JET PROPULSION LABORATORY

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To: Distribution

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Subject: A General Non-Linear Theory for Planetary Approach Guidance

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Summary

The purpose of this memo is to calculate the necessary velocity corrections which a free-fall interplanetary space vehicle must perform while moving in the graviational influence of a passing planet if the vehicle is to leave the planet's vicinity on a specified interplanetary trajectory. To achieve the greatest possible generalization we shall assume that the vehicle's approach trajectory is very bad and requires relatively large velocity changes.

I. Introduction

Suppose that an interplanetary mission requires a free-fall vehicle to pass in the vicinity of some planet denoted by P, perform some task and then proceed, using P's gravitational influence, to rendezvous with another planet or space vehicle. Now if the launch trajectory is absolutely perfect, the vehicle could conceivably fly the mission without ever having to alter its path. Since these advanced missions require extremely accurate trajectories, this situation will be highly unlikely.

Let us suppose that our vehicle approaches P on a trajectory very different from the one desired. Now the required departing trajectory can be specified by giving its departing hyperbolic asymptote along with the vehicle's hyperbolic excess velocity V_∞ and the time of closest approach T_{CA}. Although the approach trajectory is assumed to be very bad, we may however assume that its time of closest approach is approximately T_{CA}. Let T_1^* and T_2^* denote the times at which the vehicle enters and leaves the gravitational sphere of influence τ of our planet P. We wish to calculate the required velocity change $\Delta V\ (t^*)$ at some arbitrary time t^* such that $T_1^* < t^* < T_{CA}$. Consider the following figure:

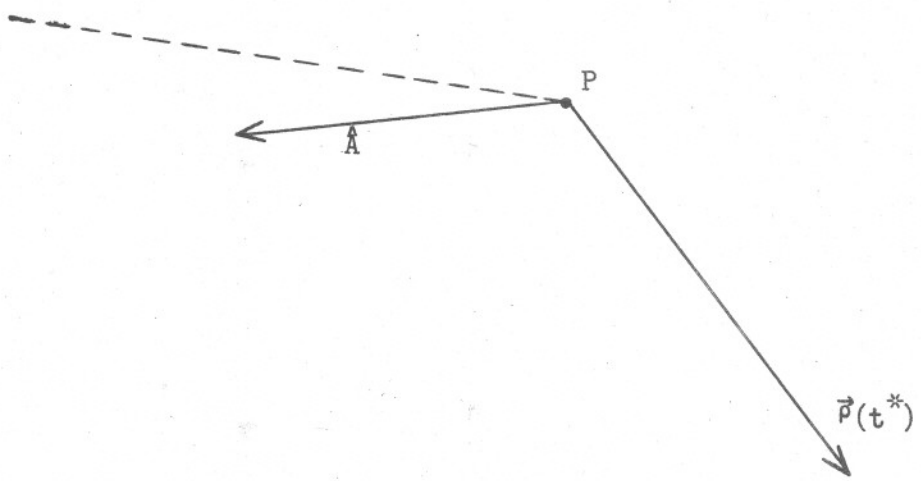


FIGURE 1.

The vector $\vec{\rho}\ (t^*)$ denotes the vehicle's position vector at the time t^*. The vector $\hat{A}$ is a unit vector parallel to the required departing hyperbolic asymptote. The dotted line denotes a line parallel to the unchanged departing hyperbolic asymptote. In general this line will not lie in the plane determined by the vectors $\hat{A}$ and $\vec{\rho}\ (t^*)$.

II The Calculation of $\vec{\Delta V}\ (t^*)$

Let a denote the value of the new semi-major axis of the corrected approach trajectory. This value is independent of t^* and can be calculated by the well known formula:

$$a = \frac{\mu}{V_\infty^2} \qquad (1)$$

where μ = GM of the planet P of mass M; G is the universal gravitational constant. Let us also denote the new trajectories $\vec{e}$ vector by $\vec{e}$. Then if we write

$$\emptyset = \measuredangle\ \hat{A},\ (-\vec{e})$$

we obtain the equation

$$\hat{A} \cdot \vec{e} + 1 = 0 \qquad (2)$$

This equation follows from the fact that

$$\cos \emptyset = \frac{1}{e}$$

Consider the unit vector $\hat{H}$ defined by

$$\hat{H} = \pm \frac{\hat{A} \times \vec{\rho}\,(t^*)}{\left| \hat{A} \times \vec{\rho}\,(t^*) \right|} \qquad (3)$$

