Transfer Between Circular Orbits

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A solution is given to the problem of transferring a rocket from a circular orbit about one planet into another about a second planet with minimum expenditure of fuel. The planetary orbits are assumed to be coplanar and the longitudes of the planets in their orbits at the instants of departure and arrival of the rocket are supposed to be specified. The case of transfer between the Earth and Mars is taken as a numerical example of the general theory.

1 The Problem

A METHOD of calculating the orbit along which a rocket may be transferred from a circular orbit about one planet into another about a second planet with a minimum fuel expenditure has been described in another paper (1).[2] In addition to specifying the optimal trajectory of transfer, the method also determines the most favorable positions of the planets in their orbits about the Sun at the instant of departure. It may be necessary to delay departure for some considerable time until these favorable positions are attained by both planets. If, however, such a delay is not acceptable, the problem arises of calculating the most satisfactory mode of transfer when the two planets are in given relative positions on the chosen date of departure. This is the problem we shall solve in this paper.

2 Solution to the Problem

We shall restrict ourselves to the two-dimensional problem of transfer between two planets moving in coplanar elliptical orbits about the Sun. Ox, Oy are fixed rectangular Cartesian axes in the plane of the orbits. The method may be generalized to deal with the realistic three-dimensional problem by introducing into the argument a third axis Oz. However, the principles upon which the solution to this more complex problem is based will be identical in character with those which lead to a solution of the more elementary one, and no advantage will be gained, therefore, at this stage, by further complicating the analysis in order to achieve complete generality. Again, absolute accuracy could only be achieved by taking into account the perturbations caused in the planets' motions by their mutual attraction and the attractions of other bodies of the solar system, but the saving in fuel which would result would be negligible by comparison with the inevitable losses which will have to be accepted on account of our inability to navigate the rocket along any prescribed track without error. The most satisfactory procedure will always be to disregard all such small effects when computing an optimal track and then to allow for these, and other errors of navigation, during passage, by small correctional thrusts from the motor applied at various check points along the transfer orbit. A method of computing such corrections has been outlined in (2).

As in (1), we shall suppose that the rocket escapes from its circular orbit about the planet of departure by means of an impulsive thrust from its motor directed along the tangent to this orbit. It then recedes along a hyperbolic trajectory, merging imperceptibly with the elliptical orbit of transfer in which it moves under the influence of the Sun alone. Coming under the influence of the planet of arrival, it falls along a hyperbolic track whose apse is at the level of the circular orbit. Upon arrival at the apse, it is transferred into the circular orbit by means of a second impulsive thrust in a direction opposing its motion, i.e., tangential to the circular orbit. Thus, apart from two short periods of thrust, the motor is inoperative.

If $(-f, -g)$ represent the x and y components of the gravitational field intensity acting upon the rocket when at the point (x, y) at time t, both f and g will be functions of the variables (x, y, t). Let $x = X(t)$, $y = Y(t)$ be the equations defining the optimal trajectory of the rocket. Along this trajectory, we define a vector called *the primer* having components (u, v) satisfying the equations

$$\left.\begin{aligned} \ddot{u} + u\frac{\partial f}{\partial X} + v\frac{\partial g}{\partial X} &= 0 \\ \ddot{v} + u\frac{\partial f}{\partial Y} + v\frac{\partial g}{\partial Y} &= 0 \end{aligned}\right\} \qquad [1]$$

It is shown in (1) and (3) that over an absolute optimal track, all motor thrusts must be impulsive in character and, in addition, the following conditions must be satisfied:

(a) $u^2 + v^2 \leq 1$ at all points.

(b) (u, v) are the direction cosines of the direction of thrust at each junction at which the motor operates.

(c) $(\dot{u}, \dot{v})$ are continuous over such a junction.

(d) $A = uf + vg + \dot{X}\dot{u} + \dot{Y}\dot{v}$ is continuous across a junction.

(e) $A = 0$ over that section of the orbit of transfer where the attraction of the planets is negligible.

The above conditions were obtained on the assumption that we were free to choose the junctions at which impulsive thrusts were to be generated to suit ourselves. In the case under consideration, this is not so, and a reconsideration of the argument of (3) will show that, if the position of any junction is fixed at the outset, condition (c) may be waived at this junction. We shall not, therefore, require this condition to be satisfied at either of the junctions of the present problem.

