letzten Kapitel ihr eigenes Forschungsinstrument beschrieben, das heute mächtigste Radio-Teleskop der Erde: jenes von Jodrell Bank, der nun elf Jahre alten radioastronomischen Forschungszentrale der Universität von Manchester.

Was die Radio-Astronomie ist, wird jeder aus diesem Werke mit wirklicher Befriedigung lernen können. Es ist auch leicht lesbar und mit geringen mathematischen Anforderungen, bei strengster Wissenschaftlichkeit, geschrieben. Die Autoren dürfen in jeder Hinsicht beglückwünscht werden!

J. Fuchs, Universitätssternwarte Innsbruck

Advances in Geophysics. Edited by H. E. Landsberg and J. van Mieghem. Vol. 5. Mit 55 Abb., X, 325 S. New York: Academic Press Inc. 1958. $ 10.00.

In dieser Serie von Berichten über die Fortschritte der Geophysik bringt der 5. Band einen an unbekannten Einzelheiten erstaunlich reichen Artikel von N. C. Gerson über die Geschichte der Idee der internationalen Polar- bzw. Geophysikalischen Jahre, in dem Karl Weyprechts Initiative erfreulich ausführlich hervorgehoben wird. Der Bericht — der in Zukunft sicher sehr viel zitiert werden wird — erfaßt auch schon Teilresultate der sieben erfolgreichen künstlichen Satelliten. Hierin wird besonders hervorgehoben, daß die beobachtete große Höhenerstreckung der Ionosphäre nunmehr endgültig auf hohe Temperaturen in ihr zurückzuführen ist (wie dies der Referent bereits 1936 in Met. Z. **53**, **41**, durch die Auswertung von ionosphärischen Grenzfrequenzausmessungen nachgewiesen hat). Beim abgelaufenen Internationalen Geophysikalischen Jahr selbst sieht Gerson eine Gefahr nur darin, daß das enorme Beobachtungsmaterial vielleicht auch nach Jahrzehnten noch nicht ausgewertet sein wird. — B. Gutenberg berichtet anschließend über die Fortschritte der Mikroseismik, die besser als Bodenunruhe zu bezeichnen wäre, da sie mit Erdbeben oder Explosionen nichts gemeinsam hat. Gutenbergs Überblick über ihren teils periodischen, teils nichtperiodischen Charakter weist in sehr klarer Weise den — überwiegend meteorologischen — Ursprung der Einzelphänomene nach. — Die modernen Methoden zur Bestimmung von Größe und Figur der Erde (z. B. astronomische Großtriangulationen, astronomische Ortsbestimmungen, Fein-Nivellements und Schweremessungen) läßt in übersichtlicher Form der Beitrag von R. A. Hirvonen Revue passieren. Das Geoid-Problem wird hierbei (eingehend informierend) diskutiert. — Das theoretische Verständnis der ozeanischen Gezeiten und den augenblicklichen Stand unseres Wissens darüber behandelt A. T. Doodson. Eindringlich wird hier (im Anschluß an eine ausgezeichnete Übersicht) auf die leider viele, noch zu leistende Arbeit hingewiesen. — Erfreulich ausführlich berichtet K. Watanabe über die ultravioletten Absorptionsprozesse in der oberen Atmosphäre. Die Wirkungsquerschnitte der in Betracht kommenden Atome und Moleküle werden eingehend besprochen und die höhenabhängige Verteilung von Temperatur, Dichte und Zusammensetzung in Ozonosphäre und Ionosphäre diskutiert. Der Artikel stellt eine Fundgrube an Einzeldaten dar. — Der ausführlichste (abschließende) Artikel ist dem aktuellen Problem der Physik der Zustandsänderungen der Wolken gewidmet, das seinen wirtschaftlich bedeutungsvollsten Ausdruck in den Versuchen zur künstlichen Steuerung der natürlichen Niederschläge (z. B. Förderung des Regens, Verhinderung von Hagel usw.) findet. Hier sollte — meint der Autor James E. McDonald — nie übersehen werden, daß unser Wissen über die Wolkenphysik noch um eine Größenordnung zu klein gegenüber dem ist, was für die Erforschung der Steuerungsprobleme nötig wäre. Zum Verständnis der gegenwärtigen Situation behandelt er anfangs die normale Wolkenphysik und die gerade in den letzten Jahren rapid entwickelten Theorien der Niederschlagsprozesse. Ergebnis: Die Sachlage ist dadurch zu kennzeichnen, daß trotz der umfangreichen Literatur ein wirklicher Erfolg dieser oder jener Methode zur künstlichen Einleitung von Niederschlägen noch nicht eindeutig festgestellt werden konnte; die gemeldeten Fälle sind statistisch derzeit noch immer ohne Überzeugungskraft.

Der Forscher wird diesen Band der „Advances in Geophysics" mit vielem Nutzen in die Hand nehmen.

J. Fuchs, Universitätssternwarte Innsbruck

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The Calculation of Minimal Orbits

By

G. C. Smith[1]

(With 1 Figure)

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Abstract — Zusammenfassung — Résumé

The Calculation of Minimal Orbits. Solutions, or approximate solutions, are found to the equations determining the mode of transfer of a rocket between two coplanar orbits, with minimal fuel expenditure, for a number of different cases; proof that a cotangential ellipse is a first order approximation to the minimal mode of transfer between two orbits of small eccentricity, is included.

Die Berechnung von „Minimalbahnen". Lösungen (oder angenäherte Lösungen) lassen sich in einer Anzahl verschiedener Fälle für die Gleichungen finden, welche die Art des Überganges einer Rakete zwischen zwei Bahnen in derselben Ebene bestimmen, wobei ein Minimum an Brennstoff verbraucht wird. Es wird der Beweis dafür erbracht, daß eine kotangentiale Ellipse eine Annäherung ersten Grades an den Minimumweg des Überganges zwischen zwei Bahnen kleiner Exzentrizität darstellt.

Le calcul des orbites minimales. Des solutions exactes ou approchées aux équations gouvernant le mode des transfert d'une fusée entre deux orbites coplanaires sont établies. Le transfert est minimal par rapport à la consommation d'ergols et comporte la preuve que pour deux orbites de faible excentricité, une ellipse cotangentielle est une approximation du premier ordre.

I. Introduction and Basic Formulae

In [1] it has been proved that the transfer of a rocket between two coplanar elliptical orbits, described in the same sense about the same centre of inverse square law of attraction, with the minimum fuel expenditure, is achieved by applying impulsive thrusts; and this problem of optimal transfer solved in the sense that, for any fixed number of impulses, the equations determining the elements of the transfer orbits and the impulses have been found. Except for one special case [2] no method of solving these equations has previously been published and this paper shows how this may be done in a number of different situations for the case of two-impulse transfer.

