The Theory of Correctional Manoeuvres in Interplanetary Space[1]

By

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(With 3 Figures)

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Abstract — Zusammenfassung — Résumé

The Theory of Correctional Manoeuvres in Interplanetary Space. The computation of correctional manoeuvres is shown to be independent of all small perturbing forces, so that the rocket track may be supposed to be a KEPLERian arc which it follows under the attraction of the Sun alone. The calculation can be reduced to a substitution of the observed divergence from a pre-computed track into given formulae, the coefficients in which can all be determined prior to departure and stored in a computer.

Über die Theorie der Bahnkorrekturen von Raumfahrzeugen. Es wird gezeigt, daß die Bestimmung von Bahnkorrekturen unabhängig von den schwachen Störungskräften erfolgen kann, so daß die Bahn des Raumfahrzeuges als KEPLER-Bahn, allein der Anziehungskraft der Sonne unterliegend, angesehen werden kann. Die Berechnung kann auf das Einsetzen der beobachteten Abweichungen von der vorausberechneten Bahn, in gegebene Formeln zurückgeführt werden. Die Koeffizienten können vorher bestimmt und in der Rechenmaschine gespeichert werden.

Corrections de manoeuvre en astronautique. Le calcul des corrections de manoeuvre est montré être indépendant des petites perturbations et la trajectoire peut être assimilée à un arc KEPLÉRien décrit sous l'attraction solaire seule. Les calculs se réduisent à la substitution dans des formules préparées de l'écart observé à une trajectoire préétablie. Les coefficients de ces formules peuvent être calculés avant le départ et incorporés dans la mémoire d'un calculateur.

I. Fundamental Theory

It will be assumed that the trajectory along which the space vehicle is to be transferred between two planets has been accurately computed prior to departure. This trajectory will be computed in three arcs,

(a) in the vicinity of the planet of departure,
(b) in interplanetary space,
(c) in the vicinity of the planet of arrival.

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It is to be expected that arc (*a*) will be calculated by adopting a frame of reference moving with the planet of departure, allowance being made for the perturbing influences of the Sun and other bodies by small corrections. Arc (*b*) will be calculated by reference to a heliocentric coordinate system and the major attraction to be taken into account will be that of the Sun. The arc of approach to the planet of arrival appears to be of minor importance, since it is likely that the approach manoeuvre will be linked to radar observations made from the vehicle over the last phase of the voyage and will not be related to a precomputed trajectory.

The first stage in pre-computing the trajectory will be to choose a suitable optimal interplanetary arc. This could be done by neglecting the finite extent of the gravitational fields of the terminal planets, as has always been done in optimal trajectory calculations in the past, and making allowance for the Sun's field alone. This method will yield a rough trajectory. The accuracy of this could then be improved as follows: Choose a point P upon it in the vicinity of the mid-point of the arc and assume the vehicle is in this position at the instant of arrival at P as predicted by the approximate calculation. Now integrate numerically along arcs backwards and forwards from P, taking account of the actual structure of the gravitational field, until the planets of departure and arrival are reached. It may then be necessary to adjust P and to reintegrate to achieve satisfactory termination of the trajectory upon circular orbits about the two planets.

Having projected the space vehicle into the arc (*a*), it seems reasonable to suppose that, unless a large discrepancy between the actual trajectory and that previously computed is found to be present, no correcting action will be taken until the rocket has entered arc (*b*) and the elements of this orbit have been established. In this paper, therefore, attention will be given to the main problem of computing any correctional thrusts which may become necessary during the motion along the arc (*b*). Immediately a divergence from the accurately pre-computed track C is observed, it becomes necessary to find a fresh track C' to the target planet. C and C' will be neighbouring trajectories in space and it is reasonable to assume that the effects of all minor perturbing influences will be identical for both. It then follows that the *divergence* of C' from C, both in relation to position and to velocity, is independent of these influences. But the appropriate correctional manoeuvre depends only upon this divergence and its calculation is accordingly independent of all small perturbing forces and may be based upon the assumption that the Sun's field alone is operative. This we proceed to demonstrate in detail.

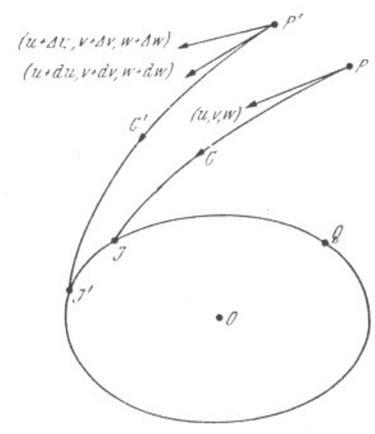

Fig. 1

Let xyz,O be rectangular cartesian axes moving with the Sun, O being the centre of this body. Let P be any point (x, y, z) and J a given point on the orbit of the planet of arrival (Fig. 1). Suppose that a rocket is at P at the instant the planet is at Q. Let T be the transit time for the planet to move from Q to J.