Over the hyperbolic trajectory of departure from the initial circular orbit, we shall neglect the effect of the Sun's attraction. Let μ_0/r_0^2 be the attraction per unit mass of the planet of departure at a distance r_0 from its center. Let (e_0, l_0) be the eccentricity and semi-latus rectum, respectively, of the hyperbolic orbit. Its polar equation is then

$$\frac{l_0}{r_0} = 1 + e_0 \cos \psi_0 \qquad [2]$$

where ψ_0 is measured from the apse. If (U_0, V_0) are the components of the primer in the directions of the radius vector r_0 and of the perpendicular to it, respectively, it is proved in (4) that the appropriate solution of Equations [1] is

$$U_0 = P_0 \cos \psi_0 + (Q_0 + H_0 l_0)e_0 \sin \psi_0 \qquad [3]$$

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[2] Numbers in parentheses indicate References at end of paper.

$$V_0 = -P_0 \sin \psi_0 + (Q_0 + H_0 I_0)(1 + e_0 \cos \psi_0) + \frac{R_0 - P_0 \sin \psi_0 + H_0 \cot \psi_0}{1 + e_0 \cos \psi_0} \quad [4]$$

where P_0, Q_0, R_0, H_0 are dimensionless constants of integration and

$$I_0 = \frac{1}{2(e_0 - 1)^2} \tan {}^1/_2\psi_0 - \frac{1}{2(e_0 + 1)^2} \cot {}^1/_2 \psi_0 - \frac{6e_0^2}{(e_0^2 - 1)^{5/2}} \tanh^{-1}\left(\sqrt{\frac{e_0 - 1}{e_0 + 1}} \tan {}^1/_2 \psi_0\right) + \frac{e_0^3}{(e_0^2 - 1)^2} \cdot \frac{\sin \psi_0}{1 + e_0 \cos \psi_0} \quad [5]$$

An impulsive thrust is applied at the apse in a direction at right angles to the radius vector. Hence, by condition (b) above, we must have $U_0 = 0$, $V_0 = 1$ at $\psi_0 = 0$. This leads to the conditions

$$P_0 - \frac{e_0}{(e_0 + 1)^2} H_0 = 0 \quad [6]$$

$$Q_0(1 + e_0) + \frac{R_0}{1 + e_0} = 1 \quad [7]$$

It should be noted that $\psi_0 = 0$ makes the expression for I_0 indeterminate. We must accordingly let $\psi_0 \rightarrow 0$ in Equation [5].

Let μ/r^2 be the attraction per unit mass of the Sun at a distance r from its center. Let the polar equation of the transfer trajectory, over which the attractions of the planets are negligible, be

$$\frac{l}{r} = 1 + e \cos \psi \quad [8]$$

where ψ is measured from perihelion. If (U, V) are the components of the primer in directions along and perpendicular to r, respectively, then, as shown in (4), since $A = 0$ (condition (e))

$$U = P \cos \psi + Qe \sin \psi \quad [9]$$

$$V = -P \sin \psi + Q(1 + e \cos \psi) + \frac{R - P \sin \psi}{1 + e \cos \psi} \quad [10]$$

where P, Q, R are constants of integration to be determined later. These equations must be identical with Equations [3] and [4] where the hyperbolic trajectory merges with the transfer orbit. The primer components being solutions of second order differential Equations [1], we can ensure this by equating the values of U_0, V_0 and their first derivatives obtained from Equations [3] and [4] with the corresponding quantities obtained from Equations [9] and [10].

If ϕ_0 is the angle made by the direction of the asymptote of the hyperbolic trajectory of departure with the perpendicular to the radius vector r drawn in the sense of ψ increasing, it is shown in (1) that the components of the time derivative of the primer vector as computed from Equations [9] and [10] are

$$(S \sin \phi_0 + T \cos \phi_0)\dot\psi \text{ along the asymptote} \quad [11]$$

$$(-S \cos \phi_0 + T \sin \phi_0)\dot\psi \text{ perpendicular to the asymptote} \quad [12]$$

where

$$S = \frac{P \sin \psi - R}{1 + e \cos \psi} - Q \quad [13]$$

$$T = \frac{Re \sin \psi - Pe - P \cos \psi}{(1 + e \cos \psi)^2} \quad [14]$$

Consider now the form taken by Equations [3] and [4] at a great distance along the asymptote of the hyperbolic orbit. At such a point, $r_0 \rightarrow \infty$ and hence $\cos \psi_0 = -1/e_0$ and $\sin \psi_0 = (e_0^2 - 1)^{1/2}/e_0$. Thus ψ_0 does not vary with t and hence

$$\dot U_0 = H_0 \dot I_0 e_0 \sin \psi_0 = H_0(e_0^2 - 1)^{1/2}\dot I_0 \quad [15]$$