Evidence of certain numerical results has suggested that the optimum transfer ellipse is closely approximated by an ellipse which is tangential to the original and terminal orbits, and, in [3], a method of determining the ellipse of this kind, for which fuel expenditure is minimal, is given. In section III of this paper it is proved that for ellipses of small eccentricity, the "best" ellipse tangential to the original and terminal ellipses, is in fact a first approximation to the optimum.

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The detailed derivation of the formulae below will be found in [1].

Using polar coordinates $(1/s, \theta)$ with pole at the centre of attraction, the elliptical orbit $s = p + q\cos(\theta - \tilde{\omega})$ will be written $(p, q, \tilde{\omega})$: orbits are always supposed to be described in an anti-clockwise direction. The angle made by the direction of thrust with the perpendicular to the radius vector, drawn in the sense that the orbit is described, will be denoted by ϕ; and if μs^2 is the attraction on unit mass, $\mu^{1/2} Z \sin\phi$ is the component of the rocket velocity perpendicular to the direction of thrust and which accordingly remains constant during the thrust.

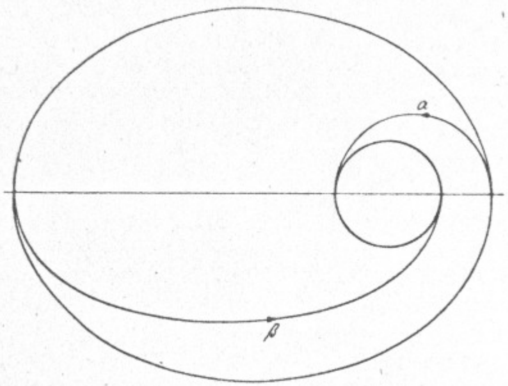

Fig. 1

If an impulse in the direction ϕ_1 applied at $(1/s_1, \theta_1)$ has the effect of transferring the rocket from orbit $(p_1, q_1, \tilde{\omega}_1)$ to $(p, q, \tilde{\omega})$, then the characteristic velocity of the manoeuvre is given by

$$c \log \frac{m_1}{m} = \mu^{1/2} s_1 (p^{-1/2} - p_1^{-1/2}) \sec\phi_1,$$

where c is the jet velocity and m_1/m is the ratio of the mass of the rocket before the impulse is applied, to the mass afterwards.

The equations determining the minimal orbit $(p, q, \tilde{\omega})$ for two-impulse transfer between coplanar orbits $(p_1, q_1, \tilde{\omega}_1)$, $(p_2, q_2, \tilde{\omega}_2)$ are then:

$$q_1 \cos(\theta_1 - \tilde{\omega}_1) = s_1 - p_1, \tag{1}$$

$$q_1 \sin(\theta_1 - \tilde{\omega}_1) = (s_1 - Z_1 p_1^{1/2}) \tan\phi_1, \tag{2}$$

$$q \cos(\theta_1 - \tilde{\omega}) = s_1 - p, \tag{3}$$

$$q \sin(\theta_1 - \tilde{\omega}) = (s_1 - Z_1 p^{1/2}) \tan\phi_1, \tag{4}$$

$$q_2 \cos(\theta_2 - \tilde{\omega}_2) = s_2 - p_2, \tag{5}$$

$$q_2 \sin(\theta_2 - \tilde{\omega}_2) = (s_2 - Z_2 p_2^{1/2}) \tan\phi_2, \tag{6}$$

$$q \cos(\theta_2 - \tilde{\omega}) = s_2 - p, \tag{7}$$

$$q \sin(\theta_2 - \tilde{\omega}) = (s_2 - Z_2 p^{1/2}) \tan\phi_2, \tag{8}$$

$$\left(Z_1 - \frac{s_1}{Z_1}\right) \sin\phi_1 = \left(Z_2 - \frac{s_2}{Z_2}\right) \sin\phi_2, \tag{9}$$

$$\left(\frac{s_1 + p}{Z_1 p^{1/2}} + 1\right) \cos\phi_1 = \left(\frac{s_2 + p}{Z_2 p^{1/2}} + 1\right) \cos\phi_2, \tag{10}$$

$$\left(1 + \frac{p^{1/2}}{Z_1}\right)(s_1 - p)\cos\phi_1 + (s_1 - Z_1 p^{1/2}) \sin\phi_1 \tan\phi_1 =$$
$$\left(1 + \frac{p^{1/2}}{Z_2}\right)(s_2 - p)\cos\phi_2 + (s_2 - Z_2 p^{1/2}) \sin\phi_2 \tan\phi_2; \tag{11}$$

$\mu^{1/2} Z_1 \sin\phi_1$, $\mu^{1/2} Z_2 \sin\phi_2$ being the velocity components normal to the at the first and second impulses respectively.

For brevity, t_1 will often be written for $\tan\phi_1$, and t_2 for $\tan\phi_2$.

II. Elliptical Orbits with Major Axes Aligned

In the expression $\mu^{1/2} s_1 (p^{-1/2} - p_1^{-1/2}) \sec\phi_1$ for the characteristic v $|\sec\phi_1| \geqslant 1$; this suggests that for the minimal orbit ϕ_1 will tend to value close to 0 or π. We consider whether $\sec\phi_1$, $\sec\phi_2$ can take th value in a minimal orbit, i.e. put $\tan\phi_1 = \tan\phi_2 = 0$; eqs. (1) — (become

$$\left.\begin{aligned} q_1 \cos(\theta_1 - \tilde{\omega}_1) &= s_1 - p_1, & q_1 \sin(\theta_1 - \tilde{\omega}_1) &= 0,\\ q \cos(\theta_1 - \tilde{\omega}) &= s_1 - p, & q \sin(\theta_1 - \tilde{\omega}) &= 0,\\ q_2 \cos(\theta_2 - \tilde{\omega}_2) &= s_2 - p_2, & q_2 \sin(\theta_2 - \tilde{\omega}_2) &= 0,\\ q \cos(\theta_2 - \tilde{\omega}) &= s_2 - p, & q \sin(\theta_2 - \tilde{\omega}) &= 0. \end{aligned}\right\}$$

Consider first the case when q_1, q_2, and q are non-zero; then we have

$$\left.\begin{aligned} \theta_1 - \tilde{\omega} &= \theta_2 - \tilde{\omega} + l\pi,\\ \theta_1 - \tilde{\omega}_1 &= \theta_2 - \tilde{\omega}_2 + m\pi, \end{aligned}\right\}$$

where l and m are integers, i.e.

$$\left.\begin{aligned} \theta_1 - \theta_2 &= l\pi,\\ \tilde{\omega}_1 - \tilde{\omega}_2 &= \theta_1 - \theta_2 - m\pi = (l - m)\pi, \end{aligned}\right\}$$

also

$$\theta_1 - \tilde{\omega}_1 = \theta_1 - \tilde{\omega} - n\pi,$$

so

$$\tilde{\omega}_1 - \tilde{\omega} = n\pi.$$

Thus we find that in the case of orbits $(p_1, q_1, \tilde{\omega}_1)$ and $(p_2, q_2, \tilde{\omega}_2)$, w have $\tan\phi_1 = 0$ and $\tan\phi_2 = 0$ only if $\tilde{\omega}_1 = \tilde{\omega}_2$ or $\tilde{\omega}_1 = \tilde{\omega}_2 + \pi$, and f that $\tilde{\omega} = 0$ or π.

