Analytic Solutions to Several Optimum Orbit Transfer Problems*

H. Munick, R. McGill and G. E. Taylor†

Abstract

The analytic solution is given for the *absolute minimum* characteristic-velocity path from a point on an elliptical orbit to a nonintersecting circular orbit. The orbits are considered coplanar and two impulses are used. Existence proofs are given followed by proofs that the absolute minimum satisfies both necessary and sufficient conditions.

More generally the complete analytic solution to the terminal-to-terminal problem as first formulated by Vargo [1][1] is given.

An interesting consequence of the analytic solution is the proof that in optimal transfer the impulses are not in general applied tangentially. Impulses are applied tangentially only at terminals having zero radial velocity.

Each point on an elliptical orbit corresponds to an arrival point of an optimum path from a lower energy circular orbit. The Hohmann path arriving at apogee is rigorously shown to be the optimum of the infinity of optimal transfer paths, for a wide class of orbits.

Symbols

u = normal component of velocity
v = radial component of velocity
x = nondimensional normal component of velocity just after first impulse
y = nondimensional radial component of velocity just after first impulse
α = initial radial distance
β = final radial distance
r = distance ratio = α/β
λ = nondimensional characteristic velocity
λ^* = characteristic velocity of transfer
μ = gravitational constant
x_2 = nondimensional normal component of arrival velocity
y_2 = nondimensional radial component of arrival velocity
subscripts as defined in text

* Presented at the 11th International Astronautical Congress, Stockholm, Sweden, August 14–20, 1960.

† Grumman Aircraft Engineering Corporation, Bethpage, New York.

[1] Numbers in brackets indicate references at end of paper.

Introduction

The problem solved in this paper is that of finding the analytic expressions for the minimum characteristic-velocity path (hereafter called the optimum path) for a class of orbital transfers. The optimum path solutions are derived for transfer between two terminals, and between a terminal and a "point." For the purposes of this paper we define a terminal as a locus specified by a radial distance and a velocity vector, i.e., a position on an ellipse with the line of apsides unspecified. A "point" is defined by a radius vector and a velocity vector. A "point" then, is a specific position on an ellipse whose line of apsides *is* specified. The class of orbits considered are those for which the major apsis of one terminal is less than the minor apsis of the other terminal. This is tantamount to saying that a minimum of two impulses is required to accomplish the transfer.

The formulation given herein is that of Ref. [2] which was pioneered by Lawden.

The assumptions of the paper are:

1) Inverse square force field
2) Two-body equations
3) All orbits are coplanar
4) Impulsive thrusting
5) Two impulses are used.

Definition of Problem

Consider the problem of optimum transfer of a space vehicle between two terminals. Let (u_0, v_0) be the velocity components of the first terminal at a distance α from the attractive center. Let (u_F, v_F) be the velocity components of the second terminal at a distance β from the attractive center. At the first terminal an impulse is applied resulting in new velocity components (u_1, v_1). The space vehicle then goes into a transfer orbit, arriving at the second terminal with velocity components (u_2, v_2). Upon arrival at the second terminal a second impulse is applied adjusting the arrival velocity to the desired terminal velocity (u_F, v_F). For such a maneuver the characteristic-velocity is given by

$$\lambda^* = \sqrt{(u_1 - u_0)^2 + (v_1 - v_0)^2} + \sqrt{(u_2 - u_F)^2 + (v_2 - v_F)^2}. \quad (1)$$

From conservation of angular momentum we have

$$u_2 = \frac{\alpha}{\beta} u_1, \quad (2)$$

and from the conservation of total energy

$$v_2^2 = \left[1 - \frac{\alpha^2}{\beta^2}\right] u_1^2 + v_1^2 + 2\mu\left[\frac{1}{\beta} - \frac{1}{\alpha}\right]. \quad (3)$$

Introduce dimensionless parameters x, y by:

$$u_1 = x\sqrt{\frac{\mu}{\alpha}}, \qquad v_1 = y\sqrt{\frac{\mu}{\alpha}}, \quad (4)$$

and in all other cases:

$$u_i = x_i\sqrt{\frac{\mu}{\alpha}}, \qquad v_i = y_i\sqrt{\frac{\mu}{\alpha}}. \quad (5)$$

Let the distance ratio be denoted by

$$r = \frac{\alpha}{\beta}. \quad (6)$$

Using (2), (4), (5), and (6), the characteristic-velocity becomes

$$\lambda^* = \sqrt{\frac{\mu}{\alpha}}\,[\sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}]. \quad (7)$$

And Eq. (3) becomes

$$y_2^2 = (1 - r^2)x^2 + y^2 + 2(r - 1). \quad (8)$$

From (7) and (8) we get

$$\lambda^* \Big/ \sqrt{\frac{\mu}{\alpha}} = \sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}. \quad (9)$$

The sign of the radical $\sqrt{(1 - r^2)x^2 + y^2 + 2(r - 1)}$ as obtained from (8), must be taken the same as y_f. This is dictated by the fact that we are minimizing characteristic-velocity.