where the plus or minus sign is chosen according to whether $0^\circ < \measuredangle \hat{A}, \vec{\rho}\,(t^*) < 180^\circ$ or $180^\circ < \measuredangle \hat{A}, \vec{\rho}\,(t^*) < 360^\circ$ respectively. Thus since $\hat{H}$ is in the direction of the new angular momentum vector $\vec{h}$ we may write

$$\hat{H} \cdot \vec{e} = 0 \qquad (4)$$

The general equation of a conic section of semi-latus rectum ℓ in polar coordinates is

$$\rho = \frac{\ell}{1 + e \cos \theta}$$

Hence, since for hyperbolic conics

$$\ell = a\,(e^2 - 1)$$

we obtain the equation

$$\rho\,(t^*) = \frac{a\,(e^2 - 1)}{1 + \hat{R} \cdot \vec{e}} \qquad (5)$$

where

$$\hat{R} = \frac{\vec{\rho}\,(t^*)}{\rho\,(t^*)}$$

Consequently by writing

$$\hat{A} = (A_1, A_2, A_3)$$

$$\hat{H} = (H_1, H_2, H_3)$$

$$\hat{R} = (R_1, R_2, R_3)$$

the unknown vector

$$\vec{e} = (e_1, e_2, e_3)$$

can be calculated by the three linearly independent equations (2), (4) and (5) which in component form become

$$A_1e_1 + A_2e_2 + A_3e_3 + 1 = 0 \qquad (6)$$

$$H_1e_1 + H_2e_2 + H_3e_3 = 0 \qquad (7)$$

$$R_1e_1 + R_2e_2\ R_3e_3 + \frac{a\,(1 - e^2)}{\rho} + 1 = 0 \qquad (8)$$

where $e^2 = e_1^2 + e_2^2 + e_2^2$

This system in general yields two distinct solutions. In order that one may understand the physical implications of these two solutions let us consider Figure 2.

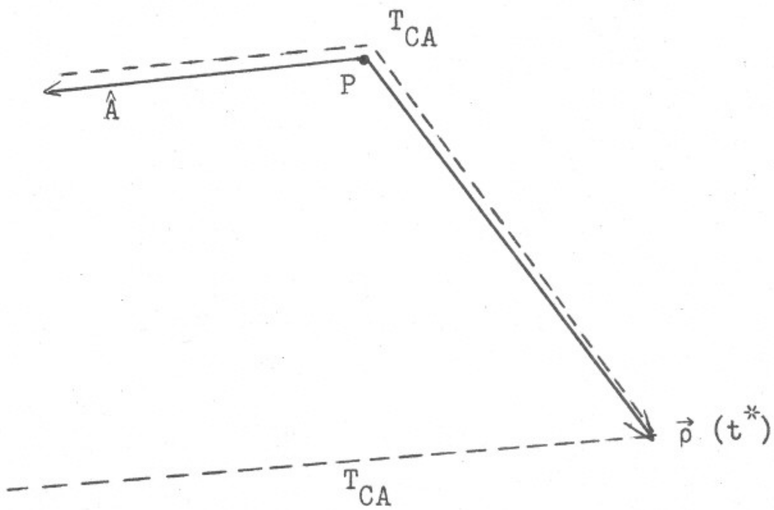


FIGURE 2

The dotted lines represent the trajectories corresponding to the two solutions of the system (6), (7) and (8). Both trajectories have their departing asymptotes parallel to $\hat{A}$ and both have the required departing hyperbolic excess velocity V_∞. The trajectory passing closest to P however, will require a much smaller velocity change. Thus the solution having the smaller value of the eccentricty e is the one chosen. Of course if

$$a\ (e - 1) < \text{Radius of P}$$

the correction could not be carried out at the time t^*. If the unchanged trajectory has a distance of closest approach greater than the planet's radius there will exist a time $T^* < T_{CA}$ such that if $T^* < t^* < T_{CA}$, the new distance of closest approach will be greater than the planet's radius.

