Optimum paths to the moon and planets

Space ships wending their way to the moon, Mars and Venus will follow trajectories designed to make the most of the fuel used, gravitational forces and relative motions of the planets

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THE WORD "OPTIMUM" tends to be used somewhat loosely. We even heard the other day of a young engineer who told his boss he had an idea that would make a certain design "more optimum." To be precise in speaking of optimum paths, we should say (1) what it is that is optimum about them and (2) what the ground rules (constraints) are for determining this optimum.

We can begin profitably with a term basic to rocketry—payload ratio, defined as initial gross weight divided by payload weight. We would like to minimize payload ratio in most rocket paths between two points. Sometimes, however, the problem becomes easier to solve if instead we try to minimize *mass ratio,* defined as initial gross weight divided by final or "all-burnt" weight. The difference, of course, lies in the amount of structure included in the final weight.

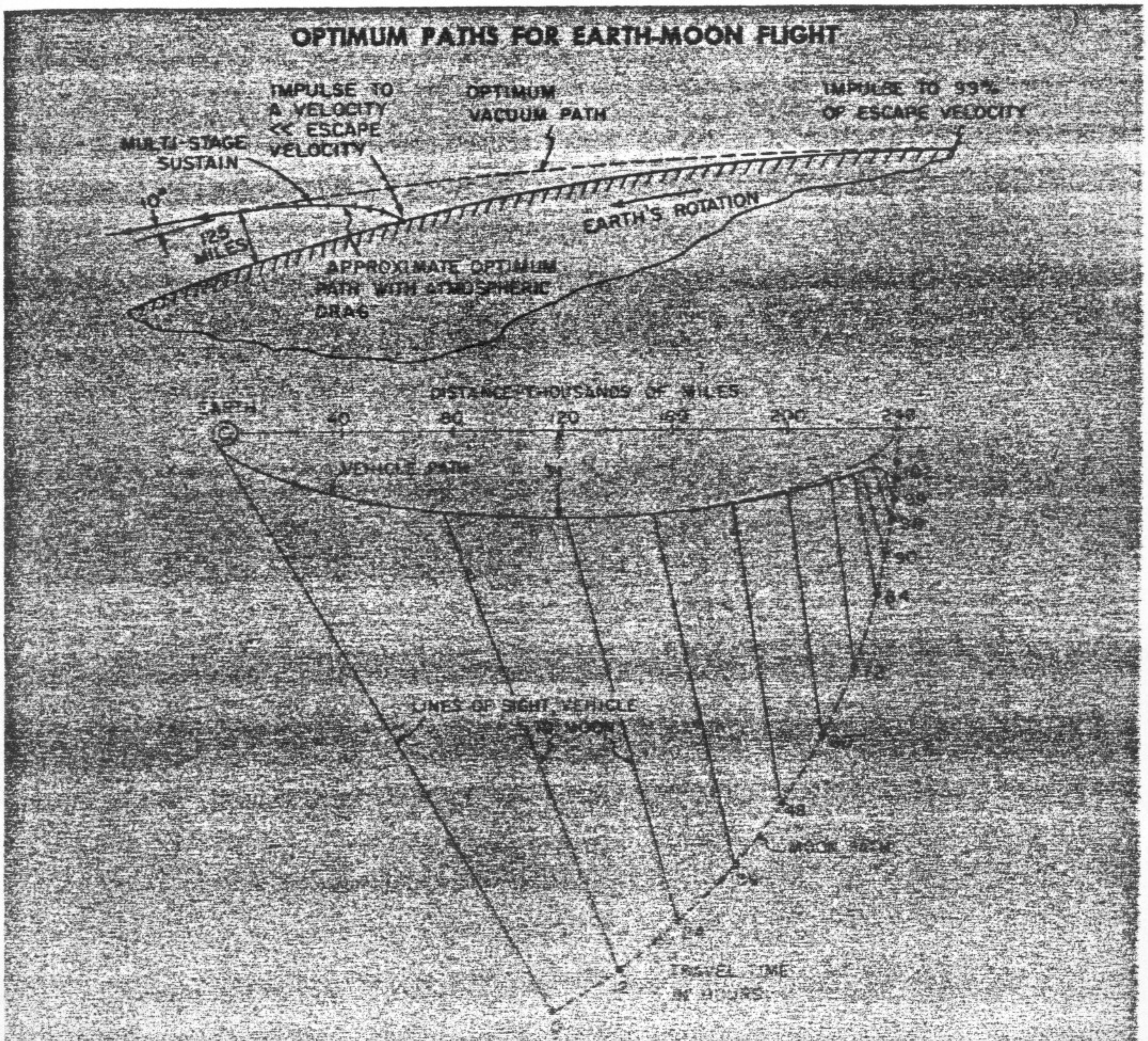
Since a large fraction of a rocket's takeoff weight is fuel, a great deal of the fuel must produce thrust simply to hold up the unburned fuel, and still more to accelerate it. For this reason it is desirable to burn fuel as rapidly as possible, converting it into kinetic energy of the vehicle, which will be exchanged for potential energy in the earth's gravitational field as the vehicle moves away from the earth. In fact, if it were possible, we would like to burn the fuel instantaneously, giving the vehicle an impulse, i.e., an instantaneous charge in speed. This statement must be modified if we consider atmospheric drag on the vehicle, because the high velocity associated with an impulse start wastes fuel in overcoming drag.