Then there is a unique ballistic arc joining P and J, along which the rocket can coast in the time T. If the rocket is to rendezvous with the planet, it must enter this arc at P. Let (u, v, w) be the components of the rocket velocity at P which will cause it to follow the arc PJ. Then we can write

$$\begin{aligned} u &= \bar{u}(x, y, z, T) + \varepsilon p(x, y, z, T), \\ v &= \bar{v}(x, y, z, T) + \varepsilon q(x, y, z, T), \\ w &= \bar{w}(x, y, z, T) + \varepsilon r(x, y, z, T), \end{aligned} \tag{1}$$

where $(\bar{u}, \bar{v}, \bar{w})$ are the forms taken by (u, v, w) in the absence of all small perturbing forces and the terms involving ε represent the corrections which must be made to allow for these. ε is small. If $\varepsilon = 0$, the arc PJ is a KEPLERian arc and $(\bar{u}, \bar{v}, \bar{w})$ are accordingly the forms taken by the functions (u, v, w) on such an arc. Similarly, if V is the velocity increment which must be given to the rocket upon arrival at J to transfer it into a circular orbit about the target planet,

$$V = V(x, y, z, T) + \varepsilon Q(x, y, z, T). \tag{2}$$

If V_0 is the velocity of approach from a great distance relative to the target planet and if V_c is the velocity in the circular orbit about this planet, we shall take

$$V = \sqrt{2 V_c^2 + V_0^2} - V_0. \tag{3}$$

This formula will be quite accurate enough for our purpose.

Now suppose that PJ is the pre-computed interplanetary arc, but that, at the instant the rocket should be at P with velocity (u, v, w), it is found to be at P' $(x + dx, y + dy, z + dz)$ with velocity $(u + \Delta u, v + \Delta v, w + \Delta w)$. Let $P'J'$ be the arc into which it is decided to transfer the rocket in order that it shall rendezvous with the planet at J'. Let dT be the additional transit time for the planet to move from J to J'. Then the time of transit over the arc $P'J'$ must be $T + dT$ and hence, if $(u + du, v + dv, w + dw)$ is the rocket velocity desired at P' to effect entry into the arc $P'J'$, from eqs. (1) it is found that

$$\begin{aligned} du = {} & \frac{\partial \bar{u}}{\partial x} dx + \frac{\partial \bar{u}}{\partial y} dy + \frac{\partial \bar{u}}{\partial z} dz + \frac{\partial \bar{u}}{\partial T} dT + \\ & + \varepsilon \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz + \frac{\partial p}{\partial T} dT \right), \end{aligned} \tag{4}$$

etc., to the first order in dx, dy, dz, dT. ε being small, the terms in the bracket are of the second order and will be neglected. At this stage, therefore, as was forecast above, the contributions made by the perturbing forces are eliminated. The derivatives $\partial \bar{u}/\partial x$, etc. are to be calculated on the assumption that the arc PJ is KEPLERian. The "bars" over the quantities $(\bar{u}, \bar{v}, \bar{w})$ will henceforward be omitted.

The velocity increment required at P' has components

$$du - \Delta u, \qquad dv - \Delta v, \qquad dw - \Delta w, \tag{5}$$

and is of magnitude

$$\begin{aligned} & [(du - \Delta u)^2 + (dv - \Delta v)^2 + (dw - \Delta w)^2]^{1/2} \\ & \qquad = [A_0 (dT)^2 + 2 A_1 dT + A_2]^{1/2}, \end{aligned} \tag{6}$$

where

$$A_0 = \left(\frac{\partial u}{\partial T}\right)^2 + \left(\frac{\partial v}{\partial T}\right)^2 + \left(\frac{\partial w}{\partial T}\right)^2, \tag{7}$$

$$\begin{aligned} A_1 = {} & \frac{\partial u}{\partial T}\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz - \Delta u\right) \\ & + \frac{\partial v}{\partial T}\left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz - \Delta v\right) \\ & + \frac{\partial w}{\partial T}\left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz - \Delta w\right), \end{aligned} \tag{8}$$

$$\begin{aligned} A_2 = {} & \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz - \Delta u\right)^2 \\ & + \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz - \Delta v\right)^2 \\ & + \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz - \Delta w\right)^2. \end{aligned} \tag{9}$$

It will be observed that A_0 is of zero order, A_1 is of the first order and A_2 is of the second order in the small quantities $dx, dy, dz, \Delta u, \Delta v, \Delta w$.

The velocity increment necessary at J' is $V + dV$. From eq. (2), neglecting the perturbation, it is found that

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz + \frac{\partial V}{\partial T} dT, \\ &= B_0 dT + B_1, \end{aligned} \tag{10}$$

where

$$B_0 = \frac{\partial V}{\partial T}, \tag{11}$$

$$B_1 = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{12}$$

Clearly, B_0 is of zero order and B_1 of the first order in the small quantities.

The net velocity increment required to transfer the rocket from P into a circular orbit about the planet at J' is now seen to be

$$W = V + B_0 dT + B_1 + [A_0 (dT)^2 + 2 A_1 dT + A_2]^{1/2}. \tag{13}$$

J' has not yet been determined and hence dT is arbitrary. This quantity will now be chosen to minimize the velocity increment (13) and hence the propellant expenditure. It will simplify the calculation if the reasonable assumption is now made that the pre-calculated trajectory is an optimal one. If this be the case, when $dx = dy = dz = \Delta u = \Delta v = \Delta w = 0$ the value of dT minimizing (13) must be zero. For zero values of the small quantities dx, etc

$$A_1 = A_2 = B_1 = 0$$

and the expression (13) reduces to

$$W = V + B_0 dT + |A_0^{1/2} dT|. \tag{14}$$

This must be a minimum for $dT = 0$ and this is so if, and only if,

$$|B_0| < A_0^{1/2}, \qquad \text{i.e. if} \qquad B_0^2 < A_0. \tag{15}$$

Here the square root is clearly to be taken positively. This convention will be assumed in all that follows.