The first two terms in expression [5] for I_0 are constant and hence make no contribution to $\dot I_0$. The last two terms become larger however, and hence must be dealt with separately. Differentiating these two terms with respect to t, we obtain

$$\frac{e_0^2(e_0^2 - 3 - 2e_0 \cos \psi_0)}{(e_0^2 - 1)^2 (1 + e_0 \cos \psi_0)^2} \dot\psi_0 \quad [16]$$

Since $\cos \psi_0 = -1/e_0$, and in view of Equation [2], this may be written

$$\frac{e_0^2}{(e_0^2 - 1)l_0^2} r_0^2 \dot\psi_0 \quad [17]$$

But $r_0^2 \dot\psi_0$ is the constant velocity moment of the rocket about the center of attraction and is accordingly equal to $(\mu_0 l_0)^{1/2}$. Thus

$$\dot I_0 = \frac{e_0^2 \mu_0^{1/2}}{(e_0^2 - 1)l_0^{3/2}} \quad [18]$$

It now follows that, at a great distance from the planet of departure

$$\dot U_0 = H_0 e_0^2 \sqrt{\frac{\mu_0}{(e_0^2 - 1)l_0^3}} \quad [19]$$

Now consider Equation [4]. It may be shown that as $\cos \psi \rightarrow -1/e_0$, $I_0(1 + e_0 \cos \psi_0) \rightarrow$ a constant. It follows that the only term which makes a nonzero contribution to $\dot V_0$ when r_0 is large is

$$\frac{R_0 - P_0 \sin \psi_0 + H_0 \cot \psi_0}{1 + e_0 \cos \psi_0} \quad [20]$$

Referring to Equation [2], we see that this may be written in the form

$$\frac{1}{l_0}(R_0 - P_0 \sin \psi_0 + H_0 \cot \psi_0)r_0 = \frac{1}{l_0}\left[R_0 - P_0 \frac{(e_0^2 - 1)^{1/2}}{e_0} - H_0 \frac{1}{(e_0^2 - 1)^{1/2}}\right] r_0 \quad [21]$$

when $\psi_0 = \cos^{-1}(-1/e_0)$. Differentiating with respect to the time, we obtain

$$\dot V_0 = \frac{1}{l_0}\left[R_0 - P_0 \frac{(e_0^2 - 1)^{1/2}}{e_0} - H_0 \frac{1}{(e_0^2 - 1)^{1/2}}\right] \dot r_0 \quad [22]$$

But $\dot r_0$ is the velocity of recession from the planet of departure. Hence

$$\dot r_0 = [\mu_0(e_0^2 - 1)/l_0]^{1/2}$$

and

$$\dot V_0 = \frac{\mu_0^{1/2}}{l_0^{3/2}}\left[R_0(e_0^2 - 1)^{1/2} - P_0 \frac{e_0^2 - 1}{e_0} - H_0\right] \quad [23]$$

Equations [19] and [23] give the components of the time derivative of the primer along and perpendicular to the asymptote of the hyperbolic orbit when the rocket has receded to a great distance from the planet of departure. Equating these quantities with the corresponding components given at [11] and [12], we obtain the equations

$$(S \sin \phi_0 + T \cos \phi_0) \frac{\dot\psi l_0^{3/2}}{\mu_0^{1/2}} = \frac{H_0 e_0^2}{(e_0^2 - 1)^{1/2}} \quad [24]$$

$$(-S \cos \phi_0 + T \sin \phi_0) \frac{\dot\psi l_0^{3/2}}{\mu_0^{1/2}} = R_0(e_0^2 - 1)^{1/2} - P_0 \frac{e_0^2 - 1}{e_0} - H_0 \quad [25]$$

But $\mu_0^{1/2}/l_0^{3/2} = (e_0 + 1)^{-3/2} \times$ angular velocity of the rocket in the circular orbit about the planet of departure and hence is large by comparison with $\dot{\psi}$, the angular velocity of the rocket about the Sun at the commencement of the transfer orbit. We shall accordingly approximate Equations [24] and [25] by the equations

$$\frac{H_0 e_0^2}{(e_0^2 - 1)^{1/2}} = R_0(e_0^2 - 1)^{1/2} - P_0 \frac{e_0^2 - 1}{e_0} - H_0 = 0 \quad [26]$$