The best compromise involves using some fuel for an instantaneous velocity charge—as rapid as possible in the practical case—and then burning fuel at a lower rate for a short time thereafter. For a single-stage vehicle, this optimum requires maximum thrust until the gravitational pull of the earth on the vehicle, W, is equal to $(1 - v/c)D$, where D is the vehicle drag, v is velocity, and c is the specific impulse of the propellant (identical to the exhaust velocity of the propellant gases). Thrust adjusted to maintain this equality gives continuous vertical acceleration until escape or other desired velocity is attained. This simple relation is only valid if the drag is proportional to the square of the velocity. The more general relation is $W = v\,\partial D/\partial v - (1 - v/c)D$.

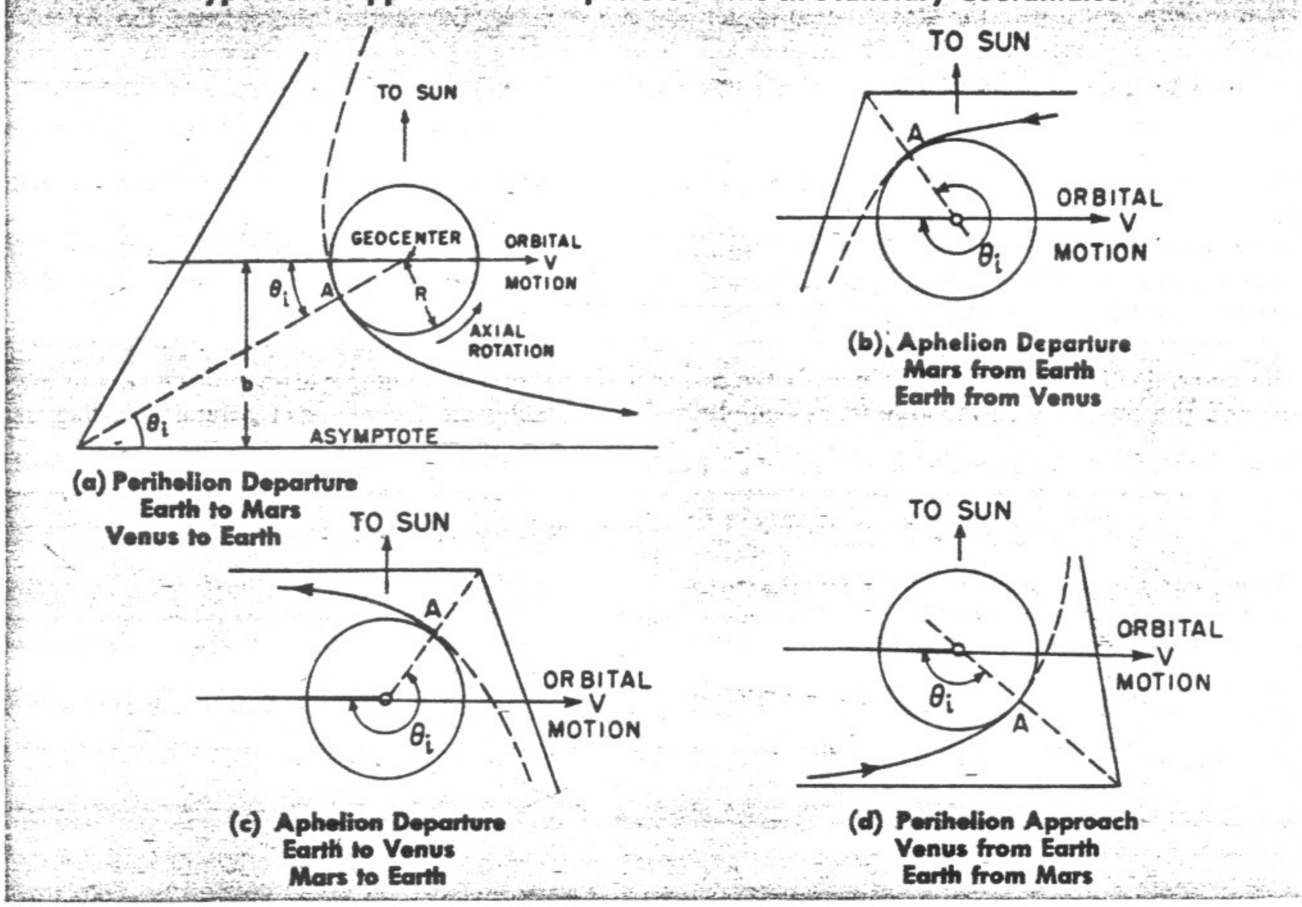
The path of least fuel expenditure between two co-planar circular orbits about a gravitational center was given by Hohmann. The path involves two impulses, one at the beginning and one at the end, the path itself being an ellipse tangent to both orbits, with the gravitational center at one focus, as shown in the top illustrations on page 21. The optimum transfer between co-planar, co-focal elliptic orbits does not appear to be so neatly solved as yet. D. Lawden has shown that two-impulse is better than one-impulse transfer for two equal intersecting elliptic orbits, and that the transfer orbit is almost tangent to both orbits. With economical use of fuel the aim, it appears best to apply impulses tangent to, or at least nearly tangent to the path.

