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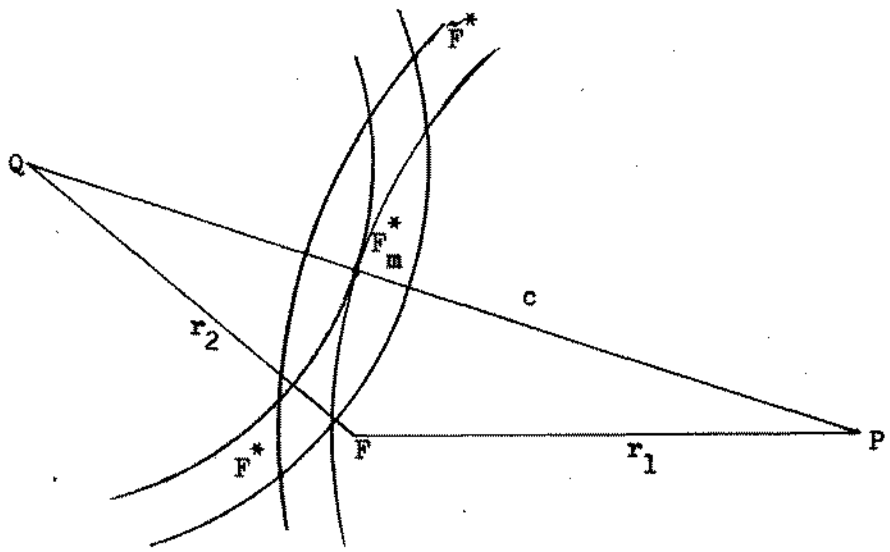
SUBJECT: An Alternative Method for the Determination of Elliptic and Hyperbolic Trajectories

DISTRIBUTION: Section 312 Engineers, J. F. Scott, W. Scholey

A bound particle moving in the gravitational field of a "stationary" celestial object will have an elliptic trajectory. Its period $\underline{P}$ is given by $\underline{P} = 2\pi\sqrt{\frac{a^3}{\mu}}$ where $\mu = GM$; G being the gravitational constant, and M being the mass of the celestial object setting up the field. It is well known that if two points, P and Q, lie on a general conic trajectory, the time required for the particle to traverse the arc $\overset{\frown}{PQ}$ is dependent only on the semi-major axis a of the conic, $\overline{FP} + \overline{FQ}$ where F is a focus of the conic and the distance between P and Q. Let us denote $r_1 = \overline{FP}$, $r_2 = \overline{FQ}$ and $c = \overline{PQ}$. Consider the problem of finding an ellipse passing through two specified points, P and Q, and one specified focus F.



Now the definition of an ellipse can be stated as the locus of points the sum of whose distances from two fixed points (called foci) is constant. We may assume without loss of generality that $r_2 > r_1$. Thus if F^* is the other focus, it must satisfy the equations $\overline{PF^*} + r_1 = \overline{QF^*} + r_2 = 2a$. Consequently, if $\overline{PF^*} = 2a - r_1$ and $\overline{QF^*} = 2a - r_2$, F^* will be a second focus. These points are easily obtained by considering families of circles about P and Q with radii $2a - r_1$ and $2a - r_2$, respectively. The intersections of these families determines a set of pairs of points $(F^*, \widetilde{F}^*)$ each of which can be the second focus. Consequently, there are two different ellipses which satisfy the conditions of the problem. It is clear that if the radii $2a-r_1$, $2a-r_2$ are too small, the circles will not intersect. Hence, there exists a minimum value of a, say a_m, such that the circles intersect in only one point. Since this intersection must occur on $\overline{PQ}$, we have $2a_m - r_1 + 2a_m - r_2 = c$. Thus letting $\frac{1}{2}(r_1 + r_2 + c) = s$, we have $2a_m = \frac{1}{2}(r_1 + r_2 + c) = s$. Since the kinetic energy of the particle at P of unit mass is $\mu(\frac{1}{r} - \frac{1}{2a})$ where $r = \overline{FP}$, it is clear that this will be minimum if $a = a_m$. Thus the unique ellipse, having a semi-major axis of $a = a_m$, can be called a minimum energy ellipse.