We are now in a position to calculate the new angular momentum vector $\vec{h}$. Its magnitude can be easily calculated by the equation

$$h = \sqrt{\mu \, a \, (e^2 - 1)} \tag{9}$$

Consequently h is obtained by

$$\vec{h} = \hat{H} \, h \tag{10}$$

Let $\vec{V}_b(t^*)$ and $\vec{V}_a(t^*)$ denote the vehicle's velocity vector before and after the instantaneous velocity change $\vec{\Delta V}\,(t^*)$ respectively so that

$$\vec{V}_a(t^*) = \vec{V}_b(t^*) + \vec{\Delta V}(t^*)$$

then since

$$\vec{V}_a(t^*) = \frac{1}{\ell}\,\vec{h} \times (\vec{e} + \hat{\rho}\,(t^*))$$

the required velocity change can be calculated by

$$\vec{\Delta V}\,(t^*) = \frac{1}{\ell}\,\vec{h} \times (\vec{e} + \hat{\rho}\,(t^*)) - \vec{V}_b\,(t^*) \tag{11}$$

Let T_b and T_a denote the times of closest approach before and after t^*. Then by employing Kepler's equation (for hyperbolic motion), T_a can be calculated by

$$T_a = K \left[e\sqrt{\chi^2 - 1} - \log\left(\chi + \sqrt{\chi^2 - 1}\right) \right] + t^* \quad (12)$$

where $K = \sqrt{\frac{a^3}{\mu}}$ and $\chi = \left(\frac{\rho(t^*)}{a} + 1 \right) \frac{1}{e}$

A program has been constructed for the IBM 1620 which calculates the required $\overrightarrow{\Delta V}\,(t^*)$ for various values of t^*. Let us now consider the results of calculations corresponding to the following example:

Calculations have shown the existence of a free-fall trajectory of the form

EARTH-VENUS-MARS

with

Launch Date = June 19, 1967, 1200 hours

Earth-Venus flight time = 96.28 days

Distance of closest approach to Venus = 311.0 km

V_∞ at Venus = 6.5575 km/sec

$\hat{A}$ (Venus) = (-.417644, -.832253, -.364438)

thus the Julian date of T_{CA} is

$$T_{CA} = 2439661.00 + 96.28 = 2439757.28$$

Now if the approach trajectory is perfect then at the time

$T_{CA} - 1.0310329$

$$\vec{\rho} = (610514., 6756., 9324.) \text{ (km)} \quad \rho = 610622. \text{ km}$$

$$\hat{V} = (-6.63785, .04232 - .05128) \text{ (km/sec)}$$

Let us suppose that at <u>this</u> time our approach trajectory is

$$\vec{\rho} = (610014., 6256., 9824.) \text{ (km)}$$

$$\vec{V} = (-6.66785, .07232, -.08128) \text{ (km/sec)}$$

consequently at this time we have

error in ρ_1 = 500 km

error in ρ_2 = 500 km

error in ρ_3 = 500 km

error in V_1 = 30 m/sec

error in V_2 = 30 m/sec

error in V_3 = 30 m/sec

error in V = 51.7 m/sec

The hyperbolic excess velocity is 6.5884 km/sec and the resulting departing asymptote has unit vector

$$\hat{A} = (-.523784, -.836194, -.162575)$$

This approach trajectory would cause the vehicle to miss Mercury by many hundreds of thousands of kilometers.

We shall suppose for our purposes that our vehicle can determine its own position and velocity vectors with infinite precision and can impart the exact $\overrightarrow{\Delta V}$ required. Thus only one correction occurs. Table 1 gives the necessary $\overrightarrow{\Delta V}$ corresponding to various values of t^*. In this table we employ the following notation:

ρ = Distance from vehicle to center of P (in km)

ΔV_i = ith component of required velocity change (in m/sec)

$\Delta V = \quad \Delta V_1^2 + \Delta V_2^2 + \Delta V_3^2$ (m/sec)

Energy = (ΔV^2) (in m^2/sec^2)

T_b = Time to closest approach before correction (days)

T_a = Time to closest approach after correction (days)

DOCA = New distance of closest approach

One may think of t^* as being either T_b or T_a. Figure 3 is a graph of Energy vs. T_a and Figure 4 is a graph of Energy vs. ρ. These graphs clearly show that the required energy for the change remains almost constant until the vehicle approaches within 1.5 million kilometers. This corresponds to about 2.6 days to T_{CA}. Consequently the velocity correction should take place before this time. The gravitational sphere of influence about Venus at this time has a radius of about 610,000 km. This distance will be reached when the vehicle is about 1.04 days to closest approach. The required energy to correct the trajectory at this time will be about twice as great than if the correction had taken place 6.8 days before closest approach.