The determination of an optimum path between earth and the moon is considerably more complicated than for orbit-to-orbit transfer, because consideration must be given to the earth-moon gravitational fields, the earth's atmosphere, the moon's motion about the earth, and the earth's rotation about its axis. Of lesser importance in determining the optimum thrust program, but important to precise determination of the path for navigation, are gravitational effects of the sun, inclination of the moon's orbit to the ecliptic, and inclination of the earth's axis to the ecliptic.

Transfer from a close satellite orbit about the earth to the more distant satellite orbit of the moon is best made by a Hohmann ellipse. This will probably be the way large payloads will be sent to the moon, using the technique of orbital refueling in



Hyperbolic Approach and Departure Paths in Planetary Coordinates



Planetary Constants

Planet	Symbol	Radius (mi)	Surface gravity (⊕ = 1.00)	Escape velocity (mi/sec)	Circular velocity (mi/sec)	Orbital velocity (mi/sec)	Orbital radius	Angular velocity (deg/day)	$\mu = g_0 R_0^2$ (mi³/sec²)
Venus	♀	3790	.845	6.46	4.57	21.75	0.723	1.606	74,460
Earth	⊕	3960	1.00	6.95	4.92	18.50	1.00	0.986	95,630
Mars	♂	2110	1.41	3.13	2.21	14.98	1.524	0.524	10,320