Let $\lambda(x, y)$ denote the nondimensional characteristic-velocity as given by the right-hand side of (9). From (8) we observe the constraint relationship that

$$(1 - r^2)x^2 + y^2 + 2(r - 1) \geqq 0. \quad (10)$$

If (10) is violated an imaginary radial velocity results. The physical interpretation of this is that the vehicle cannot achieve a distance β from the focus; hence a transfer in this case is impossible.

From (8) and (9), we can now define the mathematical problem of interest to be that of minimizing

$$\lambda(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2} + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2} \quad (11)$$

subject to

$$(1 - r^2)x^2 + y^2 + 2(r - 1) \geqq 0, \quad (12)$$

with

$$0 < r < 1. \quad (13)$$

Solution to Problem

In [2] it is rigorously proved that the function $\lambda(x, y)$ given by (11), will assume a relative minimum at an interior point bounded by the closed curves $y_2 = 0$ and $y_2 = y_F$. Since $\lambda(x, y)$ is a differentiable function in such a region, a necessary condition for the solution (x, y) of the minimum problem is given by:

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\lambda}{\partial y} = 0. \quad (14)$$

It is also shown in [2] that using the two necessary conditions given by (14) we establish for $y_2 \neq 0$

$$[(1 - r^2)x_F^2 + 2(r - 1)]K^2 - [2y_F\{(1 - r^2)rx_Fx + 2(r - 1)\}]K + [(1 - r^2)r^2y_F^2x^2 + 2(r - 1)y_F^2] = 0 \quad (15)$$

where

$$K = \sqrt{(1 - r^2)x^2 + y^2 + 2(r - 1)} = y_2.$$

Introducing the transformation $y_2' = (y_2/y_F)$ into (15), and simplifying we get

$$[(1 + r)x_F^2 - 2]y_2'^2 - 2y_2'[(1 + r)rx_Fx - 2] + [(1 + r)r^2x^2 - 2] = 0. \quad (16)$$

Re-arranging terms

$$[(1 + r)r^2]x^2 - 2x[(1 + r)rx_Fy_2'] + [(1 + r)x_F^2y_2'^2 - 2(y_2' - 1)^2] = 0. \quad (17)$$

Dividing through by $(1 + r)$ we get

$$r^2x^2 - 2rx_F\,y_2'x + \left[x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right] \times \left[x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right] = 0. \quad (18)$$

Factoring (18) yields

$$\left[rx - \left\{x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right\}\right] \times \left[rx - \left\{x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}\right\}\right] = 0. \quad (19)$$

From (19) we conclude that the minimum for $\lambda(x, y)$ must satisfy one of two possibilities:

$$rx = x_F\,y_2' + (y_2' - 1)\sqrt{\frac{2}{1 + r}}, \quad (20)$$

or

$$rx = x_F\,y_2' - (y_2' - 1)\sqrt{\frac{2}{1 + r}}. \quad (21)$$

Combining terms in (20), and using $y_2' = (y_2/y_F)$

$$rx + \sqrt{\frac{2}{1 + r}} = \frac{y_2}{y_F}\left(x_F + \sqrt{\frac{2}{1 + r}}\right). \quad (22)$$

Further simplification of (22) yields

$$\frac{y_2 - y_F}{rx - x_F} = \frac{y_F}{x_F + \sqrt{\frac{2}{1+r}}} \tag{23}$$

Using (23) and the expressions for the derivatives $(\partial\lambda/\partial x)$, $(\partial\lambda/\partial y)$ given in [2] we conclude

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 + r\sqrt{\frac{2}{1+r}}}. \tag{24}$$

Eq. (24) gives y as a linear function of x, and setting this linear function of x into (23) yields a quadratic equation in x whose coefficients depend on the given data. We then solve analytically and determine two exact values of x.