Solving these latter equations in conjunction with Equations [6] and [7], we obtain

$$P_0 = R_0 = H_0 = 0 \qquad Q_0 = \frac{1}{1 + e_0} \quad [27]$$

Thus, from Equations [3] and [4], at a great distance from the planet of departure

$$U_0 = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \qquad V_0 = 0 \quad [28]$$

This implies that the components of the primer, along and perpendicular to the radius vector r from the Sun, at the commencement of the trajectory of transfer are

$$U = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \sin \phi_0 \qquad V = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \cos \phi_0 \quad [29]$$

Identical results must be given by Equations [9] and [10] and thus

$$P \cos \psi + Qe \sin \psi = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \sin \phi_0 \quad [30]$$

$$-P \sin \psi + Q(1 + e \cos \psi) + \frac{R - P \sin \psi}{1 + e \cos \psi} = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \cos \phi_0 \quad [31]$$

where ψ takes the value appropriate to the commencement of the orbit of transfer.

The transition from the transfer orbit into the hyperbolic orbit of approach is dealt with similarly and yields the conditions

$$P \cos \psi' + Qe \sin \psi' = \sqrt{\frac{e_1 - 1}{e_1 + 1}} \sin \phi_1 \quad [32]$$

$$-P \sin \psi' + Q(1 + e \cos \psi') + \frac{R - P \sin \psi'}{1 + e \cos \psi'} = \sqrt{\frac{e_1 - 1}{e_1 + 1}} \cos \phi_1 \quad [33]$$

ψ' being the value of ψ appropriate to the end of the transfer orbit, ϕ_1 being the angle made by the asymptote of the approach orbit with the perpendicular to the radius vector r, and e_1 being the eccentricity of the hyperbola of approach.

Condition (d), above, may be applied at each junction, but yields equations involving the constants of integration which appear when Equations [1] are integrated over the circular orbits about the planets. Such equations do not limit the trajectory of transfer in any way, but only serve to fix the new constants of integration.

Elimination of the quantities P, Q, R between Equations [30] to [33] now yields the condition to be satisfied by the optimal transfer orbit, viz.

$$\begin{aligned} &E_0 \sin \phi_0 \,[(2 + e \cos \psi) \cos (\psi - \psi') - (2 + e \cos \psi')] \\ &+ E_1 \sin \phi_1 [(2 + e \cos \psi') \cos (\psi - \psi') - (2 + e \cos \psi)] \\ &+ [E_0(1 + e \cos \psi) \cos \phi_0 - E_1(1 + e \cos \psi') \cos \phi_1] \times \\ &\qquad \sin (\psi' - \psi) = 0 \quad [34] \end{aligned}$$

where

$$E_0 = \sqrt{\frac{e_0 - 1}{e_0 + 1}} \qquad E_1 = \sqrt{\frac{e_1 - 1}{e_1 + 1}} \quad [35]$$

Let (r_0, r_1) be the respective distances of the planets of departure and arrival from the Sun at the times of departure or arrival of the rocket, and let (θ_0, θ_1) be the respective longitudes of the planets at these instants. If γ is the longitude of perihelion on the transfer orbit, it then follows that at the terminals of the transfer orbit

$$\psi = \theta_0 - \gamma, \qquad \psi' = \theta_1 - \gamma \quad [36]$$

and hence, from Equation [8], that

$$\frac{l}{r_0} = 1 + e \cos (\theta_0 - \gamma) \quad [37]$$

$$\frac{l}{r_1} = 1 + e \cos (\theta_1 - \gamma) \quad [38]$$

These latter equations permit us to write the condition [34] in the form

$$\frac{E_1}{E_0} = \frac{\dfrac{l}{r_0} \sin (\theta_1 - \theta_0) \cos \phi_0 + \left[\left(1 + \dfrac{l}{r_0}\right) \cos (\theta_1 - \theta_0) - \left(1 + \dfrac{l}{r_1}\right)\right] \sin \phi_0}{\dfrac{l}{r_1} \sin (\theta_1 - \theta_0) \cos \phi_1 - \left[\left(1 + \dfrac{l}{r_1}\right) \cos (\theta_1 - \theta_0) - \left(1 + \dfrac{l}{r_0}\right)\right] \sin \phi_1} \quad [39]$$