Now take $\tilde{\omega}_1 = 0$ and consider separately the two cases

i) $\tilde{\omega}_2 = \tilde{\omega}_1 = 0$;

ii) $\tilde{\omega}_2 = \pi$, $\tilde{\omega}_1 = 0$.

Case i. The above equations now reduce to

$$\left.\begin{aligned} q_1 \cos\theta_1 &= s_1 - p_1, & \sin\theta_1 &= 0,\\ q \cos(\theta_1 - \tilde{\omega}) &= s_1 - p, & \sin\theta_2 &= 0,\\ q_2 \cos\theta_2 &= s_2 - p_2, & &\\ q \cos(\theta_2 - \tilde{\omega}) &= s_2 - p, & & \end{aligned}\right\}$$

where $\tilde{\omega}$ is either 0 or π.

Hence $\theta_1 = 0$ or π, $\theta_2 = 0$ or π; $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$ correspo rectilinear transfer which is clearly uneconomical, so we must take either and $\theta_2 = \pi$, or $\theta_1 = \pi$ and $\theta_2 = 0$, and the remaining equations are the

$$\left.\begin{aligned} \pm q_1 &= s_1 - p_1,\\ \pm q &= s_1 - p, \end{aligned}\right\} \qquad \left.\begin{aligned} \pm q_2 &= s_2 - p_2,\\ \pm q &= s_2 - p, \end{aligned}\right\}$$

where, if $\tilde{\omega} = 0$, the same sign must be taken within each bracketed pair, $\tilde{\omega} = \pi$, the opposite sign taken. But opposite signs must be attached to t as these equations derive from $q\cos(\theta_1 - \tilde{\omega}) = s_1 - p$, $q\cos(\theta_2 - \tilde{\omega}) = s$ and θ_1, θ_2 have been shown to differ by π.

Thus for definiten aking $\theta_1 = 0,\ \theta_2 = \pi,\ \tilde{\omega} = 0$ we have

$$\left.\begin{array}{l} q_1 = s_1 - p_1, \\ q = s_1 - p, \end{array}\right\} \qquad \left.\begin{array}{l} -q_2 = s_2 - p_2, \\ -q = s_2 - p, \end{array}\right\} \tag{19}$$

$$\left.\begin{array}{ll} s_1 = p_1 + q_1, & s_2 = p_2 - q_2, \\ p = \dfrac{p_1 + q_1 + p_2 - q_2}{2}, & q = \dfrac{p_1 + q_1 - p_2 + q_2}{2}, \end{array}\right\} \tag{20}$$

l changing $\tilde{\omega}$ to π only results in changing the sign of q; i.e. giving

$$q = -\left(\frac{p_1 + q_1 - p_2 + q_2}{2}\right);$$

t q is essentially positive, so we find that for $\theta_1 = 0,\ \theta_2 = \pi$ we must take
$= 0$ if $p_1 + q_1 > p_2 - q_2$, and $\tilde{\omega} = \pi$ if $p_1 + q_1 < p_2 - q_2$. The results for
$= \pi,\ \theta_2 = 0$ are similar, being as given in (20), but with the signs of q_1, q_2
nged; here $\tilde{\omega}$ is π or 0 according as $p_1 - q_1 \gtrless p_2 + q_2$.

To summarise, eqs. (1) — (8) are satisfied by $\tan \phi_1 = \tan \phi_2 = 0$ and either

$$\left.\begin{array}{l} \theta_1 = 0, \quad \theta_2 = \pi; \quad s_1 = p_1 + q_1, \quad s_2 = p_2 - q_2; \\ \tilde{\omega} = 0 \text{ or } \pi \text{ according as } p_1 + q_1 \gtrless p_2 - q_2; \\ p = \dfrac{p_1 + q_1 + p_2 - q_2}{2}, \qquad q = \dfrac{|p_1 + q_1 - p_2 + q_2|}{2}; \end{array}\right\} \tag{21}$$

$$\left.\begin{array}{l} \theta_1 = \pi, \quad \theta_2 = 0; \quad s_1 = p_1 - q_1, \quad s_2 = p_2 + q_2; \\ \tilde{\omega} = \pi \text{ or } 0 \text{ according as } p_1 - q_1 \gtrless p_2 + q_2; \\ p = \dfrac{p_1 - q_1 + p_2 + q_2}{2}, \qquad q = \dfrac{|p_1 - q_1 - p_2 - q_2|}{2}. \end{array}\right\} \tag{22}$$

These determine the transfer orbit geometrically; it remains to decide whether
ϕ_2 are 0 or π and to show that the remaining eqs. (9) — (11) can be satisfied.
diagram of the orbits quickly shows which value must be taken for ϕ_1, ϕ_2.
. (9) is satisfied identically, while (10) and (11) give

$$\left.\begin{array}{c} \left(\dfrac{s_1 + p}{Z_1 p^{1/2}} + 1\right) = \pm \left(\dfrac{s_2 + p}{Z_2 p^{1/2}} + 1\right), \\ \left(1 + \dfrac{p^{1/2}}{Z_1}\right)(s_1 - p) = \pm \left(1 + \dfrac{p^{1/2}}{Z^2}\right)(s_2 - p), \end{array}\right\} \tag{23}$$

e plus signs being taken if $\phi_1 = \phi_2$, minus if $\phi_1 = \phi_2 + \pi$. Solving these
uations for Z_1, Z_2 gives, when plus signs are taken

$$Z_1 = -\frac{2 p^{3/2}}{(s_2 + p)}, \qquad Z_2 = -\frac{2 p^{3/2}}{(s_1 + p)}, \tag{24}$$

d when the minus signs apply

$$Z_1 = \frac{2 p^{3/2} (s_2 - s_1)}{(s_1 + s_2) s_2 + p (s_1 - 3 s_2)}, \qquad Z_2 = \frac{2 p^{3/2} (s_1 - s_2)}{(s_1 + s_2) s_1 + p (s_2 - 3 s_1)}. \tag{25}$$

bstituting the appropriate values for p, s_1, s_2, we find Z_1, Z_2 and thus the
mplete solution.