Also, from its definition (6) as a sum of squares, it is necessary that

$$A_0(dT)^2 + 2\,A_1\,dT + A_2 \geqslant 0 \tag{16}$$

for all dT. It follows that

$$A_0\,A_2 - A_1^2 \geqslant 0. \tag{17}$$

First consider the special case when

$$A_0\,A_2 - A_1^2 = 0. \tag{18}$$

Then

$$A_0(dT)^2 + 2\,A_1\,dT + A_2 = A_0\left(dT + \frac{A_1}{A_0}\right)^2 \tag{19}$$

and the expression (13) for the net velocity increment reduces to

$$W = V + B_0\,dT + B_1 + \left|A_0^{1/2}\left(dT + \frac{A_1}{A_0}\right)\right|. \tag{20}$$

Since $|B_0| < A_0^{1/2}$ by (15), this expression is a minimum when

$$dT = -\,A_1/A_0 \tag{21}$$

and

$$W_{\min} = V + (A_0\,B_1 - A_1\,B_0)/A_0. \tag{22}$$

In general,

$$A_0\,A_2 - A_1^2 > 0 \tag{23}$$

and then

$$A_0(dT)^2 + 2\,A_1\,dT + A_2 > 0 \tag{24}$$

for all dT. Differentiating eq. (13) with respect to dT twice, it will be found that

$$\frac{dW}{d(dT)} = B_0 + \frac{A_0\,dT + A_1}{[A_0(dT)^2 + 2\,A_1\,dT + A_2]^{1/2}}, \tag{25}$$

$$\frac{d^2W}{d(dT)^2} = \frac{A_0\,A_2 - A_1^2}{[A_0(dT)^2 + 2\,A_1\,dT + A_2]^{3/2}}. \tag{26}$$

If the first derivative is zero, then

$$A_0\,dT + A_1 = -\,B_0\,[A_0(dT)^2 + 2\,A_1\,dT + A_2]^{1/2} \tag{27}$$

and, after squaring, this reduces to

$$A_0(dT)^2 + 2\,A_1\,dT + A_2 = \frac{A_0\,A_2 - A_1^2}{A_0 - B_0^2}. \tag{28}$$

For variable dT, the minimum value of the left-hand member of this equation is

$$\frac{1}{A_0}(A_0\,A_2 - A_1^2)$$

which is clearly less than the right-hand member. Eq. (28) accordingly possesses two real roots. Eliminating $A_0(dT)^2 + 2\,A_1\,dT + A_2$ between eqs. (27) and (28), it follows that

$$A_0\,dT + A_1 = -\,B_0\left(\frac{A_0\,A_2 - A_1^2}{A_0 - B_0^2}\right)^{1/2} \tag{29}$$

and hence that

$$dT = -\frac{B_0}{A_0}\left(\frac{A_0\,A_2 - A_1^2}{A_0 - B_0^2}\right)^{1/2} - \frac{A_1}{A_0}. \tag{30}$$

Since the second derivative (26) is positive for all values of dT, eq. (30) determines the value of dT which minimizes W. It will be found that for this value of dT,

$$W = W_{\min} = V + \frac{1}{A_0}\{[(A_0 - B_0^2)(A_0\,A_2 - A_1^2)]^{1/2} + A_0\,B_1 - A_1\,B_0\}. \tag{31}$$

It will be observed that in the special case (18), the results (30) and (31) reduce to the eqs. (21) and (22) respectively. Eqs. (30) and (31) are therefore always valid and the former determines the position for the new junction point J', if the correctional manoeuvre is to be carried out with the least possible propellant expenditure. In [1], it was assumed that $dT = 0$. The components of the velocity change necessary at P' are now obtainable from eqs. (4) and (5).

The quantities $\partial u/\partial x$, etc are all calculable immediately the interplanetary arc PJ has been selected and could be tabulated at hourly intervals along this arc prior to departure. This information could be held in the store of a computer and this device programmed to yield A_0, A_1, etc and the velocity increments (5) immediately $dx, dy, dz, \Delta u, \Delta v, \Delta w$ become available. Expressions from which these partial derivatives can be computed at any point on a KEPLERian arc are determined in the next section.

II. Correctional Formulae on a Keplerian Arc

In this section we compute, in suitable coordinates, the partial derivatives corresponding to those introduced in section I. We employ polar coordinates defined as follows: Let O be the centre of the Sun and let OZ be perpendicular to the orbital plane of the planet of arrival (Fig. 2). OR is the line of intersection of the planes of the pre-calculated rocket track and of the planet's orbit. If P is any point, ON is the line of intersection of the plane ZOP and the plane of the planet's orbit. If $r = OP$, $\chi = \angle ZOP$ and $\phi = \angle RON$, the position of P is determined by the coordinates (r, ϕ, χ).

The elements necessary to fix the position in space of any possible rocket orbit are defined as follows: Let A be the position of perihelion, and let the plane of the rocket's orbit meet the plane of the planet's orbit in the line OS. Let $\Omega = \angle ROS$, and let $\tilde{\omega} = \Omega + \angle SOA$. Let i be the angle between the orbital planes of the rocket and planet. Also let e be the eccentricity of the orbit and a its semi-major axis. Then the orbit is completely determined by the elements $(a, e, \tilde{\omega}, i, \Omega)$. The corresponding determinate elements of the planetary orbit will be denoted by $a_0, e_0, \tilde{\omega}_0$. The position in the orbit of any point P is determined by θ, where $\theta = \Omega + \angle SOP$, or, alternatively, by the eccentric anomaly E.