If w_0 is the velocity of the rocket in its circular orbit about the planet of departure and relative to this body, its velocity at infinity on the hyperbolic orbit along which it leaves the planet is $w_0(e_0 - 1)^{1/2}$. This velocity must equal the vector difference between the velocity of the planet in its orbit at the time of departure and the velocity of the rocket relative to the Sun at the commencement of the transfer orbit. If λ_0 is the semi-latus rectum, ϵ_0 is the eccentricity, and γ_0 is the longitude of perihelion of the orbit of the planet of departure, it may be shown (5) that this leads to the conditions

$$w_0(e_0 - 1)^{1/2} \cos \phi_0 = \mu^{1/2} (l^{1/2} - \lambda_0^{1/2})/r_0 \quad [40]$$

$$w_0(e_0 - 1)^{1/2} \sin \phi_0 = \mu^{1/2} \left[\frac{e}{l^{1/2}} \sin (\theta_0 - \gamma) - \frac{\epsilon_0}{\lambda_0^{1/2}} \sin (\theta_0 - \gamma_0)\right] \quad [41]$$

Whence, eliminating ϕ_0, we obtain

$$w_0^2 (e_0 - 1) = \mu \left[\frac{e^2}{l} + \frac{\epsilon_0^2}{\lambda_0} - \frac{2e\epsilon_0}{l^{1/2}\lambda_0^{1/2}} \cos (\gamma - \gamma_0) - \left(\frac{1}{l^{1/2}} - \frac{1}{\lambda_0^{1/2}}\right)^2 \left(1 + \frac{2l^{1/2}\lambda_0^{1/}}{r_0}\right)\right] \quad [42]$$

as explained in (5).

The following equations may be found similarly

$$w_1(e_1 - 1)^{1/2} \cos \phi_1 = \mu^{1/2}(\lambda_1^{1/2} - l^{1/2})/r_1 \quad [43]$$

$$w_1(e_1 - 1)^{1/2} \sin \phi_1 = \mu^{1/2} \left[\frac{\epsilon_1}{\lambda_1^{1/2}} \sin (\theta_1 - \gamma_1) - \frac{e}{l^{1/2}} \sin (\theta_1 - \gamma)\right] \quad [44]$$

$$w_1^2 (e_1 - 1) = \mu \left[\frac{e^2}{l} + \frac{\epsilon_1^2}{\lambda_1} - \frac{2e\epsilon_1}{l^{1/2}\lambda^{1/2}} \cos (\gamma - \gamma_1) - \left(\frac{1}{l^{1/2}} - \frac{1}{\lambda_1^{1/2}}\right)^2 \left(1 + \frac{2l^{1/2}\lambda_1^{1/2}}{r_1}\right)_1\right] \quad [45]$$

Eliminating ϕ_0 and ϕ_1 from the condition [39] by the use of Equations [40], [41], [43], and [44], we obtain it in the form

$$\sqrt{\frac{w_0^2\,(e_0 + 1)}{w_1^2\,(e_1 + 1)}} = \frac{L_0}{L_1} \qquad [46]$$

where

$$L_0 = \frac{l}{r_0^2}\,(l^{1/2} - \lambda_0^{1/2}) \sin(\theta_1 - \theta_0) + \left[\left(1 + \frac{l}{r_0}\right)\cos(\theta_1 - \theta_0) - \left(1 + \frac{l}{r_1}\right)\right] \times \left[\frac{e}{l^{1/2}} \sin(\theta_0 - \gamma) - \frac{\epsilon_0}{\lambda_0^{1/2}} \sin(\theta_0 - \gamma_0)\right]$$

$$L_1 = \frac{l}{r_1^2}\,(\lambda_1^{1/2} - l^{1/2}) \sin(\theta_1 - \theta_0) - \left[\left(1 + \frac{l}{r_1}\right)\cos(\theta_1 - \theta_0) - \left(1 + \frac{l}{r_0}\right)\right] \times \left[\frac{\epsilon_1}{\lambda_1^{1/2}} \sin(\theta_1 - \gamma_1) - \frac{e}{l^{1/2}} \sin(\theta_1 - \gamma)\right]$$

Equations [37], [38], [42], [45], and [46] determine the five unknowns e_0, e_1, l, e, γ and hence the optimal mode of transfer. These equations are easily reduced to one condition, since Equations [42], [45] may be employed to eliminate e_0 and e_1 from Equation [46] and then, solving Equations [37], [38] for l and e in terms of γ, these quantities also may be eliminated. We are left with a single equation for γ which must be solved numerically.

