton-Jacobi partial differential equation for the reduced system.

156. *Jacobi's equation.*

Jacobi*, in considering the motion of any number of free particles in space, which attract each other according to the Newtonian law, has introduced the function

$$\tfrac{1}{2}\sum_{i,j}\frac{m_i m_j}{M}r_{ij}^2,$$

where m_i and m_j are the masses of two typical particles of the system, r_{ij} is the distance between them at time t, M is the total mass of the particles, and the summation is extended over all pairs of particles in the system. This function, which has been used in researches concerning the stability of the system, will be called *Jacobi's function* and denoted by the symbol Φ.

We shall suppose the centre of gravity of the system to be at rest; let (x_i, y_i, z_i) be the coordinates of the particle m_i referred to fixed rectangular axes with the centre of gravity as origin. The kinetic energy of the system is

$$T=\tfrac{1}{2}\sum_i m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2),$$

and consequently we have

$$2MT=(\sum_i m_i)\times\sum_i m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2).$$

But

$$(\sum_i m_i)\times\sum_i m_i\dot{x}_i^2-(\sum_i m_i\dot{x}_i)^2=\sum_{i,j}m_i m_j(\dot{x}_i-\dot{x}_j)^2,$$

where the summation on the right-hand side is extended over every pair of particles in the system: and we have $\sum_i m_i\dot{x}_i=0$, in virtue of the properties of the centre of gravity.

Thus we have

$$T=\frac{1}{2M}\sum_{i,j}m_i m_j\{(\dot{x}_i-\dot{x}_j)^2+(\dot{y}_i-\dot{y}_j)^2+(\dot{z}_i-\dot{z}_j)^2\}$$

$$=\frac{1}{2M}\sum_{i,j}m_i m_j v_{ij}^2,$$

where v_{ij} denotes the velocity of the particle m_i relative to m

In the same way we can shew that

$$\tfrac{1}{2}\sum_i m_i(x_i^2+y_i^2+z_i^2)=\Phi.$$

If now V denotes the potential energy of the system, the arbitrary constant in V being determined by the condition that V is to be zero when the particles are at infinitely great distances from each other, we have

$$V=-\sum_{i,j}\frac{m_i m_j}{r_{ij}}.$$

The equations of motion of the particle m_i are

$$m_i\ddot{x}_i=-\frac{\partial V}{\partial x_i},\qquad m_i\ddot{y}_i=-\frac{\partial V}{\partial y_i},\qquad m_i\ddot{z}_i=-\frac{\partial V}{\partial z_i}.$$

Multiply these equations by x_i, y_i, z_i, respectively, add them, and sum for all the particles of the system: since V is homogeneous of degree -1 in the variables, we thus obtain

$$\sum_i m_i(x_i\ddot{x}_i+y_i\ddot{y}_i+z_i\ddot{z}_i)=V,$$

or

$$\frac{d^2}{dt^2}\tfrac{1}{2}\sum_i m_i(x_i^2+y_i^2+z_i^2)-2T=V,$$

or

$$\frac{d^2\Phi}{dt^2}=2T+V.$$

This is called *Jacobi's equation.*

* *Vorlesungen über Dyn.*, p. 22.

157. *Reduction to the 12th order, by use of the integrals of motion of the centre of gravity.*

We shall now proceed to carry out the reductions which have been described*. It will appear that it is possible to retain the Hamiltonian form of the equations throughout all the transformations.

Taking the equations of motion of the Problem of Three Bodies in the form obtained in § 155,

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r},\qquad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r}\qquad (r=1, 2, \ldots, 9),$$

we have first to reduce this system from the 18th to the 12th order, by use of the integrals of motion of the centre of gravity. For this purpose we perform on the variables the contact-transformation defined by the equations

$$q_r=\frac{\partial W}{\partial p_r},\qquad p_r'=\frac{\partial W}{\partial q_r'}\qquad (r=1, 2, \ldots, 9),$$

where

$$W=p_1q_1'+p_2q_2'+p_3q_3'+p_4q_4'+p_5q_5'+p_6q_6'+(p_1+p_4+p_7)q_7'$$
$$+(p_2+p_5+p_8)q_8'+(p_3+p_6+p_9)q_9'.$$

Interpreting these equations, it is easily seen that (q_1', q_2', q_3') are the coordinates of m_1 relative to m_3, (q_4', q_5', q_6') are the coordinates of m_2 relative to m_3, (q_7', q_8', q_9') are the coordinates of m_3, (p_1', p_2', p_3') are the components of momentum of m_1, (p_4', p_5', p_6') are the components of momentum of m_2, and (p_7', p_8', p_9') are the components of momentum of the system.

The differential equations now become (§ 138)

$$\frac{dq_r'}{dt}=\frac{\partial H}{\partial p_r'},\qquad \frac{dp_r'}{dt}=-\frac{\partial H}{\partial q_r'}\qquad (r=1, 2, \ldots, 9),$$

where, on substitution of the new variables for the old, we have

$$H=\left(\frac{1}{2m_1}+\frac{1}{2m_3}\right)(p_1'^2+p_2'^2+p_3'^2)+\left(\frac{1}{2m_2}+\frac{1}{2m_3}\right)(p_4'^2+p_5'^2+p_6'^2)$$
$$+\frac{1}{m_3}\{p_1'p_4'+p_2'p_5'+p_3'p_6'+\tfrac{1}{2}p_7'^2+\tfrac{1}{2}p_8'^2+\tfrac{1}{2}p_9'^2-p_7'(p_1'+p_4')$$
$$-p_8'(p_2'+p_5')-p_9'(p_3'+p_6')\}$$
$$-m_2m_3\{q_4'^2+q_5'^2+q_6'^2\}^{-\frac{1}{2}}-m_3m_1\{q_1'^2+q_2'^2+q_3'^2\}^{-\frac{1}{2}}$$
$$-m_1m_2\{(q_1'-q_4')^2+(q_2'-q_5')^2+(q_3'-q_6')^2\}^{-\frac{1}{2}}.$$