It is interesting to notice that at T_{CA} - 1.0310329 our velocity error (compared to the perfect approach trajectory) is 51.7 m/sec. However, we find from the table that at this time only about 46 m/sec is required to correct the trajectory. This difference is due to the fact that the corrected trajectory is not the perfect trajectory but one designed to yield the same departing asymptote and hyperbolic excess velocity. The resulting difference in the time of closest approach is of the order of a few minutes. If however, it turns out that these few minutes have a great effect on the accuracy of hitting Mercury then an iteration process can be set up to eliminate this discrepancy. This would involve calculating a new $\hat{A}$ vector and new V_∞ corresponding to the corrected trajectories time of closest approach. With respect to these new values of $\hat{A}$ and V_∞ the corrected trajectory must be corrected again but since the new $\hat{A}$ and V_∞ will be nearly the same as the old values very little correction would be necessary. After the second correction the time of closest approach would be different leading another time error but this discrepancy would be for less than the original. This process can be repeated until the discrepancy is smaller than some preassigned value. It is interesting to observe from the table the resulting increasing distances of closest approach resulting from velocity corrections made closer and closer to P.

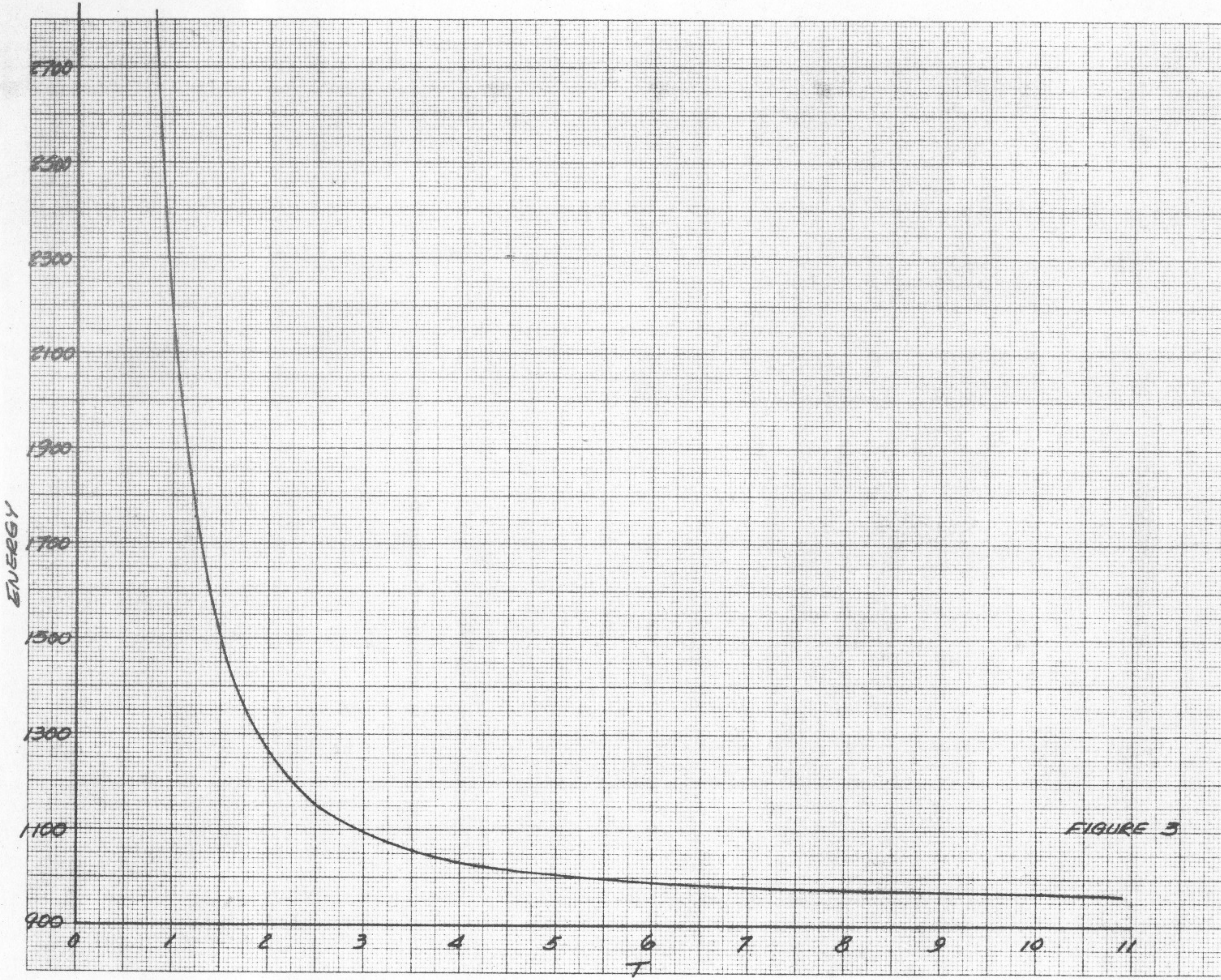
Now in reality no vehicle will be able to determine its own position and velocity vectors with infinite precision. Moreover, even if it could it can hardly be expected that the vehicle could impart the exact $\overrightarrow{\Delta V}$ required. Thus after making the first major correction (taking place before 2.6 days to closest approach) the vehicle would then approximately determine, after perhaps one day of observations, its new trajectory. Another correction would then be made. This could be repeated several times.