It can be shown (see R. Battin, The Determination of Round-Trip Planetary Reconnaissance Trajectories; ARS Journal/Space Sciences; pages 550-52) that when the vacant foci is at F^*, the time T required for the particle to traverse the elliptic arc $\widehat{PQ}$ is

(1) $$T = \frac{1}{2\pi} \underline{P} \left[(\alpha - \sin \alpha) - (\beta - \sin \beta)\right]$$

where $\sin \frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$, $\sin \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$. If the vacant focus is at $\widetilde{F}^*$, the time T can be expressed as

(2) $$\widetilde{T} = \underline{P} - \frac{1}{2\pi} \underline{P} \left[(\alpha - \sin \alpha) + (\beta - \sin \beta) \right]$$

If we set $x_1 = 1 - \frac{s}{a}$, $x_2 = 1 - \frac{s-c}{a}$ and make use of the trigonometric identities

$$\sin \frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 - \cos \theta)} \quad \text{and} \quad \sin(\cos^{-1} x) = \sqrt{1 - x^2}, \quad \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

equations (1) and (2) can be expressed as

(3) $$T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1 - x_2^2} + \sin^{-1} x_2 - \sqrt{1 - x_1^2} - \sin^{-1} x_1 \right\}$$

(4) $$\widetilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1 - x_2^2} + \sin^{-1} x_2 + \sqrt{1 - x_1^2} + \sin^{-1} x_1 \right\}$$

If $a = a_m$ the two ellipses are coincident and $T = \widetilde{T}$. This can be shown analytically by substituting $a = a_m = \frac{1}{2}s$ in (3) and (4) noting that in this case $x_1 = 1 - \frac{s}{\frac{s}{2}} = -1$ and $\sin^{-1}(-1) = -\sin^{-1} 1 = -\frac{\pi}{2}$. Thus it is clear from (3) and (4) that $T \leq \widetilde{T}$ where the equality holds only when $a = a_m$. Let the expressions on the right side of equations (3) and (4) be denoted by $f(a)$ and $\widetilde{f}(a)$, respectively, so that $T = f(a)$ and $\widetilde{T} = \widetilde{f}(a)$. Omitting the details, one easily finds

(5) $$\frac{df}{da} = \frac{3}{2}\frac{f(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

$A = \frac{1-x_1}{1+x_1}$

$B = \frac{1-x_2}{1+x_2}$

(6) $$\frac{d\tilde{f}}{da} = \frac{3}{2}\frac{\tilde{f}(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} + s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

(7) $$\frac{d^2f}{da^2} = \frac{3}{4}\frac{f(a)}{a} + \frac{1}{\sqrt{a^3\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}} + \frac{s}{\sqrt{1-x_1^2}}A - \frac{s-c}{\sqrt{1-x_2^2}}B\right\}$$

Now it is clear on physical grounds that $f(a) > 0$ and $\tilde{f}(a) > 0$. From (5) $f'(a) \to -\infty$ as $a \to a_m$ and from (6) $\tilde{f}'(a) \to +\infty$ as $a \to a_m$. Hence the two curves (T, a) and $(\tilde{T} \cdot a)$ are joined at $a = a_m$ such that the total curve C has a well defined tangent line for all values of a where f and $\tilde{f}$ are defined (i.e., $a_m < a$). In order to simplify an analytical investigation of C let us consider values of a in the closed interval $a_m \leqq a \leqq r_1 + r_2$. Since $a_m = \frac{1}{2}s = \frac{1}{4}(r_1 + r_2 + c)$ and $c \leqq r_1 + r_2$, this interval can be expressed as $\frac{1}{2}(r_1 + r_2) \leqq a \leqq r_1 + r_2$. From equation (6) since $s - c = \frac{1}{2}(r_1 + r_2 - c) \geqq 0$ $\tilde{f}'(a) > 0$. Thus this upper half of C increases with a. For the lower half of C where T is given by f(a) it is convenient to consider $\frac{d^2f}{da^2}$. Consider the expression