* The contact-transformation used in § 157 is due to Poincaré, *C.R.* CXXIII. (1896); that used in § 158 is due to the author, and was originally published in the first edition of this work (1904). It appears worthy of note from the fact that it is an extended point-transformation, which shews that the reduction could be performed on the equations in their Lagrangian (as opposed to their Hamiltonian) form, by pure point-transformations. The second transformation in the alternative reduction (§ 160) is not an extended point-transformation. Another reduction of the Problem of Three Bodies can be constructed from the standpoint of Lie's Theory of Involution-systems and Distinguished Functions: cf. Lie, *Math. Ann.* VIII. p. 282. Cf. also Woronetz, *Bull. Univ. Kief*, 1907, and Levi-Civita, *Atti del R. Ist. Veneto*, LXXIV. (1915), p. 907.

Since q_7', q_8', q_9' are altogether absent from H, they are ignorable coordinates: the corresponding integrals are

$$p_7' = \text{Constant}, \qquad p_8' = \text{Constant}, \qquad p_9' = \text{Constant}.$$

We can without loss of generality suppose these constants of integration to be zero, as this only means that the centre of gravity of the system is taken to be at rest: the reduced kinetic potential obtained by ignoration of coordinates will therefore be derived from the unreduced kinetic potential by replacing p_7', p_8', p_9' by zero, and the new Hamiltonian function will be derived from H in the same way. *The system of the 12th order, to which the equations of motion of the problem of three bodies have now been reduced, may therefore be written* (suppressing the accents to the letters)

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, 6),$$

where

$$H = \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)(p_1^2 + p_2^2 + p_3^2) + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)(p_4^2 + p_5^2 + p_6^2) + \frac{1}{m_3}(p_1 p_4 + p_2 p_5 + p_3 p_6)$$
$$- m_2 m_3 \{q_4^2 + q_5^2 + q_6^2\}^{-\frac{1}{2}} - m_3 m_1 \{q_1^2 + q_2^2 + q_3^2\}^{-\frac{1}{2}} - m_1 m_2 \{(q_1 - q_4)^2 + (q_2 - q_5)^2 + (q_3 - q_6)^2\}^{-\frac{1}{2}}.$$

This system possesses an integral of energy,

$$H = \text{Constant},$$

and three integrals of angular momentum, namely

$$\begin{cases} q_2 p_3 - q_3 p_2 + q_5 p_6 - q_6 p_5 = A_1 \\ q_3 p_1 - q_1 p_3 + q_6 p_4 - q_4 p_6 = A_2 \\ q_1 p_2 - q_2 p_1 + q_4 p_5 - q_5 p_4 = A_3 \end{cases}$$

where A_1, A_2, A_3 are constants.

158. *Reduction to the 8th order, by use of the integrals of angular momentum and elimination of the nodes.*

The system of the 12th order obtained in the last article must now be reduced to the 8th order, by using the three integrals of angular momentum and by eliminating the nodes. This may be done in the following way.

Apply to the variables the contact-transformation defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \qquad p_r' = \frac{\partial W}{\partial q_r'} \qquad (r = 1, 2, \ldots, 6),$$

where

$$W = p_1(q_1' \cos q_5' - q_2' \cos q_6' \sin q_5') + p_2(q_1' \sin q_5' + q_2' \cos q_6' \cos q_5') + p_3 q_2' \sin q_6'$$
$$+ p_4(q_3' \cos q_5' - q_4' \cos q_6' \sin q_5') + p_5(q_3' \sin q_5' + q_4' \cos q_6' \cos q_5') + p_6 q_4' \sin q_6'.$$

It is readily seen that the new variables can be interpreted physically as follows:

In addition to the fixed axes $Oxyz$, take a new set of moving axes $Ox'y'z'$; Ox' is to be the intersection or *node* of the plane Oxy with the plane of the three bodies, Oy' is to be a line perpendicular to this in the plane of the three bodies, and Oz' is to be normal to the plane of the three bodies. Then (q_1', q_2') are the coordinates of m_1 relative to axes drawn through m_3 parallel to Ox', Oy'; (q_3', q_4') are the coordinates of m_2 relative to the same axes; q_5' is the angle between Ox' and Ox; q_6' is the angle between Oz' and Oz; p_1' and p_2' are the components of momentum of m_1 relative to the axes Ox', Oy'; p_3' and p_4' are the components of momentum of m_2 relative to the same axes; p_5' and p_6' are the angular momenta of the system relative to the axes Oz and Ox' respectively.

The equations of motion in terms of the new variables are (§ 138)

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, 6),$$

where, on substitution in H of the new variables for the old, we have

$$H = \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)\Big[p_1'^2 + p_2'^2 + \frac{1}{(q_2'q_3' - q_1'q_4')^2}\{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3')q_4' \cot q_6' + p_5'q_4' \operatorname{cosec} q_6' + p_6'q_3'\}^2\Big]$$
$$+ \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)\Big[p_3'^2 + p_4'^2 + \frac{1}{(q_2'q_3' - q_1'q_4')^2}\{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3')q_2' \cot q_6' + p_5'q_2' \operatorname{cosec} q_6' + p_6'q_1'\}^2\Big]$$
$$+ \frac{1}{m_3}\Big[p_1'p_3' + p_2'p_4' - \frac{1}{(q_2'q_3' - q_1'q_4')^2}\{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3')q_4' \cot q_6' + p_5'q_4' \operatorname{cosec} q_6' + p_6'q_3'\}$$
$$\{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3')q_2' \cot q_6' + p_5'q_2' \operatorname{cosec} q_6' + p_6'q_1'\}\Big]$$
$$- m_2 m_3 (q_3'^2 + q_4'^2)^{-\frac{1}{2}} - m_3 m_1 (q_1'^2 + q_2'^2)^{-\frac{1}{2}} - m_1 m_2 \{(q_1' - q_3')^2 + (q_2' - q_4')^2\}^{-\frac{1}{2}}.$$

Now q_5' does not occur in H, and is therefore an ignorable coordinate; the corresponding integral is

$$p_5' = k, \qquad \text{where } k \text{ is a constant.}$$

The equation $dq_5'/dt = \partial H/\partial k$ can be integrated by a simple quadrature when the rest of the equations of motion have been integrated; the equations for q_5' and p_5' will therefore fall out of the system, which thus reduces to the system of the 10th order

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, 3, 4, 6),$$

where p_5' is to be replaced by the constant k wherever it occurs in H.