Using the second possibility given by (21) and proceeding as above we obtain

$$\frac{y_2 - y_F}{rx - x_F} = \frac{y_F}{x_F - \sqrt{\frac{2}{1+r}}} \tag{25}$$

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 - r\sqrt{\frac{2}{1+r}}}. \tag{26}$$

Similarly the pair of Eqs. (25), (26) can be solved analytically yielding two more exact values of x. Using (24), (26) the corresponding y values are determined.

We have now determined analytically four solutions (x, y) to the minimum problem. Using these four values of (x, y) there correspond four values of λ. The smallest of these four values of λ is the *absolute minimum* for λ. This then gives the complete solution to the terminal-to-terminal problem first formulated by Vargo [1]. It will be shown subsequently that under certain orbital conditions three of the four values of λ can be rejected analytically. We next study geometrically the optimum transfer path given by the above analytical expressions.

Replacing rx by x_2 in (23), we have the pair of equation from (23), (24)

$$\frac{y_2 - y_F}{x_2 - x_F} = \frac{y_F}{x_F + \sqrt{\frac{2}{1+r}}}$$

$$\frac{y - y_0}{x - x_0} = \frac{y_0}{x_0 + r\sqrt{\frac{2}{1+r}}}. \tag{27}$$

Eqs. (27) tells us that only in the case where the first or second terminals have radial velocities approaching zero will the transfer orbit leave or arrive tangentially. In all other cases the transfer orbit does not leave the first terminal tangentially nor arrive at the second terminal tangentially. In the interesting case of transfer between a circular orbit and a point on a nonintersecting elliptic orbit, the transfer path is tangential to the circular orbit, but is not tangent to the elliptical orbit (see Figure 1).

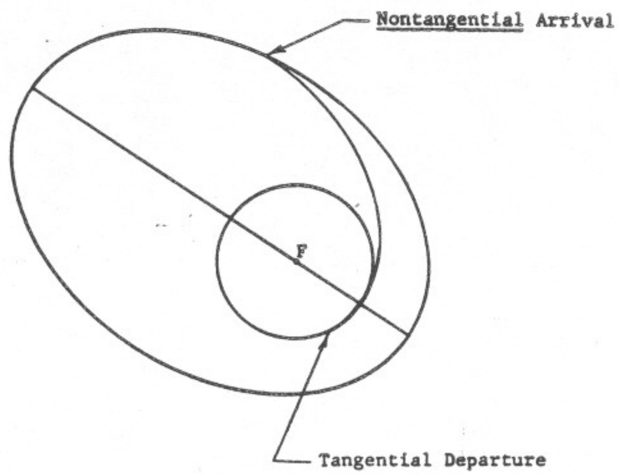

Fig. 1. Optimum Circle-to-Ellipse Transfer

Eqs. (25), (26) give the only remaining possibility, and the same tangency conditions hold for this case. We next study the case of transfer between an elliptic and a circular orbit.

Ellipse—Circle Transfer

For the case where the second terminal corresponds to a circular orbit we have

$$u_F = \sqrt{\frac{\mu}{\beta}} = \sqrt{\frac{\mu}{\alpha}}\sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{\mu}{\alpha}}\, r^{1/2}$$

$$v_F = 0.$$

Nondimensionalizing we get

$$x_F = r^{1/2}, \qquad y_F = 0.$$

Since (15) does not hold for $y_2 = 0$ we first consider transfer to an almost circular orbit

$$x_F = r^{1/2}, \qquad y_F = \epsilon$$

where ϵ is an arbitrarily small radial velocity. In this case the annular region wherein the solution lies is bounded by two ellipses arbitrarily close to one another. Also the second possibility given by (25), (26) can be eliminated for positive y_F. It is shown in [2] that y_2 is less than y_F, and rx is less than x_F. That is, the transfer orbit is itself an ellipse. Therefore the left-hand side of (25) is strictly positive. Since r must lie between zero and one we conclude

$$x_F - \sqrt{\frac{2}{1+r}} = r^{1/2} - \sqrt{\frac{2}{1+r}} < 0.$$

Therefore (25) is physically unrealizable. Eq. (23) for arbitrarily small radial velocity y_F becomes

$$\frac{y_2 - \epsilon}{rx - r^{1/2}} = \frac{\epsilon}{r^{1/2} - \sqrt{\frac{2}{1+r}}}. \tag{28}$$