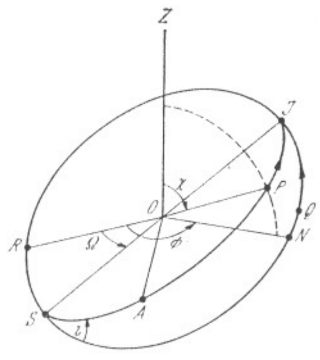

Fig. 2

Let the planet be at Q when the rocket is at P, and suppose the planet can move to J in time T. We now formulate equations which determine the unique orbit along which the rocket can move from P to J in time T. Quantities at P will be represented by plain symbols, those at J on PJ will be distinguished by a subscript 1, and those at J on QJ will be distinguished by a subscript 2. Then

$$\chi_1 = \tfrac{1}{2}\pi, \qquad \theta_1 = \theta_2 = \phi_1 = \pi + \Omega. \tag{32}$$

Also, let E_0 denote the eccentric anomaly of Q; this is a known quantity at any given time, and so is a constant in all our equations.

Applying KEPLER's equation to the two arcs PJ, QJ, we have

$$E_1 - E - e(\sin E_1 - \sin E) = \frac{\mu^{1/2} T}{a^{3/2}}, \tag{33}$$

$$E_2 - E_0 - e_0 (\sin E_2 - \sin E_0) = n_0 T, \tag{34}$$

where $n_0 = \mu^{1/2}/a_0^{3/2}$ and μ is the acceleration due to the Sun at unit distance. Also, at P we have

$$r = a(1 - e\cos E), \tag{35}$$

$$\tan \tfrac{1}{2}(\theta - \tilde{\omega}) = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} E. \tag{36}$$

At J, regarded as a point on the arc PJ, we have

$$r_1 = a(1 - e\cos E_1), \tag{37}$$

$$\tan \tfrac{1}{2}(\theta_1 - \tilde{\omega}) = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} E_1. \tag{38}$$

At J, regarded as a point on the arc QJ, we have

$$r_1 = a_0(1 - e_0 \cos E_2), \tag{39}$$

$$\tan \tfrac{1}{2}(\theta_1 - \tilde{\omega}_0) = \sqrt{\frac{1+e_0}{1-e_0}} \tan \tfrac{1}{2} E_2. \tag{40}$$

In the last of these, θ_2 has been replaced by θ_1, in virtue of eqs. (32).

Also, at P spherical trigonometry yields the equations

$$\tan(\phi - \Omega) = \tan(\theta - \Omega)\cos i, \tag{41}$$

$$\cos\chi = \sin(\theta - \Omega)\sin i. \tag{42}$$

Again, using $\Omega = \theta_1 - \pi$ from eqs. (32), eqs. (41) and (42) can be written

$$\tan(\theta_1 - \phi) = \tan(\theta_1 - \theta)\cos i, \tag{43}$$

$$\cos\chi = \sin(\theta_1 - \theta)\sin i. \tag{44}$$

The ten eqs. (33) — (40), (43) and (44) determine $a, e, \tilde{\omega}, i, \theta, \theta_1, r_1, E, E_1, E_2$ in terms of r, ϕ, χ, T. The last four quantities are the independent variables, and we shall calculate the necessary partial derivatives with respect to these. To do this, we shall totally differentiate the above equations, and then the ratio of any two differentials will give the corresponding partial derivative. As explained in section I, these derivatives are to refer to the pre-calculated rocket orbit. Hence, after differentiation, we may put $\Omega = 0$, and so, from eqs. (32), $\theta_1 = \pi$.

From eq. (33), using eqs. (35) and (37), we find

$$r_1\,dE_1 - r\,dE - a(\sin E_1 - \sin E)\,de = -\frac{3\mu^{1/2} T}{2a^{3/2}}\,da + \frac{\mu^{1/2}}{a^{1/2}}\,dT. \tag{45}$$

From eq. (34), using eq. (39), we find

$$r_1\,dE_2 = a_0 n_0\,dT. \tag{46}$$

From eqs. (35) — (40), (43) and (44), we find, in turn,

$$dr = \frac{r}{a}\,da - a\cos E\,de + ae\sin E\,dE, \tag{47}$$

$$\sec^2 \tfrac{1}{2}(\theta - \tilde{\omega})(d\theta - d\tilde{\omega}) = \frac{2\tan\frac{1}{2}E}{(1-e)(1-e^2)^{1/2}}\,de + \sqrt{\frac{1+e}{1-e}}\sec^2\tfrac{1}{2}E\,dE, \tag{48}$$

$$dr_1 = \frac{r_1}{a}\,da - a\cos E_1\,de + ae\sin E_1\,dE_1, \tag{49}$$

$$\operatorname{cosec}^2 \tfrac{1}{2}\tilde{\omega}(d\theta_1 - d\tilde{\omega}) = \frac{2\tan\frac{1}{2}E_1}{(1-e)(1-e^2)^{1/2}}\,de + \sqrt{\frac{1+e}{1-e}}\sec^2\tfrac{1}{2}E_1\,dE_1, \tag{50}$$

$$dr_1 = a_0 e_0 \sin E_2\,dE_2, \tag{51}$$

$$\operatorname{cosec}^2 \tfrac{1}{2}\tilde{\omega}_0\,d\theta_1 = \sqrt{\frac{1+e_0}{1-e_0}}\sec^2\tfrac{1}{2}E_2\,dE_2, \tag{52}$$

$$\sec^2\phi(d\theta_1 - d\phi) = \sec^2\theta(d\theta_1 - d\theta)\cos i + \tan\theta\sin i\,di, \tag{53}$$

$$\sin\chi\,d\chi = \cos\theta\sin i(d\theta_1 - d\theta) - \sin\theta\cos i\,di, \tag{54}$$