Hohmann Transfer Ellipse Constants

Trip	Semi-major axis	Half-period days	Perihelion velocity (mi/sec)	Aphelion velocity (mi/sec)	Perihelion transfer increment (mi/sec)	Aphelion transfer increment (mi/sec)
Earth-Mars	1.26	259	20.38	13.36	1.88	1.62
Earth-Venus	0.86	146	23.49	16.96	1.69	1.54

Trip Requirements (Vacuum Impulse Case)

Trip	Initial velocity increment (mps)	Launch* angle Θ_i (deg)	Departure* Asymptote distance b (kilomiles)	Planetary angle Ψ (deg)	Approach* angle Θ_i (deg)	Approach* asymptote distance (kilomiles)	Final velocity increment (mps)	Ideal total velocity requirement (mps)	Waiting period (days)	Total trip time (days)
Earth-Mars	7.21	29.3/32.2	15.2/17.1	44.4	322/321	6.16/6.39	3.52	21.5	455	973
Mars-Earth	3.52	218/219	6.16/6.39	—75	153/148	15.2/17.1	7.21			
Earth-Venus	7.08	204/207	18.3/20.6	—54.5	151/148	14.6/16.4	6.71	27.6	465	757
Venus-Earth	6.71	29.2/32.2	14.6/16.4	36	336/333	18.3/20.6	7.08			

* Surface departure or arrival/Satellite departure or arrival ($R = 5/4\ R_0$).

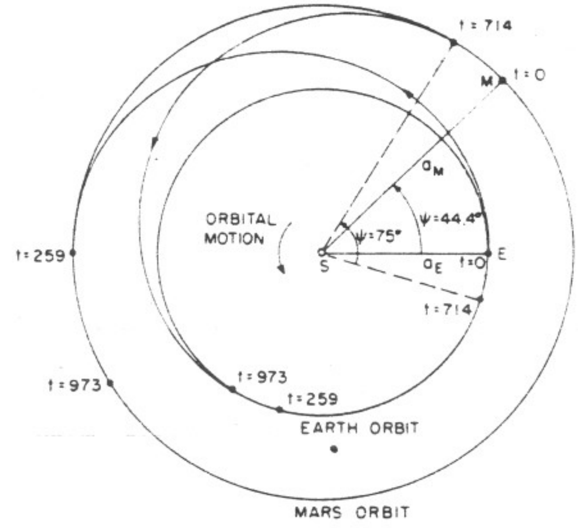
the close satellite orbit. It is not so clear that the Hohmann ellipse is optimum for a direct journey from the earth's surface to the moon. If the earth were not rotating, the "straight shot" radially outward would give a very slight (0.1 per cent) advantage over the ellipse. However, using the surface speed of the rotating earth at the equator saves roughly 0.25 mps and clearly gives the advantage to the ellipse.

Multistage Vehicle Must Be Used

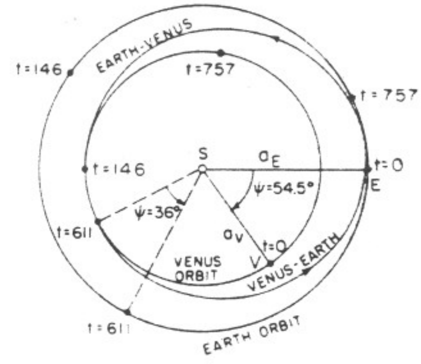
With present fuels and structural ratios, this direct shot must be a multistage affair. For a given number of stages, optimizing the multistage rocket in a vacuum is quite a problem in itself, involving a succession of impulses, which are, of course, not attainable practically. Very little work has been done for the case with drag, but the answer would probably still involve impractically high accelerations. If the structural ratio of all steps is the same, velocity increments of all stages should be equal (in the vacuum case).