We have now made use of one of the three integrals of angular momentum (namely $p_5' = k$) and the elimination of the nodes: when the other two integrals of angular momentum are expressed in terms of the new variables, they become

$$\begin{cases} (p_2'q_1' - p_1'q_2' + p_4'q_3' - p_3'q_4') \sin q_5' \operatorname{cosec} q_6' - k \sin q_5' \cot q_6' + p_6' \cos q_5' = A_1, \\ -(p_2'q_1' - p_1'q_2' + p_4'q_3' - p_3'q_4') \cos q_5' \operatorname{cosec} q_6' + k \cos q_5' \cot q_6' + p_6' \sin q_5' = A_2. \end{cases}$$

The values of the constants A_1 and A_2 depend on the position of the fixed axes $Oxyz$; we shall choose the axis Oz to be the line of resultant angular momentum of the system, so that (cf. § 69) the constants A_1 and A_2 are zero: the special xy-plane thus introduced is called the *invariable plane* of the system. The two last equations then give

$$k \cos q_6' = p_2'q_1' - p_1'q_2' + p_4'q_3' - p_3'q_4',$$
$$p_6' = 0.$$

These equations determine q_6' and p_6' in terms of the other variables, and so can be regarded as replacing the equations

$$\frac{dq_6'}{dt} = \frac{\partial H}{\partial p_6'}, \qquad \frac{dp_6'}{dt} = -\frac{\partial H}{\partial q_6'},$$

in the system. The system thus becomes

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, 3, 4),$$

where

$$\begin{aligned} H = {} & \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)\Bigg[p_1'^2 + p_2'^2 + \frac{q_4'^2}{(q_2'q_3' - q_1'q_4')^2} \\ & \qquad \{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3') \cot q_6' + k \operatorname{cosec} q_6'\}^2\Bigg] \\ & + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)\Bigg[p_3'^2 + p_4'^2 + \frac{q_2'^2}{(q_2'q_3' - q_1'q_4')^2} \\ & \qquad \{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3') \cot q_6' + k \operatorname{cosec} q_6'\}^2\Bigg] \\ & + \frac{1}{m_3}\Bigg[p_1'p_3' + p_2'p_4' - \frac{q_2'q_4'}{(q_2'q_3' - q_1'q_4')^2} \\ & \qquad \{(p_1'q_2' - p_2'q_1' + p_3'q_4' - p_4'q_3') \cot q_6' + k \operatorname{cosec} q_6'\}^2\Bigg] \\ & - m_2m_3(q_3'^2 + q_4'^2)^{-\frac{1}{2}} - m_3m_1(q_1'^2 + q_2'^2)^{-\frac{1}{2}} - m_1m_2\{(q_1' - q_3')^2 + (q_2' - q_4')^2\}^{-\frac{1}{2}}, \end{aligned}$$

and where, *after the derivates of H have been formed*, q_6' is to be replaced by its value found from the equation

$$k \cos q_6' = p_2'q_1' - p_1'q_2' + p_4'q_3' - p_3'q_4'.$$

Now let H' be the function obtained when this value of q_6' is substituted in H; then if s denotes any one of the variables $q_1', q_2', q_3', q_4', p_1', p_2', p_3', p_4'$, we have

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial q_6'}\frac{\partial q_6'}{\partial s}.$$

But since $p_6' = 0$, we have $\partial H/\partial q_6' = \dot{p}_6' = 0$, and therefore

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s};$$

in other words, we can make the substitution for q_6' in H *before* forming the derivates of H; and thus (suppressing the accents) *the equations of motion of the Problem of Three Bodies are reduced to the system of the 8th order*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, 3, 4),$$

where

$$\begin{aligned} H = {} & \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)(p_1^2 + p_2^2) + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)(p_3^2 + p_4^2) + \frac{1}{m_3}(p_1p_3 + p_2p_4) \\ & + (q_2q_3 - q_1q_4)^{-2}\left\{\left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)q_4^2 + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)q_2^2 - \frac{q_2q_4}{m_3}\right\} \\ & \qquad \{k^2 - (p_2q_1 - p_1q_2 + p_4q_3 - p_3q_4)^2\} \\ & - m_2m_3(q_3^2 + q_4^2)^{-\frac{1}{2}} - m_3m_1(q_1^2 + q_2^2)^{-\frac{1}{2}} - m_1m_2\{(q_1 - q_3)^2 + (q_2 - q_4)^2\}^{-\frac{1}{2}}. \end{aligned}$$

Many of the quantities occurring in H have simple physical interpretations: thus $(q_2q\ - q_1q_4)$ is twice the area of the triangle formed by the bodies: and

$$\frac{2m_1m_2m_3}{m_1 + m_2 + m_3}\left\{\left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)q_4^2 + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)q_2^2 - \frac{1}{m_3}q_2q_4\right\}$$

is the moment of inertia of the three bodies about the line in which the plane of the bodies meets the invariable plane through their centre of gravity.

It is also to be noted that this value of H differs from the value of H when k is zero by terms which do not involve the variables p_1, p_2, p_3, p_4: these terms in k can therefore be regarded as part of the potential energy, and we can say that the system differs from the corresponding system for which k is zero only by certain modifications in the potential energy. It may easily be shewn that when k is zero the motion takes place in a plane.