Eqs. (46) and (52) together give

$$d\theta_1 = \alpha\,dT, \tag{55}$$

while eqs. (46) and (51) give

$$dr_1 = \beta\,dT, \tag{56}$$

where

$$\alpha = \frac{a_0 n_0}{r_1}\sqrt{\frac{1+e_0}{1-e_0}}\sin^2\tfrac{1}{2}\tilde{\omega}_0\sec^2\tfrac{1}{2}E_2,$$

$$\beta = \frac{a_0^2 e_0 n_0 \sin E_2}{r_1}.$$

In the case $E_2 = \pi$, $\sec^2\frac{1}{2}E_2$ is infinite, and α is apparently infinite. However, from eq. (40) we have on the pre-calculated orbit,

$$\cot\tfrac{1}{2}\tilde{\omega}_0 = \sqrt{\frac{1+e_0}{1-e_0}}\tan\tfrac{1}{2}E_2.$$

From this

$$\cos\tfrac{1}{2}E_2\cos\tfrac{1}{2}\tilde{\omega}_0 = \sqrt{\frac{1+e_0}{1-e_0}}\sin\tfrac{1}{2}E_2\sin\tfrac{1}{2}\tilde{\omega}_0.$$

Hence, if $E_2 = \pi$, it follows that $\tilde{\omega}_0 = 0$. We can rewrite the last equation as

$$\sin\tfrac{1}{2}\tilde{\omega}_0\sec\tfrac{1}{2}E_2 = \sqrt{\frac{1-e_0}{1+e_0}}\cos\tfrac{1}{2}\tilde{\omega}_0\operatorname{cosec}\tfrac{1}{2}E_2.$$

When $E_2 = \pi$, $\tilde{\omega}_0 = 0$, the right-hand member of this equation has the value $[(1-e_0)/(1+e_0)]^{1/2}$, and the corresponding value of α is

$$\frac{a_0 n_0}{r_1}\sqrt{\frac{1-e_0}{1+e_0}}.$$

Any other apparent infinity can be dealt with in a similar way.

Eq. (41) applied to the pre-calculated rocket orbit, i.e. putting $\Omega = 0$, gives

$$\tan\phi = \tan\theta\cos i. \tag{57}$$

We now solve eqs. (53), (54) and (55), for di, $d\theta$ in terms of $d\phi$, $d\chi$, dT. Using eq. (57) to effect simplification, we find that

$$di = \alpha_1\,d\phi + \beta_1\,d\chi + \gamma_1\,dT, \tag{58}$$

$$d\theta = \alpha_2\,d\phi + \beta_2\,d\chi + \gamma_2\,dT, \tag{59}$$

where

$$\alpha_1 = -\sin i\cot\theta,$$
$$\beta_1 = -\cos i\sec^2\theta\operatorname{cosec}\theta\cos^2\phi\sin\chi,$$
$$\gamma_1 = -\alpha\alpha_1,$$
$$\alpha_2 = \cos i,$$
$$\beta_2 = -\sin i\sec\theta\cos^2\phi\sin\chi,$$
$$\gamma_2 = \alpha(1 - \cos i).$$

Using eqs. (55), (56), (58), (59) and performing some slight manipulation, we can now write eqs. (47), (45), (48), (49), (50), in that order, in the matrix form

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{pmatrix} \begin{pmatrix} da \\ de \\ d\tilde{\omega} \\ dE \\ dE_1 \end{pmatrix} = \begin{pmatrix} f_1\,dr \\ k_2\,dT \\ g_3\,d\phi + h_3\,d\chi + k_3\,dT \\ k_4\,dT \\ k_5\,dT \end{pmatrix} \qquad (60)$$

where

$$a_1 = r/a, \qquad a_2 = \frac{3\,\mu^{1/2}\,T}{2\,a^{3/2}},$$

$$b_1 = -\,a\cos E, \qquad b_2 = -\,a\,(\sin E_1 - \sin E),$$

$$c_1 = 0, \qquad c_2 = 0,$$

$$d_1 = a\,e\sin E, \qquad d_2 = -\,r,$$

$$e_1 = 0, \qquad e_2 = r_1,$$

$$f_1 = 1, \qquad k_2 = \mu^{1/2}/a^{1/2},$$

$$a_3 = 0, \qquad a_4 = r_1/a, \qquad a_5 = 0,$$

$$b_3 = \frac{\sin E\cos^2\tfrac{1}{2}(\theta - \tilde{\omega})}{(1-e)\,(1-e^2)^{1/2}}, \qquad b_4 = -\,a\cos E_1, \qquad b_5 = \frac{\sin E_1\sin^2\tfrac{1}{2}\tilde{\omega}}{(1-e)\,(1-e^2)^{1/2}},$$

$$c_3 = \cos^2\tfrac{1}{2}E, \qquad c_4 = 0, \qquad c_5 = \cos^2\tfrac{1}{2}E_1,$$

$$d_3 = \sqrt{\frac{1+e}{1-e}}\cos^2\tfrac{1}{2}(\theta - \tilde{\omega}), \qquad d_4 = 0, \qquad d_5 = 0,$$

$$e_3 = 0, \qquad e_4 = ae\sin E_1, \qquad e_5 = \sqrt{\frac{1+e}{1-e}}\sin^2\tfrac{1}{2}\tilde{\omega},$$