3 Transfer Between Circular Orbits

The orbits of the principal bodies of the solar system being very nearly circular, the simplified forms taken by the equations of the last section when it is assumed that the orbits of the planets of departure and arrival are exactly circular are of some importance. In this case, $\epsilon_0 = \epsilon_1 = 0$ and λ_0, λ_1 are the radii of the planetary orbits. Also $r_0 = \lambda_0$, $r_1 = \lambda_1$. Thus Equations [42], [45] can now be written

$$w_0^2\,(e_0 - 1) = \mu\left(\frac{e^2 - 1}{l} + \frac{3}{r_0} - \frac{2l^{1/2}}{r_0^{3/2}}\right) \qquad [47]$$

$$w_1^2\,(e_1 - 1) = \mu\left(\frac{e^2 - 1}{l} + \frac{3}{r_1} - \frac{2l^{1/2}}{r_1^{3/2}}\right) \qquad [48]$$

Condition [46] is then equivalent to

$$\sqrt{\frac{e^2 - 1 + \frac{3l}{r_0} - 2\left(\frac{l}{r_0}\right)^{3/2} + \frac{2w_0^2}{\mu}\,l}{e^2 - 1 + \frac{3l}{r_1} - 2\left(\frac{l}{r_1}\right)^{3/2} + \frac{2w_1^2}{\mu}\,l}} = \frac{G_0}{G_1} \qquad [49]$$

where

$$G_0 = \left[\left(\frac{l}{r_0}\right)^2 - \left(\frac{l}{r_0}\right)^{3/2}\right] \sin(\theta_1 - \theta_0) + \left[\left(1 + \frac{l}{r_0}\right)\cos(\theta_1 - \theta_0) - 1 - \frac{l}{r_1}\right] e \sin(\theta_0 - \gamma)$$

$$G_1 = \left[\left(\frac{l}{r_1}\right)^2 - \left(\frac{l}{r_1}\right)^{3/2}\right] \sin(\theta_0 - \theta_1) + \left[\left(1 + \frac{l}{r_1}\right)\cos(\theta_0 - \theta_1) - 1 - \frac{l}{r_0}\right] e \sin(\theta_1 - \gamma).$$

This latter equation, together with Equations [37] and [38], specify the optimal trajectory.

If the rocket is to be transferred from a circular orbit just outside the Earth's atmosphere into another close to the surface of Mars, the following values must be substituted into Equation [49]

$$r_0 = 1.497 \times 10^{13}, \qquad r_1 = 2.280 \times 10^{13}$$

$$\mu = 1.33 \times 10^{26}, \qquad w_0 = 7.912 \times 10^5,\; w_1 = 3.557 \times 10^5$$

the units being cgs. In this case, we have solved Equation [49] for γ when $(\theta_1 - \theta_0)$ takes the values shown in Table 1. The corresponding values of l and e are also given, together with the net characteristic velocity V (km/sec) of the maneuver and the time of transit T (days). The amount $\Delta\omega$ by which the longitude of Mars exceeds that of the Earth at the instant of departure is shown in the last column.

When $\theta_1 - \theta_0 = 180$ deg, the optimal transfer orbit is tangential to both the planetary orbits. This is the well-known "Hohmann" case and corresponds to an over-all optimum.

For values of $\Delta\omega$ other than those within the range of the table, the characteristic velocity is far too large to be practicable, or more than one complete circuit of the transfer orbit must be made so that the waiting time is greater than that which must elapse to bring Mars into a more favorable position. There will accordingly be periods during which transfer between the Earth and Mars is not practicable. This phenomenon has been discussed by Preston-Thomas (6).

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Table 1

$\theta_1 - \theta_0$	γ	l	e	V	T	$\Delta\omega$
45°	−108.5°	1.259×10^{13}	0.5010	18.58	80	3.2°
90°	−62.5°	1.696×10^{13}	0.2889	9.61	136	18.7°
135°	−30°	1.786×10^{13}	0.2241	6.38	198	31.5°
180°	0°	1.807×10^{13}	0.2076	5.75	259	44.4°
225°	30°	1.786×10^{13}	0.2241	6.38	317	58.8°
270°	62.5°	1.696×10^{13}	0.2889	9.61	366	78.2°
315°	108.5°	1.259×10^{13}	0.5010	18.58	354	129.5°