Since ϵ can be chosen arbitrarily small we conclude from (28) that in transfer to a circular orbit y_2 becomes arbitrarily small. We then say

$$y_2^2 = (1 - r^2)x^2 + y^2 + 2(r - 1) = 0 \quad (29)$$

and from (24)

$$\frac{y - y_0}{x - x_0} = \tan \varphi_0 \quad (30)$$

where

$$\tan \varphi_0 = \frac{y_0}{x_0 + r\sqrt{\frac{2}{1+r}}}. \quad (31)$$

To determine the possible roots we first introduce the following

Let $A = \tan \varphi_0$

$$B = y_0 - x_0 \tan \varphi_0 = r\sqrt{\frac{2}{1+r}} \tan \varphi_0. \quad (32)$$

Using these values of A, B we get from (29)

$$(1 - r^2 + A^2)x^2 + 2ABx + [B^2 - 2(1 - r)] = 0. \quad (33)$$

Solving for x

$$x = \frac{-AB}{1 - r^2 + A^2} \pm \frac{\{A^2B^2 - (1 - r^2 + A^2)[B^2 - 2(1-r)]\}^{\frac{1}{2}}}{1 - r^2 + A^2}. \quad (34)$$

From (32), (34) and some simplification we get

$$x = \frac{\tan^2 \varphi_0 r \sqrt{\frac{2}{1+r}} \pm (1 - r)\sqrt{2(1+r)} \sec \varphi_0}{r^2 - \sec^2 \varphi_0}. \quad (35)$$

Since the denominator in (35) is strictly negative we choose the negative possibility for the numerator to insure that x is given as a positive quantity. That x must be positive is determined by the fact that in this case x_0 and x_F have the same sign (both orbital rotations taken in same sense). For the retrograde case x_0 and x_F would have opposite signs and x could be negative. This gives

$$x = \frac{\tan^2 \varphi_0 r \sqrt{\frac{2}{1+r}} - (1 - r)\sqrt{2(1+r)} \sec \varphi_0}{r^2 - \sec^2 \varphi_0}. \quad (36)$$

In general the nondimensional characteristic-velocity can be written as

$$\lambda = (x - x_0)\sqrt{1 + \left(\frac{y - y_0}{x - x_0}\right)^2} + (x_F - x_2)\sqrt{1 + \left(\frac{y_2 - y_F}{x_2 - x_F}\right)^2}. \quad (37)$$

For the ellipse to circle case using (23), (24), (36) and (37) the absolute minimum for λ is given by

$$\lambda_{\substack{\text{absolute}\\ \text{minimum}}} = r^{1/2} + \sqrt{\frac{2}{1+r}} - \sqrt{\left(x_0 + r\sqrt{\frac{2}{1+r}}\right)^2 + y_0^2}. \quad (38)$$

For the special case of $x_0 = 0$, i.e., the initial velocity all radial, this result reduces to that given by Lawden [3]. Also for the case wherein $y_0 = 0$, circle-to-circle, Eq. (38) reduces to the Hohmann result which is hereby clearly shown to be an absolute minimum.

An analogous result may be obtained for the case of circle-to-ellipse transfer.

Optimum Apogee Arrival Paths

In [2] it is demonstrated that for optimum circle-to-ellipse transfer the first impulse must be applied tangentially. For this case the nondimensional characteristic-velocity is found to be

$$\lambda = x - 1 + \sqrt{(rx - x_F)^2 + (y_2 - y_F)^2}. \quad (39)$$

From (39)

$$\lambda \geqq x - 1 + x_F - rx, \quad (40)$$

the nondimensional velocity x_F, in terms of the orbital elements, is given by

$$x_F = r\sqrt{\frac{1 + e_F}{r_p}} \quad (41)$$

where: e_F = eccentricity of elliptic orbit

r_p = distance ratio at perigee.