(8) $$\frac{s-c}{s} \cdot \frac{\sqrt{\frac{1-x_2}{1+x_2}} - \frac{1}{\sqrt{1-x_2^2}}}{\sqrt{\frac{1-x_1}{1+x_1}} - \frac{1}{\sqrt{1-x_1^2}}}$$

Since $x_1 = 1 - \frac{s}{a}$ and $x_2 = 1 - \frac{s-c}{a}$ we obtain $s = a(1 - x_1)$ and $s - c = a(1 - x_2)$. Thus (8) may be written as

$$\frac{1-x_2}{1-x_1} \cdot \frac{\sqrt{\frac{1-x_2}{1+x_2}} - \frac{1}{\sqrt{1-x_2}\,\sqrt{1+x_2}}}{\sqrt{\frac{1-x_1}{1+x_1}} - \frac{1}{\sqrt{1-x_1}\,\sqrt{1+x_1}}} = \frac{1-x_2}{1-x_1} \cdot \frac{\sqrt{1-x_2} - \frac{1}{\sqrt{1-x_2}}}{\sqrt{1-x_1} - \frac{1}{\sqrt{1-x_1}}} \cdot \frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}$$

(9) $$= \frac{1-x_2}{1-x_1} \cdot \frac{1-x_2-1}{1-x_1-1} \cdot \frac{\sqrt{1-x_1}}{\sqrt{1-x_2}} \cdot \frac{1+x_1}{1+x_2} = \frac{x_2}{x_1} \cdot \frac{1-x_2}{1-x_1} \cdot \frac{\sqrt{1-x_1^2}}{\sqrt{1-x_2^2}}$$

Now $1 - x^2_2 = 1 - (1 - \frac{s-c}{a})^2 = 1 - (x_1 + \frac{c}{a})^2 = 1 - x^2_1 - \frac{c}{a}(2x_1 + \frac{c}{a})$.

$2x_1 + \frac{c}{a} = 2 - \frac{2s}{a} + \frac{c}{a} = 2 - \frac{r_1 + r_2 + c}{a} + \frac{c}{a} = 2 - \frac{r_1 + r_2}{a}$. Hence

$\max(2x_1 + \frac{c}{a}) = 2 - \frac{r_1 + r_2}{r_1 + r_2} = 1, \quad \min(2x_1 + \frac{c}{a}) = 2 - \frac{r_1 + r_2}{\frac{r_1 + r_2}{2}} = 0$

Thus $2x_1 + \frac{c}{a} \geqq 0$ and we conclude that

$$\sqrt{\frac{1 - x_1^2}{1 - x_2^2}} = \sqrt{\frac{1 - x_1^2}{(1-x_1^2) - \frac{c}{a}(2x_1 + \frac{c}{a})}} \geqq 1 \tag{10}$$

From $1 \geqq 2 - \frac{1}{a}(r_1 + r_2) \geqq 0$ we write using above results

$$1 \geqq 2 - \frac{1}{a}(2s - c) = 1 - \frac{s-c}{a} + 1 - \frac{s}{a} = x_2 + x_1$$

$$\therefore \quad x_2 - x_1 \geqq (x_2 + x_1)(x_2 - x_1) = x^2_2 - x^2_1$$

$$\therefore \quad x_2 - x^2_2 \geqq x_1 - x^2_1$$

$$\therefore \quad x_2(1 - x_2) \geqq x_1(1 - x_1)$$

$$\therefore \quad \frac{x_2}{x_1} \cdot \frac{1 - x_2}{1 - x_1} \geq 1$$

With this result and (10) we obtain, since (8) is equal to (9), the important inequality

$$(s - c)\sqrt{\frac{1 - x_2}{1 + x_2}} - (s - c)\frac{1}{\sqrt{1-x^2_2}} \geqq s\,\frac{1-x_1}{1+x_1} - s\,\frac{1}{\sqrt{1 - x^2_1}}$$