Ca . Where $\tilde{\omega}_1 = 0,\ \tilde{\omega}_2 = \pi$; we find in a similar manner the soluti

$$\left.\begin{array}{l} \theta_1 = 0, \quad \theta_2 = \pi; \quad s_1 = p_1 + q_1, \quad s_2 = p_2 + q_2; \\ \tilde{\omega} = 0 \text{ or } \pi \text{ according as } p_1 + q_1 \gtrless p_2 + q_2; \\ p = \dfrac{p_1 + q_1 + p_2 + q_2}{2}, \qquad q = \dfrac{|p_1 + q_1 - p_2 - q_2|}{2}; \end{array}\right\} \tag{26}$$

$$\left.\begin{array}{l} \theta_1 = \pi, \quad \theta_2 = 0; \quad s_1 = p_1 - q_1, \quad s_2 = p_2 - q_2; \\ \tilde{\omega} = \pi \text{ or } 0 \text{ according as } p_1 - q_1 \gtrless p_2 - q_2; \\ p = \dfrac{p_1 - q_1 + p_2 - q_2}{2}, \qquad q = \dfrac{|p_1 - q_1 - p_2 + q_2|}{2}; \end{array}\right\} \tag{27}$$

and ϕ_1, ϕ_2, Z_1, Z_2 are determined as before.

Thus we have proved that for orbits with major axes aligned there are two minimal orbits of transfer, the impulses being applied at the apses and each transfer orbit touches the initial and final orbit at their apses. An example follows.

Example. Consider the orbits

$$p_1 = 1.5, \quad q_1 = 1, \quad \tilde{\omega}_1 = 0,$$
$$p_2 = 4, \quad q_2 = 1, \quad \tilde{\omega}_2 = 0.$$

Since $\tilde{\omega}_1 = \tilde{\omega}_2 = 0$, solutions (21), (22) are appropriate. Using (21), (transfer orbit α in diagram)

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad s_1 = 2.5, \quad s_2 = 3,$$
$$p = 2.75, \quad q = 0.25,$$
$$\tilde{\omega} = \pi \quad \text{as} \quad p_1 + q_1 = 2.5 < 3 = p_2 - q_2.$$

As each impulse slows down the rocket, $\phi_1 = \phi_2 = \pi$, thus Z_1, Z_2 are given by (24), i.e.

$$Z_1 = -1.586, \qquad Z_2 = -1.747.$$

The characteristic velocity is therefore

$$\{2.5\,|(2.75)^{-1/2} - (1.5)^{-1/2}| + 3\,|(2.75)^{-1/2} - 0.5|\}\,\mu^{1/2} = 0.8429\,\mu^{1/2}.$$

Using (22), (transfer orbit β)

$$\theta_1 = \pi, \quad \theta_2 = 0, \quad s_1 = 0.5, \quad s_2 = 5,$$
$$p = 2.75, \quad q = 2.25,$$
$$\tilde{\omega} = \pi \quad \text{as} \quad p_1 - q_1 = 0.5 < 5 = p_2 + q_2;$$

also $\phi_1 = \phi_2 = \pi$, and using (24) we find

$$Z_1 = -1.176, \qquad Z_2 = -2.806.$$

The characteristic velocity in this case is

$$\{0.5\,|(2.75)^{-1/2} - (1.5)^{-1/2}| + 5\,|(2.75)^{-1/2} - 0.5|\}\,\mu^{1/2} = 0.6227\,\mu^{1/2}.$$

Orbit β therefore is that of absolute minimal fuel expenditure.

Previously we have supposed q_1, q_2, and q all non-zero. The results, however, remain true when one, or both, of q_1 and q_2 are zero. For if $q_2 = 0$, then eqs. (1) — (8) become

$$\begin{array}{ll} q_1 \sin \theta_1 = 0, & q_1 \cos \theta_1 = s_1 - p_1, \\ q \sin (\theta_1 - \tilde{\omega}) = 0, & 0 = s_2 - p_2, \\ q \sin (\theta_2 - \tilde{\omega}) = 0, & q \cos (\theta_1 - \tilde{\omega}) = s_1 - p, \\ & q \cos (\theta_2 - \tilde{\omega}) = s_2 - p. \end{array}$$

Supposing $q \neq 0$ these give $\theta_1 = n\pi$, $\theta_1 - \theta_2 = \pi$ and $\tilde{\omega} = 0$ or π i.e., by the same argument as before, either

$$\theta_1 = 0, \qquad \theta_2 = \pi; \qquad s_1 = p_1 + q_1, \qquad s_2 = p_2;$$
$$\tilde{\omega} = 0 \text{ or } \pi \text{ according as } p_1 + q_1 \gtrless p_2;$$
$$p = \frac{p_1 + q_1 + p_2}{2}, \qquad q = \frac{|p_1 + q_1 - p_2|}{2};$$

or $\theta_1 = \pi$ and $\theta_2 = 0$, with similar results. These formulae are the same as those of (21) and (22) with $q_2 = 0$.

When $q_1 = q_2 = 0$ we have the case of transfer between two circular orbits; a similar discussion shows that

$$s_1 = p_1, \qquad s_2 = p_2; \qquad p = \frac{p_1 + p_2}{2}, \qquad q = \frac{|p_1 - p_2|}{2}; \tag{28}$$

θ_1, θ_2 and $\tilde{\omega}$ are now indeterminate, but θ_1 and θ_2 differ by π. This orbit of transfer is the HOHMANN ellipse.

III. Initial and Terminal Orbits of Small Eccentricity

Consider now the case when q_1, q_2 are small, so that the orbits are very nearly concentric circles. Then, to a zero order of approximation, taking $\tilde{\omega}_1 = 0$ and $p_1 > p_2$, we have the case of transfer between two circular orbits, the solution for which is

$$\left.\begin{aligned} &\theta_1 - \theta_2 = \pi; \qquad s_1 = p_1, \qquad s_2 = p_2; \\ &p = \frac{p_1 + p_2}{2}, \qquad q = \frac{p_1 - p_2}{2}; \end{aligned}\right\} \tag{29}$$

and as $\phi_1 = \phi_2 = 0$, by (24)

$$Z_1 = -\frac{2\,p^{3/2}}{s_2 + p}, \qquad Z_2 = -\frac{2\,p^{3/2}}{s_1 + p},$$

where s_1, s_2, p have the above values.