159. *Reduction to the 6th order.*

The equations of motion can now be reduced further from the 8th to the 6th order, by making use of the integral of energy

$$H = \text{Constant},$$

and eliminating the time. The theorem of § 141 shews that in performing this reduction the Hamiltonian form of the differential equations can be conserved. As the actual reduction is not required subsequently, it will not be given here in detail.

The Hamiltonian system of the 6th order thus obtained is, in the present state of our knowledge, the ultimate reduced form of the equations of motion of the general Problem of Three Bodies.

160. *Alternative reduction of the problem from the 18th to the 6th order.*

We shall now give another reduction* of the general problem of three bodies to a Hamiltonian system of the 6th order.

Let the original Hamiltonian system of equations of motion (§ 155) be transformed by the contact-transformation

$$q_r' = \frac{\partial W}{\partial p_r'}, \quad p_r = \frac{\partial W}{\partial q_r} \qquad (r = 1, 2, \ldots, 9),$$

where

$$\begin{aligned} W = {} & p_1'(q_4 - q_1) + p_2'(q_5 - q_2) + p_3'(q_6 - q_3) \\ & + p_4'\left(q_7 - \frac{m_1 q_1 + m_2 q_4}{m_1 + m_2}\right) + p_5'\left(q_8 - \frac{m_1 q_2 + m_2 q_5}{m_1 + m_2}\right) \\ & + p_6'\left(q_9 - \frac{m_1 q_3 + m_2 q_6}{m_1 + m_2}\right) + p_7'(m_1 q_1 + m_2 q_4 + m_3 q_7) \\ & + p_8'(m_1 q_2 + m_2 q_5 + m_3 q_8) + p_9'(m_1 q_3 + m_2 q_6 + m_3 q_9). \end{aligned}$$

The integrals of motion of the centre of gravity, when expressed in terms of the new variables, can be written

$$q_7' = q_8' = q_9' = p_7' = p_8' = p_9' = 0,$$

and consequently the transformed system is only of the 12th order: suppressing the accents in the new variables, it is

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, 6),$$

where

$$\begin{aligned} H = {} & \frac{1}{2\mu}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2\mu'}(p_4^2 + p_5^2 + p_6^2) - m_1 m_2 (q_1^2 + q_2^2 + q_3^2)^{-\frac{1}{2}} \\ & - m_1 m_3 \left\{q_4^2 + q_5^2 + q_6^2 + \frac{2m_2}{m_1 + m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) \right. \\ & \qquad \left. + \left(\frac{m_2}{m_1 + m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2)\right\}^{-\frac{1}{2}} \\ & - m_2 m_3 \left\{q_4^2 + q_5^2 + q_6^2 - \frac{2m_1}{m_1 + m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) \right. \\ & \qquad \left. + \left(\frac{m_1}{m_1 + m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2)\right\}^{-\frac{1}{2}}, \end{aligned}$$

and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \mu' = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}.$$

The new variables may be interpreted physically in the following way: Let G be the centre of gravity of m_1 and m_2. Then (q_1, q_2, q_3) are the

* Due to Radau, *Annales de l'Éc. Norm. Sup.* v. (1868), p. 311.

projections of $m_1 m_2$ on the fixed axes, and (q_4, q_5, q_6) are the projections of Gm_3 on the axes. Further

$$\mu \frac{dq_r}{dt} = p_r \quad (r = 1, 2, 3), \quad \text{and} \quad \mu' \frac{dq_r}{dt} = p_r \quad (r = 4, 5, 6).$$

The new Hamiltonian system clearly represents the equations of motion of two particles, one of mass μ at a point whose coordinates are (q_1, q_2, q_3), and the other of mass μ' at a point whose coordinates are (q_4, q_5, q_6); these particles being supposed to move freely in space under the action of forces derivable from a potential energy represented by the terms in H which are independent of the p's. We have therefore replaced the Problem of Three Bodies by the problem of two bodies moving under this system of forces. This reduction, though substantially contained in Jacobi's* paper of 1843, was first explicitly stated by Bertrand† in 1852.

We shall suppose the axes so chosen that the plane of xy is the invariable plane for the motion of the particles μ and μ', i.e. so that the angular momentum of these particles about any line in the plane Oxy is zero.

Let the Hamiltonian system of the 12th order be transformed by the contact-transformation which is defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \quad p_r' = \frac{\partial W}{\partial q_r'} \qquad (r = 1, 2, \ldots, 6),$$

where

$$\begin{aligned} W = {} & (p_2 \sin q_5' + p_1 \cos q_5')\, q_1' \cos q_3' + q_1' \sin q_3' \{(p_2 \cos q_5' - p_1 \sin q_5')^2 + p_3^2\}^{\frac{1}{2}} \\ & + (p_5 \sin q_6' + p_4 \cos q_6')\, q_2' \cos q_4' + q_2' \sin q_4' \{(p_5 \cos q_6' - p_4 \sin q_6')^2 + p_6^2\}^{\frac{1}{2}}. \end{aligned}$$

The new variables are easily seen to have the following physical interpretations: q_1' is the length of the radius vector from the origin to the particle μ, q_2' is the radius from the origin to μ', q_3' is the angle between q_1' and the intersection (or *node*) of the invariable plane with the plane through two consecutive positions of q_1' (which we shall call the *plane of instantaneous motion* of μ), q_4' is the angle between q_2' and the node of the invariable plane on the plane of instantaneous motion of μ', q_5' is the angle between Ox and the former of these nodes, q_6' is the angle between Ox and the latter of these nodes, p_1' is $\mu \dot{q}_1'$, p_2' is $\mu' \dot{q}_2'$, p_3' is the angular momentum of μ round the origin, p_4' is the angular momentum of μ' round the origin, p_5' is the angular momentum of μ round the normal at the origin to the invariable plane, and p_6' is the angular momentum of μ' round the same line.