The velocity necessary to get to the moon is only 1 per cent less than escape velocity, namely, 6.89 mps. Thus, the Hohmann ellipse near the earth looks almost identical to the parabola for escape. The path to the moon is shown by the two illustrations of paths on page 19. Because of the atmosphere, the optimum path will obviously not start off tangent to the earth's surface. In fact, it probably starts off almost vertically (CONTINUED ON PAGE 80)

Paths for Round Trips to Mars and Venus



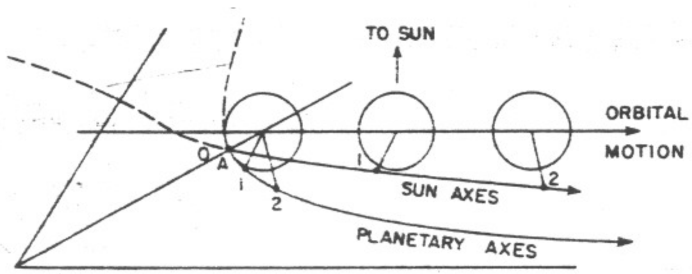
(a) Earth-Mars-Earth



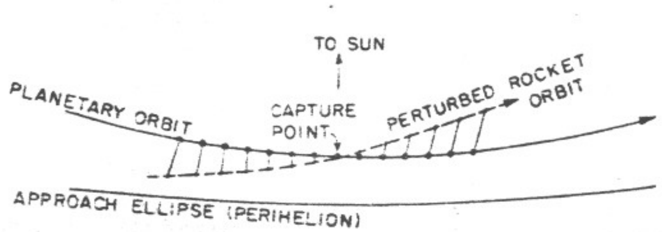
(b) Earth-Venus-Earth

Note: Transfer ellipses between planetary orbits show boundary conditions and elapsed times. Relative positions of planets are to scale.

Perihelion Departures in Sun Coordinates



(a) Departure



(b) Approach

Optimum Paths

(CONTINUED FROM PAGE 21)

with a large impulse, like the sounding rocket, then bends over slowly with finite sustain thrust, smaller impulses at the beginning of each step followed by sustain periods, and, lastly, coasts after last-step burnout until the path comes tangent to the vacuum ellipse with the right velocity.

Oberth calls this his "synergic path" and estimates about 5–10 per cent increase in fuel to overcome drag. Precise determination of the path involves solving a problem in the calculus of variations. At present the curve is determined, for reasons of practicality, by thrust and acceleration limits.

The shape of the path near the moon is, of course, perturbed by the moon's gravitational field. To impact or come close to the moon, we must aim ahead of it and let it "run into" our vehicle. The vehicle's apogee velocity, if unperturbed by the moon, is 0.14 mps, whereas the moon's orbital velocity around the earth is 0.64 mps. Thus the minimum speed one could expect relative to the moon while still far away from it is about 0.50 mps.

In the moon's reference frame, neglecting the earth's influence, the vehicle will describe a hyperbola in falling toward the moon. How close the vehicle comes to the moon for this minimum initial velocity depends, in a most sensitive manner, on the velocity of launching and somewhat less sensitively on the angle of launching at the earth.

L. G. Walters gives ± 15 fps (out of 36,400 fps) and ± 0.8 deg as the tolerances to achieve moon impact. He also shows that a slight increase in initial velocity (300 fps) reverses the accuracy requirements for impact, making them ± 100 fps and ± 0.2 deg.

George Gamow and Krafft Ehricke have given vehicle paths for passing under the sun's influence very near the moon's surface. They show that to "ricochet" back toward the earth will require fantastically precise navigation. It can be assumed that a moon rocket should have some provision for homing and thrust control if any degree of accuracy for impact point is desired.

Use of Fuel in Braking

There is also the question of whether to use fuel to brake a vehicle returning to earth. Rather than carry precious fuel for a braking maneuver, we should like to take advantage of atmospheric drag to decelerate us. It seems likely that this re-entry problem will be solved in time, so that a returning vehicle will be able deliberately to graze the atmosphere on its first pass, penetrating deep enough to decelerate itself into an elliptical orbit around the earth, and in successive passes, perhaps five or six, reduce the size of this orbit until some aerodynamic lift device enables it to land directly or parachute to the surface.