Writing

$$\left.\begin{aligned} &\frac{q_1}{p_1} = \lambda_1 \varepsilon, && \frac{q_2}{p_2} = \lambda_2 \varepsilon, \\ &s_1 = p_1 (1 + \varrho_1 \varepsilon), && s_2 = p_2 (1 + \varrho_2 \varepsilon), \\ &p = \bar{p} (1 + \omega \varepsilon), && q = \bar{q} (1 + \nu \varepsilon), \end{aligned}\right\} \tag{30}$$

where

$$\bar{p} = \frac{p_1 + p_2}{2}, \qquad \bar{q} = \frac{p_1 - p_2}{2}; \tag{31}$$

also taking

$$t_1 = \tau_1 \varepsilon, \qquad t_2 = \tau_2 \varepsilon, \tag{32}$$

we have $\sin \phi_1 = \tau_1 \varepsilon$, $\sin \phi_2 = \tau_2 \varepsilon$, $\cos \phi_1 = 1$, $\cos \phi_2 = 1$ to the first order in ε; further write

$$Z_1 = -\frac{2\,\bar{p}^{3/2}}{p_2 + \bar{p}}, \qquad Z_2 = -\frac{2\,\bar{p}^{3/2}}{p_1 + \bar{p}}. \tag{33}$$

From eqs. (1), (5), (2), (6), neglecting quantities $O(\varepsilon^2)$

$$\left.\begin{aligned} &\lambda_1 \cos \theta_1 = \varrho_1, && \lambda_2 \cos (\theta_2 - \tilde{\omega}_2) = \varrho_2, \\ &\lambda_1 \sin \theta_1 = l_1 \tau_1, && \lambda_2 \sin (\theta_2 - \tilde{\omega}_2) = l_2 \tau_2, \end{aligned}\right\}$$

where

$$l_1 = 1 - Z_1 p_1^{-1/2}, \qquad l_2 = 1 - Z_2 p_2^{-1/2}.$$

From eqs. (4), (8), it is clear that $\sin (\theta_1 - \tilde{\omega})$ and $\sin (\theta_2 - \tilde{\omega})$ are $O($ $\cos (\theta_1 - \tilde{\omega}) = 1$ and $\cos (\theta_2 - \tilde{\omega}) = -1$ to the first order in ε. Eq (3), (7) then give to the first order in ε

$$\left.\begin{aligned} \sin (\theta_1 - \tilde{\omega}) &= m_1 \tau_1 \varepsilon, \\ \sin (\theta_2 - \tilde{\omega}) &= m_2 \tau_2 \varepsilon, \end{aligned}\right\}$$

where

$$m_1 = \frac{(p_1 - Z_1 \bar{p}^{1/2})}{\bar{q}}, \qquad m_2 = \frac{(p_2 - Z_2 \bar{p}^{1/2})}{\bar{q}},$$

and

$$\left.\begin{aligned} \bar{q} + \bar{q}\,\nu\,\varepsilon &= (p_1 - \bar{p}) + (p_1 \varrho_1 - \bar{p}\,\omega)\,\varepsilon, \\ -\bar{q} - \bar{q}\,\nu\,\varepsilon &= (p_2 - \bar{p}) + (p_2 \varrho_2 - \bar{p}\,\omega)\,\varepsilon; \end{aligned}\right\}$$

i.e. since $\bar{q} = p_1 - \bar{p}$, $-\bar{q} = p_2 - \bar{p}$,

$$\left.\begin{aligned} \bar{q}\,\nu &= p_1 \varrho_1 - \bar{p}\,\omega, \\ -\bar{q}\,\nu &= p_2 \varrho_2 - \bar{p}\,\omega, \end{aligned}\right\}$$

which give

$$\bar{p}\,\omega = \frac{p_1 \varrho_1 + p_2 \varrho_2}{2}, \qquad \bar{q}\,\nu = \frac{p_1 \varrho_1 - p_2 \varrho_2}{2}.$$

Now, from (34), we have

$$\left.\begin{aligned} \lambda_1^2 &= \varrho_1^2 + l_1^2 \tau_1^2, \\ \lambda_2^2 &= \varrho_2^2 + l_2^2 \tau_2^2. \end{aligned}\right\}$$

From $\cos (\theta_1 - \tilde{\omega}) = 1$, $\cos (\theta_2 - \tilde{\omega}) = -1$ and (36) we deduce th

$$\theta_1 - \tilde{\omega} = m_1 \tau_1 \varepsilon, \qquad \theta_2 - \tilde{\omega} = \pi - m_2 \tau_2 \varepsilon,$$

and so to the first order in ε

$$\theta_1 - \theta_2 = (m_1 \tau_1 + m_2 \tau_2)\,\varepsilon - \pi;$$

substituting these results [and using (34)] in the identity

$$\cos (\theta_1 - \theta_2) \cos \tilde{\omega}_2 - \sin (\theta_1 - \theta_2) \sin \tilde{\omega}_2 =$$
$$= \cos \theta_1 \cos (\theta_2 - \tilde{\omega}_2) + \sin \theta_1 \sin (\theta_2 - \tilde{\omega}_2),$$

we find, neglecting terms in ε,

$$\varrho_1 \varrho_2 + l_1 l_2 \tau_1 \tau_2 = \lambda_1 \lambda_2 \cos (-\pi) \cos \tilde{\omega}_2.$$

Squaring and using (41) this reduces to

$$\left(\frac{l_1}{\lambda_1}\right)^2 \tau_1^2 + 2 \left(\frac{l_1}{\lambda_1}\right) \left(\frac{l_2}{\lambda_2}\right) \tau_1 \tau_2 \cos \tilde{\omega}_2 + \left(\frac{l_2}{\lambda_2}\right)^2 \tau_2^2 = \sin^2 \tilde{\omega}_2.$$

Also from (9), to the first order in ε we find

$$k_1 \tau_1 = k_2 \tau_2,$$

where

$$k_1 = \frac{p_1}{Z_1} - Z_1, \qquad k_2 = \frac{p_2}{Z_2} - Z_2.$$

and (46) give

$$\tau_1 = \pm \frac{\lambda_1}{l_1} \frac{c \sin \tilde\omega_2}{\Delta}, \qquad \tau_2 = \pm \frac{\lambda_2}{l_2} \frac{b \sin \tilde\omega_2}{\Delta}, \tag{48}$$

re

$$\left. \begin{aligned} b &= \frac{\lambda_1 k_1}{l_1} \varepsilon, \qquad c = \frac{\lambda_2 k_2}{l_2} \varepsilon, \\ \Delta &= (b^2 + 2\, b\, c \cos \tilde\omega_2 + c^2)^{1/2}; \end{aligned} \right\} \tag{49}$$

e (41) gives

$$\varrho_1 = \pm \frac{\lambda_1 (b + c \cos \tilde\omega_2)}{\Delta}, \qquad \varrho_2 = \pm \frac{\lambda_2 (c + b \cos \tilde\omega_2)}{\Delta}. \tag{50}$$

on substituting for Z_1, Z_2 it follows immediately that k_1, k_2, l_1, l_2 are positive, thus that b, c are positive, and τ_1, τ_2 are either both positive or both negative. (44) we have

$$\varrho_1 \varrho_2 = - \lambda_1 \lambda_2 \left(\cos \tilde\omega_2 + \frac{l_1 \tau_1}{\lambda_1} \frac{l_2 \tau_2}{\lambda_2} \right), \tag{51}$$