The equations of motion in their new form are (§ 138)

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \quad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, \ldots, 6),$$

* *Journal für Math.* XXVI. p. 115. † *Journal de math.* XVII. p. 393.

where H is supposed expressed in terms of the new variables. Let this system be transformed by the contact-transformation

$$p_r'' = \frac{\partial W}{\partial q_r''}, \qquad q_r' = \frac{\partial W}{\partial p_r'} \qquad (r = 1, 2, \ldots, 6),$$

where

$$W = q_5''(p_5' - p_6') + q_6''(p_5' + p_6') + q_1''p_1' + q_2''p_2' + q_3''p_3' + q_4''p_4'.$$

The equations of motion now become

$$\frac{dq_r''}{dt} = \frac{\partial H}{\partial p_r''}, \qquad \frac{dp_r''}{dt} = -\frac{\partial H}{\partial q_r''} \qquad (r = 1, 2, \ldots, 6).$$

But H does not involve q_6'', as may be seen either by expressing H in terms of the new variables, or by observing that q_6'' depends on the arbitrarily chosen position of the axis Ox, while none of the other coordinates depend on this quantity. We have therefore

$$\dot{p}_6'' = -\partial H/\partial q_6'' = 0, \qquad \text{so } p_6'' = k,$$

where k is a constant; this is really one of the three integrals of angular momentum. Substituting k for p_6'' in H, the equation

$$\dot{q}_6'' = \partial H/\partial k$$

can be integrated by a quadrature when the rest of the equations have been solved: so the equations for p_6'' and q_6'' can be separated from the system, which reduces to the 10th order system

$$\frac{dq_r''}{dt} = \frac{\partial H}{\partial p_r''}, \qquad \frac{dp_r''}{dt} = -\frac{\partial H}{\partial q_r''} \qquad (r = 1, 2, \ldots, 5).$$

We have still to use the two remaining integrals of angular momentum; these, when expressed in terms of the new variables, are readily found to be represented by

$$q_5'' = 90^\circ, \qquad kp_5'' = p_3''^2 - p_4''^2;$$

no arbitrary constants of integration enter, owing to the fact that the plane of xy is the invariable plane.

The system may therefore be replaced by these two equations and the equations

$$\frac{dq_r''}{dt} = \frac{\partial H}{\partial p_r''}, \qquad \frac{dp_r''}{dt} = -\frac{\partial H}{\partial q_r''} \qquad (r = 1, 2, 3, 4),$$

where, in this last set, q_5'' can be replaced by 90° before the derivates of H have been formed, and p_5'' is to be replaced by $(p_3''^2 - p_4''^2)/k$ after the derivates of H have been formed. Let H' denote the function derived from H by making this substitution for p_5'', and let s denote any one of the variables $q_1'', q_2'', q_3'', q_4'', p_1'', p_2'', p_3'', p_4''$; then we have

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial p_5''}\frac{\partial p_5''}{\partial s} = \frac{\partial H}{\partial s} + \dot{q}_5''\frac{\partial p_5''}{\partial s} = \frac{\partial H}{\partial s},$$

and it is therefore allowable to substitute for p_5'' in H *before* the derivates of H have been formed. The equations of motion are thus reduced to a system of the 8th order, which (suppressing the accents) may be written in the form

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, 3, 4),$$

where, effecting in H the transformations which have been indicated, we have

$$H = \frac{1}{2\mu}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) + \frac{1}{2\mu'}\left(p_3^2 + \frac{p_4^2}{q_3^2}\right) - m_1m_2q_1^{-1}$$
$$- m_1m_3\left\{q_3^2 - \frac{2m_2q_1q_3}{m_1+m_2}\left(\cos q_2\cos q_4 - \frac{k^2 - p_2^2 - p_4^2}{2p_2p_4}\sin q_2\sin q_4\right) + \frac{m_2^2}{(m_1+m_2)^2}q_1^2\right\}^{-\frac{1}{2}}$$
$$- m_2m_3\left\{q_3^2 + \frac{2m_1q_1q_3}{m_1+m_2}\left(\cos q_2\cos q_4 - \frac{k^2 - p_2^2 - p_4^2}{2p_2p_4}\sin q_2\sin q_4\right) + \frac{m_1^2}{(m_1+m_2)^2}q_1^2\right\}^{-\frac{1}{2}}.$$

The equations of motion may further be reduced to a system of the 6th order by the method of § 141, using the integral of energy

$$H = \text{Constant}$$

and eliminating the time. As the reduction is not required subsequently, it will not be given in detail here.

161. *The problem of three bodies in a plane.*

The motion of the three particles may be supposed to take place in a plane, instead of in three-dimensional space; this will obviously happen if the directions of the initial velocities of the bodies are in the plane of the bodies.

This case is known as the *problem of three bodies in a plane*: we shall now proceed to reduce the equations of motion to a Hamiltonian system of the lowest possible order.

Let (q_1, q_2) be the coordinates of m_1, (q_3, q_4) the coordinates of m_2, and (q_5, q_6) the coordinates of m_3, referred to any fixed axes Ox, Oy in the plane of the motion; and let $p_r = m_k\dot{q}_r$, where k denotes the greatest integer in $\frac{1}{2}(r+1)$. The equations of motion are (as in § 155)

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, \ldots, 6),$$

where

$$H = \frac{1}{2m_1}(p_1^2 + p_2^2) + \frac{1}{2m_2}(p_3^2 + p_4^2) + \frac{1}{2m_3}(p_5^2 + p_6^2) - m_2m_3\{(q_3 - q_5)^2 + (q_4 - q_6)^2\}^{-\frac{1}{2}}$$
$$- m_3m_1\{(q_5 - q_1)^2 + (q_6 - q_2)^2\}^{-\frac{1}{2}} - m_1m_2\{(q_1 - q_3)^2 + (q_2 - q_4)^2\}^{-\frac{1}{2}}.$$