TABLE 1

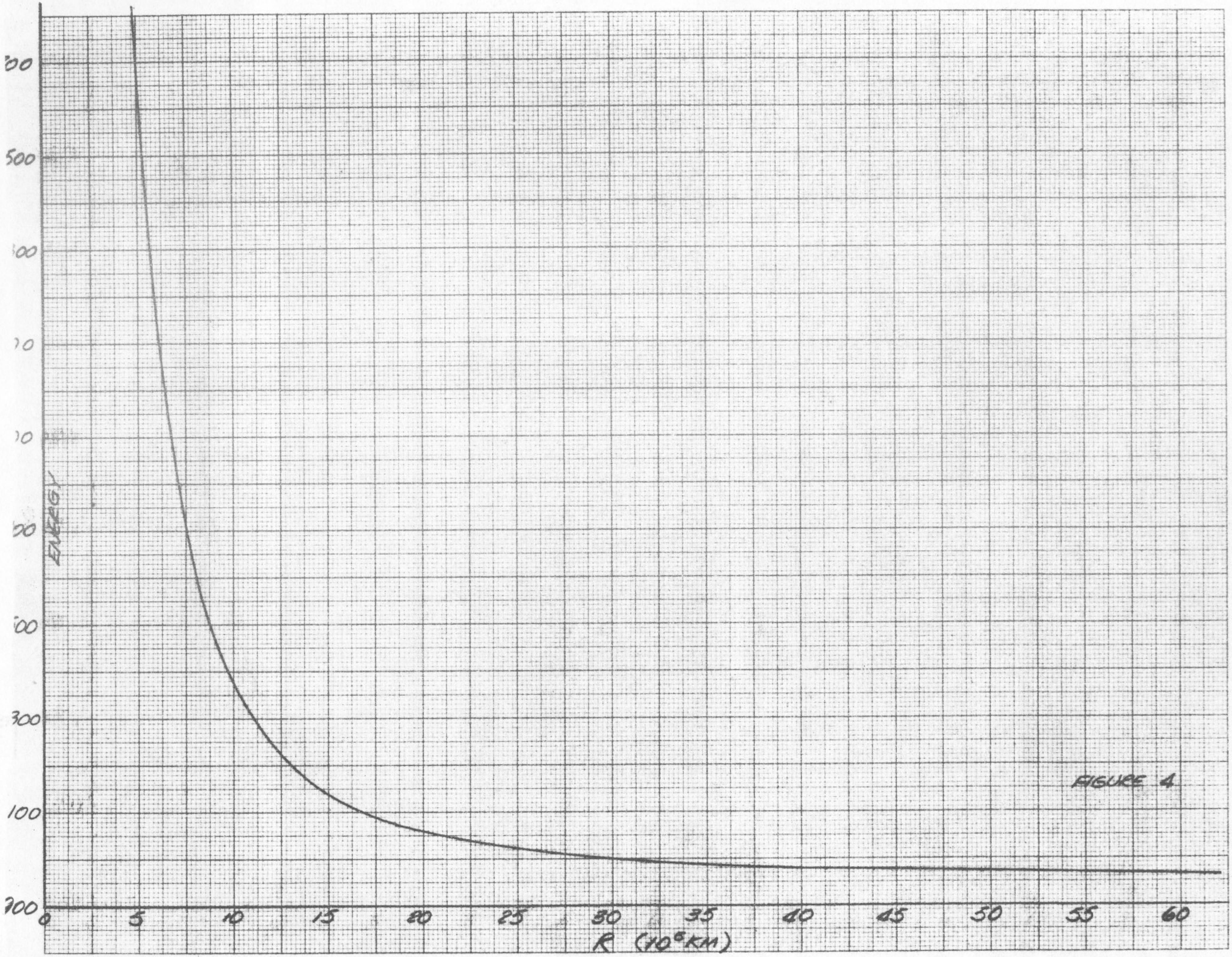
Necessary Velocity Corrections Corresponding to Various Values of t^*