as τ_1, τ_2, b, c are all positive this becomes

$$\begin{aligned} \varrho_1 \varrho_2 &= - \lambda_1 \lambda_2 \left(\cos \tilde\omega_2 + \frac{b\, c \sin^2 \tilde\omega_2}{\Delta^2} \right) \\ &= - \lambda_1 \lambda_2 \frac{(b \cos \tilde\omega_2 + c)(c \cos \tilde\omega_2 + b)}{\Delta^2}; \end{aligned} \tag{52}$$

hence in (50) opposite signs must be attached to the expressions for ϱ_1 and ϱ_2. Thus, to summarise, we have two solutions which are determined by:

$$\left. \begin{aligned} s_1 &= p_1 \pm \frac{b + c \cos \tilde\omega_2}{\Delta} \lambda_1 \varepsilon, \qquad s_2 = p_2 \mp \frac{c + b \cos \tilde\omega_2}{\Delta} \lambda_2 \varepsilon \\ p &= \frac{p_1 + p_2}{2} \pm \frac{(b + \cos \tilde\omega_2)\, p_1 \lambda_1 \varepsilon - (c + b \cos \tilde\omega_2)\, p_2 \lambda_2 \varepsilon}{2 \Delta}, \end{aligned} \right\} \tag{53}$$

ng either all upper signs or all lower signs; the remaining quantities are then y deduced

$$\left. \begin{aligned} q &= \frac{p_1 - p_2}{2} \pm \frac{(b + c \cos \tilde\omega_2)\, p_1 \lambda_1 \varepsilon + (c + b \cos \tilde\omega_2)\, p_2 \lambda_2 \varepsilon}{2 \Delta}, \\ \tan \tilde\omega &= \pm \frac{c \sin \tilde\omega_2}{b + c \cos \tilde\omega_2}, \\ t_1 &= \pm \frac{c \sin \tilde\omega_2}{\Delta} \frac{\lambda_1 \varepsilon}{l_1}, \qquad t_2 = \pm \frac{b \sin \tilde\omega_2}{\Delta} \frac{\lambda_2 \varepsilon}{l_2}, \\ \tan \theta_1 &= \pm \frac{c \sin \tilde\omega_2}{b + c \cos \tilde\omega_2}, \qquad \tan (\theta_2 - \tilde\omega_2) = \pm \frac{b \sin \tilde\omega_2}{c + b \cos \tilde\omega_2}; \end{aligned} \right\} \tag{54}$$

Z_1, Z_2 follow from eqs. (10) and (11); to the first order in ε these reduce to

$$\frac{s_1 + p}{Z_1 p^{1/2}} + 1 = \pm \left(\frac{s_2 + p}{Z_2 p^{1/2}} + 1 \right),$$

$$\left(1 + \frac{p^{1/2}}{Z_1} \right) (s_1 - p) = \pm \left(1 + \frac{p^{1/2}}{Z_2} \right) (s_2 - p);$$

taki[ng] the positive signs if ϕ_1, ϕ_2 are both near 0 or π and the negative sign if they [di]ffer by π. Solving gives the formulae (24) or (25) for Z_1, Z_2.

Since the transfer orbit is completely determined when any three quantities are known, it is more satisfactory to calculate s_1, s_2, p by the above formulae and then return to eqs. (1) — (8) to obtain the remaining quantities.

This result is, in fact, the cotangential ellipse of least fuel expenditure. For it has been shown in [3], p. 286, formulae (26), (29), that the cotangential ellipse of least fuel expenditure has

$$p - \bar p = \frac{1}{2} q_1 \cos (\tilde\omega - \tilde\omega_1) - \frac{1}{2} q_2 \cos (\tilde\omega - \tilde\omega_2), \tag{55}$$

where

$$\tan \tilde\omega = \frac{b \sin \tilde\omega_1 + c \sin \tilde\omega_2}{b \cos \tilde\omega_1 + c \cos \tilde\omega_2}. \tag{56}$$

Putting $\tilde\omega_1 = 0$ and substituting for $\tan \tilde\omega$, we obtain

$$p = \frac{p_1 + p_2}{2} \pm \frac{q_1}{2} \frac{b + c \cos \tilde\omega_2}{\Delta} \mp \frac{q_2}{2} \frac{c + b \cos \tilde\omega_2}{\Delta}, \tag{57}$$

which is the same as the result obtained above at eq. (53); similarly it may be verified that q, $\tilde\omega$ are also the same.

IV. One Orbit of Small Eccentricity

We consider now the case of transfer between two elliptical orbits, one only of which has small eccentricity.

Suppose $\tilde\omega_1 = 0$, $p_1 - q_1 > p_2 + q_2$, and write

$$\frac{q_2}{p_2} = \varepsilon. \tag{58}$$

To zero order in ε, we have transfer from the ellipse $(p_1, q_1, 0)$ to the circle $s_2 = p_2$; the solution for this is given by (22)

$$\left. \begin{aligned} &\theta_1 = \pi, \quad \theta_2 = 0; \quad s_1 = p_1 - q_1, \quad s_2 = p_2; \\ &\tilde\omega = \pi \text{ as } p_1 - q_1 > p_2 + q_2; \\ &p = \frac{p_1 - q_1 + p_2}{2}, \qquad q = \frac{p_1 - q_1 - p_2}{2}; \end{aligned} \right\} \tag{59}$$

and as $\phi_1 = \phi_2 = 0$, from (24) and the formulae above

$$Z_1 = - \frac{\sqrt{2}\, (p_1 - q_1 + p_2)^{3/2}}{p_1 - q_1 + 3\, p_2}, \qquad Z_2 = - \frac{\sqrt{2}\, (p_1 - q_1 + p_2)^{3/2}}{3\, p_1 - 3\, q_1 + p_2}. \tag{60}$$

Denoting the above values by $\bar s_1, \bar s_2, \bar p, \bar q, \bar Z_1, \bar Z_2$, write

$$\left. \begin{aligned} s_1 &= \bar s_1 (1 + \varrho_1 \varepsilon), \qquad s_2 = \bar s_2 (1 + \varrho_2 \varepsilon), \\ p &= \bar p\, (1 + \omega \varepsilon), \qquad q = \bar q\, (1 + \nu \varepsilon), \\ t_1 &= \tau_1 \varepsilon, \qquad t_2 = \tau_2 \varepsilon. \end{aligned} \right\} \tag{61}$$

Substituting these expressions in eqs. (1), (2), (5), (6) gives to the first order in ε

$$\left. \begin{aligned} \cos \theta_1 &= -1 + \frac{\bar s_1 \varrho_1}{q_1} \varepsilon, \qquad \sin \theta_1 = \frac{x_1 \tau_1}{q_1} \varepsilon, \\ \cos (\theta_2 - \tilde\omega_2) &= \varrho_2, \qquad \sin (\theta_2 - \tilde\omega_2) = \frac{x_2 \tau_2}{p_2}, \end{aligned} \right\} \tag{62}$$

where

$$x_1 = \bar s_1 - Z_1 p_1^{1/2}, \qquad x_2 = \bar s_2 - Z_2 p_2^{1/2}. \tag{63}$$