These equations will now be reduced from the 12th to the 8th order, by using the four integrals of motion of the centre of gravity. Perform on the variables the contact-transformation defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \qquad p_r' = \frac{\partial W}{\partial q_r'} \qquad (r = 1, 2, \ldots, 6),$$

where

$$W = p_1 q_1' + p_2 q_2' + p_3 q_3' + p_4 q_4' + (p_1 + p_3 + p_5)\, q_5' + (p_2 + p_4 + p_6)\, q_6'.$$

It is easily seen that (q_1', q_2') are the coordinates of m_1 relative to axes through m_3 parallel to the fixed axes, (q_3', q_4') are the coordinates of m_2 relative to the same axes, (q_5', q_6') are the coordinates of m_3 relative to the original axes, (p_1', p_2') are the components of momentum of m_1, (p_3', p_4') are the components of momentum of m_2, and (p_5', p_6') are the components of momentum of the system.

As in § 157, the equations for q_5', q_6', p_5', p_6' disappear from the system; and (suppressing the accents in the new variables) the equations of motion reduce to the system of the 8th order,

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, 3, 4),$$

where

$$H = \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)(p_1^2 + p_2^2) + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)(p_3^2 + p_4^2) + \frac{1}{m_3}(p_1 p_3 + p_2 p_4)$$
$$- m_2 m_3 (q_3^2 + q_4^2)^{-\frac{1}{2}} - m_3 m_1 (q_1^2 + q_2^2)^{-\frac{1}{2}} + m_1 m_2 \{(q_1 - q_3)^2 + (q_2 - q_4)^2\}^{-\frac{1}{2}}.$$

Next, we shall shew that this system possesses an ignorable coordinate, which will make possible a further reduction through two units.

Perform on the system the contact-transformation defined by the equations

$$q_r = \frac{\partial W}{\partial p_r}, \qquad p_r' = \frac{\partial W}{\partial q_r'} \qquad (r = 1, 2, 3, 4),$$

where

$$W = p_1 q_1' \cos q_4' + p_2 q_1' \sin q_4' + p_3 (q_2' \cos q_4' - q_3' \sin q_4') + p_4 (q_2' \sin q_4' + q_3' \cos q_4').$$

The physical interpretation of this transformation is as follows: q_1' is the distance $m_1 m_3$; q_2' and q_3' are the projections of $m_3 m_2$ on, and perpendicular to, $m_1 m_3$; q_4' is the angle between $m_3 m_1$ and the axis of x; p_1' is the component of momentum of m_1 along $m_3 m_1$; p_2' and p_3' are the components of momentum of m_2 parallel and perpendicular to $m_3 m_1$; and p_4' is the angular momentum of the system.

The differential equations, when expressed in terms of the new variables, become

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2, 3, 4),$$

where

$$H = \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)\left\{p_1'^2 + \frac{1}{q_1'^2}(p_3' q_2' - p_2' q_3' - p_4')^2\right\} + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)(p_2'^2 + p_3'^2)$$
$$+ \frac{1}{m_3}\left\{p_1' p_2' - \frac{p_3'}{q_1'}(p_3' q_2' - p_2' q_3' - p_4')\right\} - m_2 m_3 (q_2'^2 + q_3'^2)^{-\frac{1}{2}}$$
$$- m_3 m_1 q_1'^{-1} - m_1 m_2 \{(q_1' - q_2')^2 + q_3'^2\}^{-\frac{1}{2}}.$$

Since q_4' is not contained in H, it is an ignorable coordinate; the corresponding integral is $p_4' = k$, where k is a constant; this can be interpreted as the integral of angular momentum of the system. The equation $\dot{q}_4' = \partial H / \partial p_4'$ can be integrated by a quadrature when the rest of the equations have been integrated; and thus the equations for p_4' and q_4' disappear from the system.

Suppressing the accents on the new variables, the equations can therefore be written

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2, 3),$$

where

$$H = \left(\frac{1}{2m_1} + \frac{1}{2m_3}\right)\left\{p_1^2 + \frac{1}{q_1^2}(p_3 q_2 - p_2 q_3 - k)^2\right\} + \left(\frac{1}{2m_2} + \frac{1}{2m_3}\right)(p_2^2 + p_3^2)$$
$$+ \frac{1}{m_3}\left\{p_1 p_2 - \frac{p_3}{q_1}(p_3 q_2 - p_2 q_3 - k)\right\} - m_2 m_3 (q_2^2 + q_3^2)^{-\frac{1}{2}}$$
$$- m_3 m_1 q_1^{-1} - m_1 m_2 \{(q_1 - q_2)^2 + q_3^2\}^{-\frac{1}{2}}.$$

This is a system of the 6th order; it can be reduced to the 4th order by the process of § 141, making use of the integral of energy and eliminating the time.

162. *The restricted problem of three bodies.*

Another special case of the problem of three bodies, which has occupied a prominent place in recent researches, is the *restricted problem of three bodies*; this may be enunciated as follows:

Two bodies S and J revolve round their centre of gravity, O, in circular orbits, under the influence of their mutual attraction. A third body P, without mass (i.e. such that it is attracted by S and J, but does not influence their motion), moves in the same plane as S and J; the restricted problem of three bodies is to determine the motion of the body P, which is generally called the *planetoid*.