$$g_3 = \alpha_2\cos^2\tfrac{1}{2}E, \qquad k_4 = \beta, \qquad k_5 = \alpha\cos^2\tfrac{1}{2}E_1.$$

$$h_3 = \beta_2\cos^2\tfrac{1}{2}E,$$

$$k_3 = \gamma_2\cos^2\tfrac{1}{2}E.$$

Let D be the determinant $|a_1\,b_2\,c_3\,d_4\,e_5|$ and let DA_i, DB_i, DC_i $(i = 1, 2, 3, 4, 5)$ denote the cofactors of a_i, b_i, c_i respectively in D. Eqs. (60) then give

$$\begin{aligned} da &= f_6\,dr + g_6\,d\phi + h_6\,d\chi + k_6\,dT, \\ de &= f_7\,dr + g_7\,d\phi + h_7\,d\chi + k_7\,dT, \\ d\tilde{\omega} &= f_8\,dr + g_8\,d\phi + h_8\,d\chi + k_8\,dT, \end{aligned} \qquad (61)$$

where

$$\begin{aligned} &f_6 = A_1 f_1, \quad &g_6 = A_3 g_3, \quad &h_6 = A_3 h_3, \quad &k_6 = A_i k_i, \\ &f_7 = B_1 f_1, \quad &g_7 = B_3 g_3, \quad &h_7 = B_3 h_3, \quad &k_7 = B_i k_i, \\ &f_8 = C_1 f_1, \quad &g_8 = C_3 g_3, \quad &h_8 = C_3 h_3, \quad &k_8 = C_i k_i. \end{aligned} \qquad (62)$$

In the last column, $i = 2, 3, 4, 5$ and the repeated subscript summation convention is operative.

Let u_r, v_θ denote the radial and transverse components of velocity. Then, at P,

$$u_r = \sqrt{\frac{\mu}{a\,(1-e^2)}}\,e\sin(\theta - \tilde{\omega}), \qquad v_\theta = \frac{\sqrt{\mu\,a\,(1-e^2)}}{r}. \qquad (63)$$

Let (u, v, w) be the components of velocity corresponding to the polar coordinates (r, ϕ, χ) in the senses shown in Fig. 3 and let ψ be the angle between the transverse direction at P and the plane of the planet of arrival. Then, we have,

$$\begin{aligned} u &= u_r, \\ v &= v_\theta\cos\psi, \\ w &= v_\theta\sin\psi, \end{aligned} \qquad (64)$$

where, by spherical trigonometry,

$$\tan\psi = \cos(\theta - \Omega)\tan i. \qquad (65)$$

Employing eqs. (32), (41), (42), eq. (65) can be written

$$\tan\psi = -\cot(\theta_1 - \phi)\cos\chi. \qquad (66)$$

Differentiating this equation and using eq. (55), we find that

$$\sec^2\psi\,d\psi = -\operatorname{cosec}^2\phi\cos\chi\,d\phi - \cot\phi\sin\chi\,d\chi + \alpha\operatorname{cosec}^2\phi\cos\chi\,dT. \qquad (67)$$

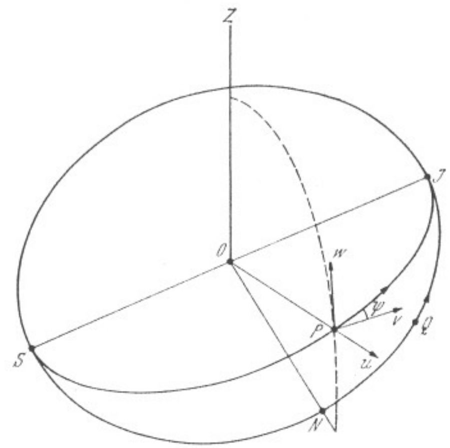

Fig. 3

From eqs. (63), (64) we obtain

$$du = -\frac{u}{2\,a}\,da + \frac{u}{e\,(1-e^2)}\,de + u\cot(\theta - \tilde{\omega})\,(d\theta - d\tilde{\omega}), \qquad (68)$$

$$dv = \frac{v}{2\,a}\,da - \frac{v\,e}{1-e^2}\,de - \frac{v}{r}\,dr - v_\theta\sin\psi\,d\psi, \qquad (69)$$

$$dw = \frac{w}{2\,a}\,da - \frac{w\,e}{1-e^2}\,de - \frac{w}{r}\,dr + v_\theta\cos\psi\,d\psi. \qquad (70)$$

We now write

$$l_1 = -\frac{u}{2\,a}, \qquad m_1 = \frac{u}{e\,(1-e^2)}, \qquad n_1 = u\cot(\theta - \tilde{\omega}),$$

$$l_2 = \frac{v}{2\,a}, \qquad m_2 = -\frac{v\,e}{1-e^2},$$

$$l_3 = \frac{w}{2\,a}, \qquad m_3 = -\frac{w\,e}{1-e^2},$$

$$\begin{aligned} p_2 &= v_\theta\sin\psi\cos^2\psi\operatorname{cosec}^2\phi\cos\chi, \\ q_2 &= v_\theta\sin\psi\cos^2\psi\cot\phi\sin\chi, \\ p_3 &= -\,p_2\cot\psi, \\ q_3 &= -\,q_2\cot\psi. \end{aligned}$$