The dynamic problems of interplanetary motion are essentially those encountered in an earth-moon journey, except that the sun's gravitational attraction acts as the primary controlling force, with the planetary forces providing perturbations at both ends of the flight. The interplanetary trip can thus be divided into three almost discrete steps: (1) Departure from earth, either from the surface or from a satellite orbit, along a hyperbolic path relative to the earth where the earth's attraction is dominant, (2) travel along an ellipse about the sun's attraction is dominant, and (3) approach to the target planet along a hyperbolic path relative to the planet where planetary attraction is dominant. The planetary orbits will be approximated as circular and co-planar.

To effect an Earth-Mars transfer, the space vehicle must be accelerated to perihelion velocity of the Earth-Mars Hohmann ellipse along a path tangent to the earth's orbit, as indicated in top diagram (a) on page 21. Since the earth's orbital velocity around the sun is 18.5 mps, the vehicle needs only an additional 1.88 mps once it is free of the earth's attraction.

The vehicle must therefore be accelerated to 7.21 mps tangent to the earth's surface (square root of the sum of the squares of 6.95 mps escape velocity and the 1.88 mps velocity increment) or to 4.78 mps if takeoff is from a satellite whose orbital radius is $^{5}/_{4}$ of the earth's radius. A satellite velocity of 2.18 mps will increase this 4.78 to 6.96 mps, which is the velocity necessary to escape from satellite orbit and enter the transfer ellipse.

Departing tangentially from the earth's surface allows a reduction in the velocity requirements, since the rotation of the earth about its axis provides a small velocity increment. In practice, departure will be along a "synergic curve" with slightly higher velocity requirements.

Diagram (a) on page 20 shows the departure from earth for Mars in terms of a nonrotating geocentric coordinate system. The assumption is made that the vehicle reaches final velocity in a very short time. This cannot be accomplished in practice, but the effect on trajectory will be small. An excellent discussion of finite acceleration time in taking off from a satellite orbit is given by Von Braun in his book, *The Mars Project* (University of Illinois Press, 1953).

As seen in figure (a), page 20, the vehicle leaves from "A" and moves away in a hyperbolic path, relative to the earth, that approaches asymptotically a line perpendicular to a radius vector from the sun. In sun coordinates this departure path appears to be only a small perturbation from the transfer ellipse, as in diagram (a) bottom of page 21.

After 259 days of travel along the transfer ellipse, the vehicle arrives in the vicinity of Mars. Here the attracting force of Mars perturbs the path so that, relative to Mars, the path appears hyperbolic. Since Mars is traveling faster than the vehicle, the latter appears to approach in a direction opposite to Mars' orbital motion, as shown in the illustration (b) on page 20. A retarding thrust must be applied here at "A" if the vehicle is to remain with the planet, that is, unless atmospheric braking is utilized.

The tables on page 20 give the velocity increments and other pertinent data referring to this transfer. The return trip is similar except for the departure and approach paths, which are shown in the illustrations (c and d), respectively, on page 20. The figures also apply to the Earth-Venus trip, whose constants are included in the tables. Calculations in the tables are based on data in *Astronomy* by John Charles Duncan (Fourth Edition; Harpers, New York, 1946).

A Major Disadvantage

One major disadvantage of the Hohmann ellipse transfer is the requirement that time be spent waiting at Mars for the correct angular relationship between it and earth for the return trip. The waiting period for the Mars-Earth trip is approximately 455 days. The relative positions of the planets at arrival and departure are to scale in the top figures on page 21. M. Vertregt has worked out the requirements for nonoptimum transfer between two co-focal circular orbits such that, by following indirect routes, there are no rigid requirements on the initial positions of the planets or waiting periods, and transfer times are of the order of 100 days. The total energy requirement for this transfer runs one to three times the optimum value.

When the time for an actual Mars or Venus flight draws near, the eccentricities of the planetary orbits and their inclinations to the elliptic must be taken into account in the predicted path calculations. Since the astronomical parameters vary slowly with time, there is no point in making such calculations until they are needed.