Let m_1 and m_2 be the masses of S and J, and write

$$F = \frac{m_1}{SP} + \frac{m_2}{JP}.$$

Take any fixed rectangular axes OX, OY, through O, in the plane of the motion; let (X, Y) be the coordinates, and (U, V) the components of velocity, of P. The equations of motion are

$$\frac{d^2X}{dt^2} = \frac{\partial F}{\partial X}, \qquad \frac{d^2Y}{dt^2} = \frac{\partial F}{\partial Y},$$

or in the Hamiltonian form,

$$\frac{dX}{dt} = \frac{\partial H}{\partial U}, \qquad \frac{dY}{dt} = \frac{\partial H}{\partial V}, \qquad \frac{dU}{dt} = -\frac{\partial H}{\partial X}, \qquad \frac{dV}{dt} = -\frac{\partial H}{\partial Y},$$

where $$H = \tfrac{1}{2}(U^2 + V^2) - F.$$

Since F is a function not only of X and Y but also of t, the equation $H =$ Constant is *not* an integral of the system.

Perform on the variables the contact-transformation which is defined by the equations

$$X = \frac{\partial W}{\partial U}, \qquad Y = \frac{\partial W}{\partial V}, \qquad u = \frac{\partial W}{\partial x}, \qquad v = \frac{\partial W}{\partial y},$$

where $\quad W = U(x \cos nt - y \sin nt) + V(x \sin nt + y \cos nt)$,

and n is the angular velocity of SJ. The equations become

$$\frac{dx}{dt} = \frac{\partial K}{\partial u}, \qquad \frac{dy}{dt} = \frac{\partial K}{\partial v}, \qquad \frac{du}{dt} = -\frac{\partial K}{\partial x}, \qquad \frac{dv}{dt} = -\frac{\partial K}{\partial y},$$

where (§ 138)

$$K = H - \frac{\partial W}{\partial t}$$
$$= \tfrac{1}{2}(u^2 + v^2) + n(uy - vx) - F;$$

it is at once seen that x and y are the coordinates of the planetoid referred to the moving line OJ as axis of x, and a line perpendicular to this through O as axis of y. F is now a function of x and y only, so K does not involve t explicitly, and

$$K = \text{Constant}$$

is an integral of the system; it is called the *Jacobian integral** of the restricted problem of three bodies.

Another form of the equations of motion is obtained by applying to the last system the contact-transformation

$$x = \frac{\partial W}{\partial u}, \qquad y = \frac{\partial W}{\partial v}, \qquad p_1 = \frac{\partial W}{\partial q_1}, \qquad p_2 = \frac{\partial W}{\partial q_2},$$

where $\quad W = q_1 (u \cos q_2 + v \sin q_2)$.

The new variables may be defined directly by the equations

$$q_1 = OP, \qquad q_2 = POJ, \qquad p_1 = \frac{d}{dt}(OP), \qquad p_2 = OP^2 \frac{d}{dt}(POX),$$

and the equations of motion become

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2),$$

where

$$H = \tfrac{1}{2}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) - np_2 - F.$$

Another form† is obtained by applying to these equations the contact-transformation

$$p_r = \frac{\partial W}{\partial q_r}, \qquad q_r' = \frac{\partial W}{\partial p_r'} \qquad (r = 1, 2),$$

where

$$W = p_2' q_2 + \int_{p_1'\{p_1' - (p_1'^4 - p_2'^2)^{\frac{1}{2}}\}}^{q_1} \left\{-\frac{p_2'^2}{u^2} + \frac{2}{u} - \frac{1}{p_1'^2}\right\}^{\frac{1}{2}} du,$$

* Jacobi, *Comptes Rendus*, III. (1836), p. 59.

† Adopted by Poincaré in his *Nouvelles méthodes de la Méc. Céleste.*

where u denotes a current variable of integration. These equations may be written

$$p_1 = \left(-\frac{p_2'^2}{q_1^2} + \frac{2}{q_1} - \frac{1}{p_1'^2}\right)^{\frac{1}{2}}, \qquad p_2 = p_2',$$

$$q_1' = \arccos\left\{\frac{1 - \dfrac{q_1}{p_1'^2}}{\left(1 - \dfrac{p_2'^2}{p_1'^2}\right)^{\frac{1}{2}}}\right\} - \left(-\frac{p_2'^2}{p_1'^2} + \frac{2q_1}{p_1'^2} - \frac{q_1^2}{p_1'^4}\right)^{\frac{1}{2}}, \quad q_2' = q_2 - \arccos\left\{\frac{\dfrac{p_2'^2}{q_1} - 1}{\left(1 - \dfrac{p_2'^2}{p_1'^2}\right)^{\frac{1}{2}}}\right\},$$

and it is easily seen that q_1' is the mean anomaly of the planetoid in the ellipse which it would describe about a fixed body of unit mass at O, if projected from its instantaneous position with its instantaneous velocity; q_2' is the longitude of the apse of this ellipse, measured from OJ; p_1' is $a^{\frac{1}{2}}$, and p_2' is $\{a(1 - e^2)\}^{\frac{1}{2}}$, where a is the semi-major axis and e is the eccentricity of this ellipse. H does not involve t explicitly, so $H =$ Constant is an integral of the equations of motion, which are now

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \qquad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \qquad (r = 1, 2).$$

If we take the sum of the masses of S and J to be the unit of mass, and denote these masses by $1 - \mu$ and μ respectively, we have

$$H = \tfrac{1}{2}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) - np_2 - \frac{1 - \mu}{SP} - \frac{\mu}{JP}.$$

This is an analytic function of p_1', p_2', q_1', q_2', μ, which is periodic in q_1' and q_2', with the period 2π. Moreover, to find the term independent of μ in H, we suppose μ to be zero; since SP now becomes q_1, we have

$$H = \tfrac{1}{2}\left(\frac{2}{q_1} - \frac{1}{p_1'^2}\right) - np_2' - \frac{1}{q_1} = -\frac{1}{2p_1'^2} - np_2'.$$

Thus finally, discarding the accents, *the equations of motion of the restricted problem of three bodies may be taken in the form*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \qquad (r = 1, 2),$$

where H can be expanded as a power-series in μ in the form

$$H = H_0 + \mu H_1 + \mu^2 H_2 + \ldots,$$

and

$$H_0 = -\frac{1}{2p_1^2} - np_2,$$

while H_1, H_2, ... are periodic in q_1 and q_2, with the period 2π.