ρ	ΔV_1	ΔV_2	ΔV_3	ΔV	ENERGY	T_b	T_a	DOCA
6215688.	30.786	-2.639	2.742	31.020	962.3	10.8442	10.8935	289.3
5736059.	30.837	-2.924	2.901	31.111	967.9	10.0027	10.0480	289.6
4968865.	30.811	-3.301	3.301	31.162	971.1	8.6568	8.6958	289.9
4657172.	30.825	-3.539	3.480	31.222	974.8	8.1101	8.1465	290.2
4382110.	30.811	-3.730	3.679	31.253	976.7	7.6277	7.6619	290.3
4137726.	30.775	-3.888	3.893	31.263	977.4	7.1991	7.2313	290.4
3918902.	30.752	-4.066	4.098	31.289	979.0	6.8154	6.8458	290.6
3722057.	30.774	-4.299	4.280	31.366	983.8	6.4703	6.4990	290.8
2475208.	30.693	-6.291	6.254	31.949	1020.7	4.2852	4.3037	292.5
1851764.	30.623	-8.315	8.232	32.782	1074.7	3.1938	3.2072	294.3
1477710.	30.569	-10.358	10.221	33.856	1146.2	2.5396	2.5499	296.1
1228335	30.498	-12.393	12.232	35.119	1233.4	2.1039	2.1122	297.9
1050218.	30.419	-14.415	14.268	36.560	1336.7	1.7930	1.7999	299.6
916627.	30.347	-16.456	16.315	38.183	1457.9	1.5601	1.5659	301.3
812731.	30.291	-18.527	18.368	39.977	1598.2	1.3791	1.3841	303.1
729616.	30.226	-20.593	20.443	41.900	1755.6	1.2345	1.2388	304.9
661613.	30.157	-22.662	22.536	43.941	1930.8	1.1163	1.1200	306.6
604947.	30.089	-24.733	24.648	46.093	2124.6	1.0179	1.0211	308.4
557001.	30.034	-26.839	26.765	48.361	2338.7	.9347	.9376	310.2
515906.	29.972	-28.946	28.902	50.710	2571.5	.8635	.8660	311.9
480293.	29.866	-30.987	31.087	53.090	2818.6	.8018	.8041	313.6
449134.	29.842	-33.157	33.239	55.630	3094.7	.7479	.7500	315.4
421641.	29.739	-35.219	35.457	58.154	3381.9	.7004	.7022	317.1
397205.	29.697	-37.381	37.653	60.803	3697.0	.6582	.6599	318.9
375343.	29.667	-39.560	39.863	63.515	4034.2	.6205	.6220	320.7
293379.	29.369	-50.386	51.269	77.652	6029.8	.4795	.4804	329.4
239705.	29.277	-61.698	63.028	92.932	8636.4	.3877	.3882	338.6
201841.	25.724	-68.554	77.325	106.492	11340.4	.3232	.3235	339.2
151984.	28.762	-96.539	101.584	143.061	20466.4	.2389	.2389	364.9
120656.	28.816	-121.141	129.958	180.168	32460.4	.1864	.1862	382.7
99174.	29.112	-147.545	160.838	220.195	48486.0	.1508	.1505	400.4
83549.	29.804	-175.201	194.450	263.428	69394.4	.1251	.1248	418.2
71691.	30.938	-204.481	231.140	310.153	96195.1	.1058	.1055	436.0
62400.	32.601	-235.572	271.275	360.759	130147.2	.0908	.0905	453.7
54938.	34.897	-268.657	315.294	415.698	172804.5	.0789	.0786	471.3
48825.	38.005	-304.026	363.664	475.529	225127.6	.0692	.0689	488.7
43737.	42.074	-341.949	416.961	540.885	292556.4	.0612	.0609	505.9
39445.	47.330	-382.705	475.854	612.515	375174.7	.0544	.0542	523.0
35786.	54.038	-426.684	541.108	691.311	477910.7	.0487	.0485	539.7

MM/vp



ENERGY
2700
2500
2300
2100
1900
1700
1500
1300
1100
900
0
1
2
3
4
5
6
7
8
9
10
11
T
FIGURE 3


ENERGY
R (10⁶ KM)
0
5
10
15
20
25
30
35
40
45
50
55
60
100
300
500
FIGURE 4