Then, from eqs. (68) — (70) and using eqs. (61) and (67), it will be found that

$$\begin{aligned} du &= F_1\,dr + G_1\,d\phi + H_1\,d\chi + K_1\,dT, \\ dv &= F_2\,dr + G_2\,d\phi + H_2\,d\chi + K_2\,dT, \\ dw &= F_3\,dr + G_3\,d\phi + H_3\,d\chi + K_3\,dT, \end{aligned} \qquad (71)$$

where

$$\begin{aligned}
F_1 &= l_1 f_6 + m_1 f_7 - n_1 f_8,\\
G_1 &= l_1 g_6 + m_1 g_7 + n_1 (\alpha_2 - g_8),\\
H_1 &= l_1 h_6 + m_1 h_7 + n_1 (\beta_2 - h_8),\\
K_1 &= l_1 k_6 + m_1 k_7 + n_1 (\gamma_2 - k_8),\\
F_2 &= l_2 f_6 + m_2 f_7 - v/r,\\
G_2 &= l_2 g_6 + m_2 g_7 + p_2,\\
H_2 &= l_2 h_6 + m_2 h_7 + q_2,\\
K_2 &= l_2 k_6 + m_2 k_7 - \alpha p_2,\\
F_3 &= l_3 f_6 + m_3 f_7 - w/r,\\
G_3 &= l_3 g_6 + m_3 g_7 + p_3,\\
H_3 &= l_3 h_6 + m_3 h_7 + q_3,\\
K_3 &= l_3 k_6 + m_3 k_7 - \alpha p_3.
\end{aligned}$$

At J on PJ, the radial and transverse components of velocity are u_1, v_1 where

$$u_1 = \sqrt{\frac{\mu}{a(1-e^2)}}\, e \sin(\theta_1 - \tilde\omega), \qquad v_1 = \frac{\sqrt{\mu a (1-e^2)}}{r_1}. \tag{72}$$

Differentiating these two equations, we obtain

$$du_1 = -\frac{u_1}{2a} da + \frac{u_1}{e(1-e^2)} de - u_1 (d\theta_1 - d\tilde\omega) \cot\tilde\omega, \tag{73}$$

$$dv_1 = \frac{v_1}{2a} da - \frac{v_1 e}{1-e^2} de - \frac{v_1}{r_1} dr_1. \tag{74}$$

We now write

$$l_4 = -\frac{u_1}{2a}, \qquad m_4 = \frac{u_1}{e(1-e^2)}, \qquad n_4 = -u_1 \cot\tilde\omega,$$

$$l_5 = \frac{v_1}{2a}, \qquad m_5 = -\frac{v_1 e}{1-e^2}.$$

Then, from eqs. (73) and (74), using eqs. (55), (56) and (61), we find that

$$\begin{aligned}
du_1 &= F_4\, dr + G_4\, d\phi + H_4\, d\chi + K_4\, dT,\\
dv_1 &= F_5\, dr + G_5\, d\phi + H_5\, d\chi + K_5\, dT,
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
F_4 &= l_4 f_6 + m_4 f_7 - n_4 f_8,\\
G_4 &= l_4 g_6 + m_4 g_7 - n_4 g_8,\\
H_4 &= l_4 h_6 + m_4 h_7 - n_4 h_8,\\
K_4 &= l_4 k_6 + m_4 k_7 + n_4 (\alpha - k_8),\\
F_5 &= l_5 f_6 + m_5 f_7,\\
G_5 &= l_5 g_6 + m_5 g_7,\\
H_5 &= l_5 h_6 + m_5 h_7,\\
K_5 &= l_5 k_6 + m_5 k_7 - \frac{\beta v_1}{r_1}.
\end{aligned}$$

At J on QJ, the radial and transverse components of velocity are u_2, v_2, where

$$u_2 = \sqrt{\frac{\mu}{a_0(1-e_0^2)}}\, e_0 \sin(\theta_1 - \tilde\omega_0), \qquad v_2 = \frac{\sqrt{\mu a_0 (1-e_0^2)}}{r_1}. \tag{76}$$

From these we find that

$$du_2 = -u_2 \cot\tilde\omega_0\, d\theta_1,$$

$$dv_2 = -\frac{v_2}{r_1} dr_1,$$

and so, using eqs. (55) and (56), we have

$$\begin{aligned}
du_2 &= K_6\, dT,\\
dv_2 &= K_7\, dT,
\end{aligned} \tag{77}$$

where

$$K_6 = -\alpha u_2 \cot\tilde\omega_0, \qquad K_7 = -\beta v_2/r_1.$$

At J, let V_0 be the magnitude of the velocity of the rocket relative to the planet as it approaches from infinity. Then

$$V_0^2 = u_1^2 + v_1^2 + u_2^2 + v_2^2 - 2u_1u_2 - 2v_1v_2\cos i.$$

From planetary theory, we have

$$u_1^2 + v_1^2 = \mu\left(\frac{2}{r_1} - \frac{1}{a}\right),$$

$$u_2^2 + v_2^2 = \mu\left(\frac{2}{r_1} - \frac{1}{a_0}\right).$$

Hence

$$V_0^2 = \mu\left(\frac{4}{r_1} - \frac{1}{a} - \frac{1}{a_0}\right) - 2u_1u_2 - 2v_1v_2\cos i. \tag{78}$$

Differentiating this, we obtain

$$\begin{aligned}
V_0\, dV_0 = &-\frac{2\mu}{r_1^2} dr_1 + \frac{\mu}{2a^2} da - u_1\, du_2 - u_2\, du_1 +\\
&+ v_1 v_2 \sin i\, di - v_1 \cos i\, dv_2 - v_2 \cos i\, dv_1.
\end{aligned} \tag{79}$$