The equations of this 4th order system may be reduced to a Hamiltonian system of the second order by use of the integral $H =$ Constant and elimination of the time, as in § 141.

163. *Extension to the problem of n bodies.*

Many of the transformations which have been used in the present chapter in the reduction of the problem of three bodies can be extended so as to apply to the general problem of n bodies which attract each other according to the Newtonian law. In their original form, the equations of motion of the n bodies constitute a system of the $6n$th order; this can be reduced to the $(6n-12)$th order, by using the six integrals of motion of the centre of gravity, the three integrals of angular momentum, the integral of energy, the elimination of the time, and the elimination of the nodes.

The reduction has been performed by T. L. Bennett, *Mess. of Math.* (2) XXXIV. (1904), p. 113.

Miscellaneous Examples.

1. If in the problem of three bodies the units are so chosen that the energy integral is

$$\tfrac{1}{2}(v_1^2+v_2^2+v_3^2)=\frac{1}{r_{23}}+\frac{1}{r_{31}}+\frac{1}{r_{12}}-\frac{1}{r},$$

where r_{12} is the distance between the bodies whose velocities are v_1 and v_2, and if r is a positive constant, shew that the greatest possible value of the angular momentum of the system about its centre of gravity is $\tfrac{3}{2}\sqrt{2r}$.

(Camb. Math. Tripos, Part I, 1893.)

2. In the problem of three bodies, let Φ be Jacobi's function, let Ω be the angle between any fixed line in the invariable plane and the node of the plane of the three bodies on the invariable plane, let i be the inclination of the plane of the three bodies to the invariable plane, and let η be the area of the triangle formed by the three bodies. Shew that

$$\frac{d\Omega}{dt}=\frac{k}{\Phi},$$

$$\frac{1}{\sin i}\frac{di}{dt}=k\left\{\frac{M}{m_1m_2m_3\eta^2}-\frac{1}{\Phi^2}\right\}^{\frac{1}{2}},$$

where k is the angular momentum of the system round the normal to the invariable plane.

(De Gasparis.)

3. Let the problem of three bodies be replaced by the problem of two bodies μ and μ' as in § 160: let q_1 and q_2 be the distances of μ and μ' from the origin: let q_3 and q_4 be the angles made by q_1 and q_2 respectively with the intersection of the plane through the bodies and the invariable plane: let p_1 and p_2 denote $\mu\dot{q}_1$ and $\mu'\dot{q}_2$ respectively; and let p_3 and p_4 be the components of angular momentum of μ and μ' respectively, in the plane through the bodies and the origin. Shew that the equations of motion may be written

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r},\qquad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r}\qquad (r=1,2,3,4),$$

where $H=$ Constant is the integral of energy.

(Bour.)

4. Apply the contact-transformation defined by the equations

$$q_1'=\{(q_4-q_7)^2+(q_5-q_8)^2+(q_6-q_9)^2\}^{\frac{1}{2}},$$
$$q_2'=\{(q_7-q_1)^2+(q_8-q_2)^2+(q_9-q_3)^2\}^{\frac{1}{2}},$$
$$q_3'=\{(q_1-q_4)^2+(q_2-q_5)^2+(q_3-q_6)^2\}^{\frac{1}{2}},$$
$$q_4'=b_1(q_1+iq_2)+b_2(q_4+iq_5)+b_3(q_7+iq_8),$$
$$q_5'=c_1q_3+c_2q_6+c_3q_9,$$
$$q_6'=m_1q_1+m_2q_4+m_3q_7,$$
$$q_7'=m_1q_2+m_2q_5+m_3q_8,$$
$$q_8'=m_1q_3+m_2q_6+m_3q_9,$$
$$q_0'=\frac{a_1(q_1+iq_2)+a_2(q_4+iq_5)+a_3(q_7+iq_8)}{b_1(q_1+iq_2)+b_2(q_4+iq_5)+b_3(q_7+iq_8)},$$
$$p_r=\sum_{k=0}^{8}p_k'\frac{\partial q_k'}{\partial q_r}\qquad (r=0,1,2,\ldots,8),$$

(where i stands for $\sqrt{-1}$ and $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are any nine constants which satisfy the equations

$$a_1+a_2+a_3=0,\quad b_1+b_2+b_3=0,\quad c_1+c_2+c_3=0,\quad a_2b_3-a_3b_2=1),$$

to the Hamiltonian system of the 18th order which (§ 155) determines the motion of the three bodies.

Shew that the integrals of motion of the centre of gravity are

$$q_6'=q_7'=q_8'=p_6'=p_7'=p_8'=0.$$

Shew further that when the invariable plane is taken as plane of xy, the variable p_5' is zero, and that the integral of angular momentum round the normal to the invariable plane is

$$p_4'q_4'=k,\qquad \text{where } k \text{ is a constant.}$$

Hence shew that the equations reduce to the 8th order system

$$\frac{dq_r'}{dt}=\frac{\partial H}{\partial p_r'},\qquad \frac{dp_r'}{dt}=-\frac{\partial H}{\partial q_r'}\qquad (r=0,1,2,3),$$

where

$$H=\Sigma p_1'^2q_1'^2\cdot\frac{m_2+m_3}{2m_2m_3}+\Sigma\frac{p_2'p_3'}{q_2'q_3'}\cdot\frac{q_2'^2+q_3'^2-q_1'^2}{2m_1}$$
$$+\Sigma\frac{1}{m_1}\{p_0'(a_1-b_1q_0')+kb_1\}\left\{\frac{p_3'}{q_3'}(a_3-b_3q_0')-\frac{p_2'}{q_2'}(a_2-b_2q_0')\right\}-\Sigma\frac{m_2m_3}{q_1'}.$$

Reduce this to a system of the 6th order, by the theorem of § 141.

(Bruns.)