Squaring and adding (62), gives to the first order in ε

$$\left.\begin{aligned} \varrho_1 &= 0, \\ \varrho_2{}^2 + \frac{x_2{}^2\,\tau_2{}^2}{p_2{}^2} &= 1. \end{aligned}\right\} \tag{64}$$

Eqs. (4), (8), (3), (7), give similarly

$$\sin(\theta_1 - \tilde{\omega}) = \frac{y_1\,\tau_1}{\bar{q}}\,\varepsilon, \qquad \sin(\theta_2 - \tilde{\omega}) = \frac{y_2\,\tau_2}{\bar{q}}\,\varepsilon, \tag{65}$$

where

$$y_1 = \bar{s}_1 - Z_1\,\bar{p}^{1/2}, \qquad y_2 = \bar{s}_2 - Z_2\,\bar{p}^{1/2}; \tag{66}$$

also

$$\left.\begin{aligned} \cos(\theta_1 - \tilde{\omega}) &= 1, & \cos(\theta_2 - \tilde{\omega}) &= -1, \\ \bar{q}\,\nu &= -\bar{p}\,\omega, & -\bar{q}\,\nu &= \bar{s}_2\,\varrho_2 - \bar{p}\,\omega; \end{aligned}\right\} \tag{67}$$

i.e.

$$\bar{p}\,\omega = -\bar{q}\,\nu = \frac{\bar{s}_2\,\varrho_2}{2}. \tag{68}$$

From (65)

$$\theta_2 - \theta_1 = \pi - \left(\frac{y_1\,\tau_1}{\bar{q}} + \frac{y_2\,\tau_2}{\bar{q}}\right)\varepsilon. \tag{69}$$

On substituting from (62), (67) in the identity

$$\begin{aligned} \cos\theta_1 \cos(\theta_2 - \tilde{\omega}_2) + \sin\theta_1 \sin(\theta_2 - \tilde{\omega}_2) &= \\ = \cos(\theta_1 - \theta_2)\cos\tilde{\omega}_2 - \sin(\theta_1 - \theta_2)\sin\tilde{\omega}_2, \end{aligned}$$

we find

$$(-1)\,\varrho_2 + \frac{x_1\,x_2\,\tau_1\,\tau_2}{p_2\,q_1}\,\varepsilon = (-1)\cos\tilde{\omega}_2 + \left(\frac{y_1\,\tau_1}{\bar{q}} + \frac{y_2\,\tau_2}{\bar{q}}\right)\varepsilon\sin\tilde{\omega}_2, \tag{70}$$

i.e. neglecting terms in ε

$$\varrho_2 = \cos\tilde{\omega}_2. \tag{71}$$

Hence from (64) we have

$$\tau_2{}^2 = \frac{p_2{}^2}{x_2{}^2}\,(1 - \cos^2\tilde{\omega}_2),$$

i.e.

$$\tau_2 = \pm\,\frac{p_2 \sin\tilde{\omega}_2}{x_2}. \tag{72}$$

Also from (9) we have

$$z_1\,\tau_1 = z_2\,\tau_2, \tag{73}$$

where

$$z_1 = Z_1 - \frac{\bar{s}_1}{Z_1}, \qquad z_2 = Z_2 - \frac{\bar{s}_2}{Z_2}; \tag{74}$$

and thus

$$\tau_1 = \pm\,\frac{z_2\,p_2 \sin\tilde{\omega}_2}{z_1\,x_2}. \tag{75}$$

To summarise, we have approximately

$$\left.\begin{aligned} s_1 &= p_1 - q_1, \\ s_2 &= p_2 + q_2\cos\tilde{\omega}_2, \\ p &= \frac{p_1 - q_1 + p_2}{2} + \frac{q_2}{2}\cos\tilde{\omega}_2, \\ q &= \frac{p_1 - q_1 - p_2}{2} - \frac{q_2}{2}\cos\tilde{\omega}_2, \\ t_1 &= \pm\,q_2\,\frac{z_2}{z_1\,x_2}\sin\tilde{\omega}_2, \qquad t_2 = \pm\,\frac{q_2}{x_2}\sin\tilde{\omega}_2. \end{aligned}\right\}$$

θ_1, θ_2 and $\tilde{\omega}$ are found from (62) and (65), and Z_1, Z_2 by substituting values for s_1, s_2, p in (24). Alternatively, having found s_1, s_2, p as remaining quantities may be found from eqs. (1) — (8).

V. Ellipses with Axes Inclined at a Small Angle

Consider the case of orbits which have their major axes inclined angle; suppose $\tilde{\omega}_1 = 0$, $\tilde{\omega}_2 = \varepsilon$. For the zero order approximation w case of ellipses with axes aligned. In particular suppose the solutio

$$\theta_1 = 0, \qquad \theta_2 = \pi; \qquad s_1 = p_1 + q_1, \qquad s_2 = p_2 - q_2;$$

$$\therefore\; p = \frac{p_1 + q_1 + p_2 - q_2}{2}, \qquad q = \frac{p_1 + q_1 - p_2 + q_2}{2};$$

$$\tilde{\omega} = 0, \text{ supposing } p_1 + q_1 > p_2 - q_2,$$

$$\phi_1 = 0, \qquad \phi_2 = 0;$$

$$Z_1 = -\frac{\sqrt{2}\,(p_1 + q_1 + p_2 - q_2)^{3/2}}{p_1 + q_1 + 3\,p_2 - 3\,q_2}, \qquad Z_2 = -\frac{\sqrt{2}\,(p_1 + q_1 + p_2 - q_2)^{3}}{3\,p_1 + 3\,q_1 + p_2 - q_2}$$

Write

$$\left.\begin{aligned} t_1 &= \tau_1\,\varepsilon, & t_2 &= \tau_2\,\varepsilon, & \tilde{\omega} &= \lambda\,\varepsilon; \\ p &= \bar{p}\,(1 + \omega\,\varepsilon), & & & q &= \bar{q}\,(1 + \nu\,\varepsilon), \\ s_1 &= \bar{s}_1\,(1 + \varrho_1\,\varepsilon), & & & s_2 &= \bar{s}_2\,(1 + \varrho_2\,\varepsilon), \end{aligned}\right\}$$

where $\bar{p}$, $\bar{q}$, $\bar{s}_1$, $\bar{s}_2$, Z_1, Z_2 denote the values given in (77).