Differentiating eq. (3), it will be found that

$$dV = -\frac{V}{V+V_0} dV_0.$$

Writing

$$\lambda = \frac{V}{V_0(V+V_0)}, \qquad l_8 = -\frac{\lambda\mu}{2a^2}, \qquad m_8 = \lambda u_2,$$

$$p_8 = -\lambda v_1 v_2 \sin i, \qquad n_8 = \lambda v_2 \cos i,$$

it follows from eq. (79), after employing eqs. (56), (58), (75) and (77), that

$$dV = F_8\, dr + G_8\, d\phi + H_8\, d\chi + K_8\, dT, \tag{80}$$

where

$$\begin{aligned}
F_8 &= l_8 f_6 + m_8 F_4 + n_8 F_5,\\
G_8 &= l_8 g_6 + m_8 G_4 + n_8 G_5 + p_8 \alpha_1,\\
H_8 &= l_8 h_6 + m_8 H_4 + n_8 H_5 + p_8 \beta_1,\\
K_8 &= l_8 k_6 + m_8 K_4 + n_8 K_5 + p_8 \gamma_1 + \lambda\left(\frac{2\mu\beta}{r_1^2} + u_1 K_6 + v_1 K_7 \cos i\right).
\end{aligned}$$

Any required partial derivative can now be obtained from the appropriate equation. For example, from eqs. (71), we have

$$\frac{\partial u}{\partial r} = F_1, \qquad \frac{\partial u}{\partial \phi} = G_1, \qquad \text{etc.}$$

From this point onwards $dr, d\phi, d\chi$ are to be understood as observed quantities such that if, in Fig. 1, the coordinates of P are (r, ϕ, χ), then the coordinates of P' are $(r + dr,\ \phi + d\phi,\ \chi + d\chi)$. We now write

$$P_1 = F_1\,dr + G_1\,d\phi + H_1\,d\chi - \Delta u,$$
$$P_2 = F_2\,dr + G_2\,d\phi + H_2\,d\chi - \Delta v,$$
$$P_3 = F_3\,dr + G_3\,d\phi + H_3\,d\chi - \Delta w.$$

Then, from eqs. (7), (8), (9) and (71), we have

$$A_0 = K_1^2 + K_2^2 + K_3^2,$$
$$A_1 = K_1 P_1 + K_2 P_2 + K_3 P_3,$$
$$A_2 = P_1^2 + P_2^2 + P_3^2.$$

Similarly, from eqs. (11), (12) and (80), we obtain

$$B_0 = K_8,$$
$$B_1 = F_8\,dr + G_8\,d\phi + H_8\,d\chi.$$

Then dT is determined from eq. (30). Using this value of dT, from eq. (5), the components of the velocity change required at P' are

$$P_1 + K_1\,dT, \qquad P_2 + K_2\,dT, \qquad P_3 + K_3\,dT,$$

and the total velocity increment required is given by eq. (31).

Reference

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Isotherme Düsenströmungen[1,2]

Von

V. Stingelin[3]

(Mit 7 Abbildungen)

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Zusammenfassung — Abstract — Résumé

Isotherme Düsenströmungen. Die Möglichkeit, mit Hilfe der isothermen Expansion hohe Geschwindigkeiten zu erzeugen, wurde rechnerisch untersucht. Dabei wurden folgende Fälle behandelt: Konstante Temperaturdifferenz zwischen Wand- und mittlerer Gastemperatur, konstante Wandtemperatur und gegebene Wärmeproduktion längs der Achse. Der Einfluß der Reibung und der Temperaturgrenzschicht wurde berücksichtigt und für den turbulenten Wärmeübergang die REYNOLDS-Analogie zu Hilfe gezogen.

Isothermal Nozzle Flows. The possibility of generating high velocities by means of isothermal expansion was numerically investigated. The following cases were treated: Constant difference between wall temperature and average gas temperature, and constant wall temperature with given heat production along the axis. The influence of friction and of the temperature boundary layer was considered and for the turbulent heat transfer the REYNOLDS analogy was applied.

Ecoulement isotherme en tuyère. Une investigation numérique sur la possibilité d'atteindre de hautes températures par expansion isotherme. Les cas suivants ont été traités: écart constant entre température moyenne du gaz et température de paroi, température constante de paroi et apport donné de chaleur le long de l'axe. L'influence de la viscosité et de la couche limite thermique ont été considérées; l'analogie de REYNOLDS étant utilisée pour la transmission de chaleur en régime turbulent.

I. Einleitung

Es ist naheliegend, Reaktoren als Energiequellen zur Erzeugung von großen spezifischen Impulsen in Betracht zu ziehen. — Ihre große spezifische Wärmeproduktion kann jedoch nicht ohne weiteres in Form von hoher spezifischer Energie des Kühlmittels weggeführt werden. Erhitzen wir das Gas isobar (abgesehen von Reibungsdruckabfall), so bleibt im Normalfall die Austrittstemperatur des Kühlmittels spürbar unter der Oberflächentemperatur des Reaktors. Bei an-

[1] Vorgetragen an der Frühjahrstagung der Schweizerischen Physikalischen Gesellschaft in Brugg [2].

[2] Für die Anregung zu dieser Arbeit und für die wertvollen Ratschläge bei ihrer Durchführung möchte ich Herrn Professor Dr. J. ACKERET meinen besten Dank aussprechen.

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