Since departure from the circular orbit is tangential ($y = 0$) it follows that

$$x \geqq \sqrt{\frac{2}{1+r}}. \quad (42)$$

This may be deduced from a study of the annular region containing the solution (see Ref. [2]). Combining (39), (40) and (41):

$$\lambda \geqq \sqrt{\frac{2}{1+r}}(1 - r) + r\sqrt{\frac{1 + e_F}{r_p}} - 1. \quad (43)$$

The least value of the derivative with respect to r of the right-hand side of (43) is

$$-\frac{3}{2}\sqrt{2} + \frac{1}{r_p}.$$

If the least value of this derivative is positive then all values of the derivative are positive. Therefore the right-hand side of (43) is monotonically increasing in r provided

$$r_p \leqq \frac{1}{4.5}. \quad (44)$$

It is shown in [2] that at apogee or perigee the equality in (43) holds. For the class of orbits given by (44) the right-hand side of (43) assumes its minimum value at the smallest r, i.e., at apogee. Since from (43) it is impossible for λ to assume a smaller value than the minimum of the right-hand side it follows that λ assumes its least value at the smallest value of r, which occurs at apogee.

Starting from the circular orbit, the methods given in this paper give the optimum path to any given point on the elliptical orbit. Therefore to every point on the elliptical orbit there corresponds an optimum transfer path. Of the infinity of optimal paths there is an optimum transfer path. It is rigorously demonstrated above that for the class of orbits defined by (44) the Hohmann path arriving at apogee is such a least fuel path.

References

[1] Vargo, L. G., "Optimal Transfer Between Two Coplanar Terminals in a Gravitational Field," Presented at the Western Regional Meeting, August 18–19, 1958, AAS Preprint No. 58–20.

[2] Munick, H., McGill, R. and Taylor, G. E., "Minimization of Characteristic Velocity for Transfer Between Arbitrary Terminals in an Inverse Square Field Using Two Impulses," Presented at the ARS 14th Annual Meeting, Washington, D.C., Nov. 16–20, 1959, ARS Preprint No. 959–59. See also abridged version ARS Journal July, 1960.

[3] Lawden, D. F., "Entry Into Circular Orbits—1," J.B.I.S., 10(1), 5–17, Jan. 1951.

Universities Receiving THE JOURNAL OF THE ASTRONAUTICAL SCIENCES

University of Alabama
University, Alabama

University of Arizona
Tucson, Arizona

Adelphi College
Garden City, New York

Boston University
Boston, Massachusetts

California Inst. of Technology
Pasadena, California

University of California
Berkeley, Los Angeles, Calif.

Carnegie Inst. of Technology
Pittsburgh, Pennsylvania

Case Inst. of Technology
Cleveland, Ohio

Chipola Junior College
Marianna, Florida

Clemson Agricultural College
Clemson, South Carolina

Colorado State University
Fort Collins, Colorado

Cornell University
Ithaca, New York

Fairleigh Dickinson University
Teaneck, New Jersey

Florence State College
Florence, Alabama

Florida State University
Tallahassee, Florida

Georgia Inst. of Technology
Atlanta, Georgia

Illinois Inst. of Technology
Chicago, Illinois

University of Illinois
Chicago, Illinois

R.P.I. Hartford Graduate Ctr.
East Windsor Hill, Connecticut

Iowa State University
Iowa City, Iowa

Purdue University
Lafayette, Indiana

University of South Carolina
Columbia, South Carolina

Southern Methodist University
Dallas, Texas

Stanford Research Institute
Menlo Park, California

Stanford University
Stanford, California

University of Texas
Austin, Texas

Union College
Schenectady, New York

USMA
West Point, New York

USNA
Annapolis. Maryland

Johns Hopkins University
Applied Physics Laboratory
Silver Spring, Maryland

Lafayette College
Easton, Pennsylvania

Lehigh University
Bethlehem, Pennsylvania

Los Angeles City College
Los Angeles, California

Los Angeles Valley College
Van Nuys, California

Lowell Technological Institute
Lowell, Massachusetts

University of Michigan
Ann Arbor, Michigan

University of Minnesota
Minneapolis, Minnesota

New York University
New York, New York

North Carolina State College
Raleigh, North Carolina

Northwestern Technological Institute
Evanston, Illinois

Ohio State University
Columbus, Ohio

Pan American College
Edinberg, Texas

University of Pennsylvania
Philadelphia, Pennsylvania

Polytechnic Inst. of Brooklyn
Brooklyn, New York

Princeton University
Princeton, New Jersey

San Francisco State College
San Francisco, California

San Jose State College
San Jose, California

U. S. Navy Postgraduate School
Monterey, California

University of Utah
Salt Lake City, Utah

University of Washington
Seattle, Washington

University of Wisconsin
Madison, Wisconsin

College of Aeronautics
Cranfield Bletchley Bucks,
England

Istituto Universitario Navale
Napoli, Italy

University of Tokyo
Chiba-Ken, Japan