Eqs. (1), (2), (5), (6), give to the first order in ε

$$\cos\theta_1 = 1 + \frac{\bar{s}_1\,\varrho_1\,\varepsilon}{q_1}, \qquad \sin\theta_1 = x_1\,\tau_1\,\varepsilon,$$

$$\cos\theta_2 + \varepsilon\sin\theta_2 = -1 + \frac{\bar{s}_2\,\varrho_2\,\varepsilon}{q_2}, \qquad \sin\theta_2 - \varepsilon\cos\theta_2 = x_2\,\tau_2$$

where

$$x_1 = \frac{(\bar{s}_1 - Z_1\,p_1{}^{1/2})}{q_1}, \qquad x_2 = \frac{(\bar{s}_2 - Z_2\,p_2{}^{1/2})}{q_2}.$$

On squaring and adding (79), neglecting terms of the second and hi in ε, we find

$$\varrho_1 = 0, \qquad \varrho_2 = 0.$$

Eqs. (3), (7), (4), (8), now give

$$\bar{q}\,\nu = -\bar{p}\,\omega, \qquad -\bar{q}\,\nu = -\bar{p}\,\omega,$$

i.e.

$$\nu = \omega = 0;$$

also

$$\sin\theta_1 - \lambda\varepsilon\cos\theta_1 = y_1\tau_1\varepsilon, \qquad \sin\theta_2 - \lambda\varepsilon\cos\theta_2 = y_2\tau_2\varepsilon, \tag{83}$$

where

$$y_1 = \frac{\bar{s}_1 - Z_1\bar{p}^{1/2}}{\bar{q}}, \qquad y_2 = \frac{\bar{s}_2 - Z_2\bar{p}^{1/2}}{\bar{q}}. \tag{84}$$

From eq. (9), to the first order in ε, we have

$$z_1\tau_1 = z_2\tau_2, \tag{85}$$

where

$$z_1 = Z_1 - \frac{\bar{s}_1}{Z_1}, \qquad z_2 = Z_2 - \frac{\bar{s}_2}{Z_2}; \tag{86}$$

eliminating λ from (83):

$$\sin(\theta_1 - \theta_2) = (y_1\tau_1\cos\theta_2 - y_2\tau_2\cos\theta_1)\,\varepsilon. \tag{87}$$

Hence using (79) to eliminate θ_1 and θ_2, and neglecting terms $O(\varepsilon^2)$

$$(x_1 - y_1)\,\tau_1 + (x_2 - y_2)\,\tau_2 - 1 = 0. \tag{88}$$

Eqs. (85) and (88) then give

$$\left.\begin{aligned} \tau_1 &= z_2\,[z_1(x_2 - y_2) + z_2(x_1 - y_1)]^{-1}, \\ \tau_2 &= z_1\,[z_1(x_2 - y_2) + z_2(x_1 - y_1)]^{-1}, \end{aligned}\right\} \tag{89}$$

while from (83)

$$x_1\tau_1\varepsilon - \lambda\varepsilon = y_1\tau_1\varepsilon, \tag{90}$$

i.e.

$$\lambda = (x_1 - y_1)\,\tau_1. \tag{91}$$

As $\varrho_1, \varrho_2, \omega$, are all zero, it is clear that the terms of the first order in ε appearing in the expressions for Z_1, Z_2, must also vanish.

To summarise; to the first order in ε, the solution is:

$$\left.\begin{aligned} &\theta_1 = 0, \quad \theta_2 = \pi; \quad s_1 = p_1 + q_1, \quad s_2 = p_2 - q_2; \\ &p = \frac{p_1 + q_1 + p_2 - q_2}{2}, \quad q = \frac{p_1 + q_1 - p_2 + q_2}{2}, \\ &\tilde{\omega} = z_2(x_1 - y_1)\,\varepsilon\,[z_1(x_2 - y_2) + z_2(x_1 - y_1)]^{-1}; \\ &t_1 = z_2\,\varepsilon\,[z_1(x_2 - y_2) + z_2(x_1 - y_1)]^{-1}, \\ &t_2 = z_1\,\varepsilon\,[z_1(x_2 - y_2) + z_2(x_1 - y_1)]^{-1}, \\ &Z_1 = -\frac{\sqrt{2}\,(p_1 + q_1 + p_2 - q_2)^{3/2}}{p_1 + q_1 + 3p_2 - 3q_2}, \quad Z_2 = -\frac{\sqrt{2}\,(p_1 + q_1 + p_2 - q_2)^{3/2}}{3p_1 + 3q_1 + p_2 - q_2}. \end{aligned}\right\} \tag{92}$$

VI. The Attraction of Large Eccentricity

We now return to the case where the initial and terminal ellipses have small eccentricity, in order to exhibit an interesting rule relating to the value taken by $\tilde{\omega}$.

From eqs. (54),

$$\pm\tan\tilde{\omega} = \frac{c\sin\tilde{\omega}_2}{b + c\cos\tilde{\omega}_2}; \tag{93}$$

in section III, where this result is found, we have taken $\tilde{\omega}_1 = 0$; for the following work it is more convenient to abandon this convention, and in consequence the equivalent result is easily seen to be

$$\pm\tan\tilde{\omega} = \frac{b\sin\tilde{\omega}_1 + c\sin\tilde{\omega}_2}{b\cos\tilde{\omega}_1 + c\cos\tilde{\omega}_2}. \tag{94}$$

Now writing e_1, e_2 for the eccentricity of the given orbits i.e.

$$\frac{q_1}{p_1} = e_1, \qquad \frac{q_2}{p_2} = e_2,$$

and using the notation of section III, we have

$$b = \left(\frac{k_1}{l_1}\right) e_1, \qquad c = \left(\frac{k_2}{l_2}\right) e_2.$$

Hence (94) becomes

$$\pm\tan\tilde{\omega} = \frac{P_1 e_1\sin\tilde{\omega}_1 + P_2 e_2\sin\tilde{\omega}_2}{P_1 e_1\cos\tilde{\omega}_1 + P_2 e_2\cos\tilde{\omega}_2},$$

where

$$P_1 = \frac{k_1}{l_1}, \qquad P_2 = \frac{k_2}{l_2}.$$

Reference to eqs. (35), (47), (33), (31) shows that P_1, P_2 are functi
p_1, p_2 only.

Thus, orbits for which $p_1, p_2, \tilde{\omega}_1, \tilde{\omega}_2$ are given, but with q_1, q_2 varying
$\tilde{\omega}$ determined as a function of e_1, e_2. Further we see that if e_1 is sufficiently
so that $P_1 e_1 \ll P_2 e_2$, we have

$$\pm\tan\tilde{\omega} \doteqdot \tan\tilde{\omega}_2$$

i.e.

$$\tilde{\omega} \doteqdot \tilde{\omega}_2 \quad \text{or} \quad \tilde{\omega}_2 + \pi.$$

This result is described as *the rule of attraction of large eccentricity*, i
axis of the transfer orbit is 'attracted' to lie close to that of the more ec
of the initial and terminal orbits. The results obtained in section IV also
this phenomenon, and numerical evidence suggests that is is true generall

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