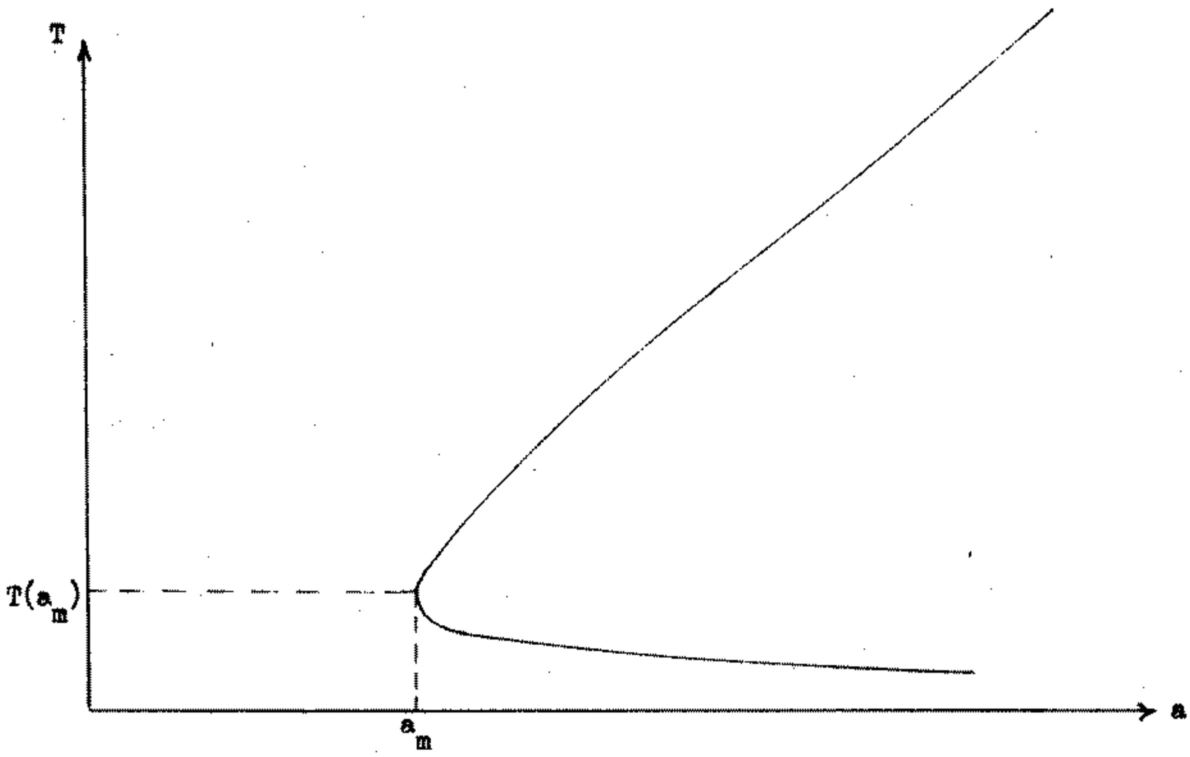
which can be written as

$$\left\{(s - c)\sqrt{\frac{1 - x_2}{1 + x_2}} - s\sqrt{\frac{1 - x_1}{1 + x_1}} + \frac{s}{\sqrt{1-x^2_1}} - \frac{s - c}{\sqrt{1-x^2_2}}\right\} \geqq 0$$

Employing this result in equation (7) we find

$$\frac{d^2f}{da^2} > 0 \qquad a_m \leqq a \leqq r_1 + r_2$$

Since $\frac{df}{da} \rightarrow -\infty$ as $a \rightarrow a_m$ we may now conclude that the lower half of C for values of a in $a_m \leqq a \leqq r_1 + r_2$ will be convex from below. Thus the curve C will take on the general shape of



Suppose a particle is moving in an elliptic trajectory about a gravitating body of mass M (i.e., the particle is in free fall motion but bound in an orbit about the body). If one specifies two points $\vec{r}_1 = \vec{OP}$ and $\vec{r}_2 = \vec{OQ}$, which lies on the trajectory, and the time T taken for the particle to pass from $\vec{r}_1$ to $\vec{r}_2$, one--and only one--trajectory exists which satisfies these conditions, (provided T is greater than some minimum value T_o). We consider two possible cases:

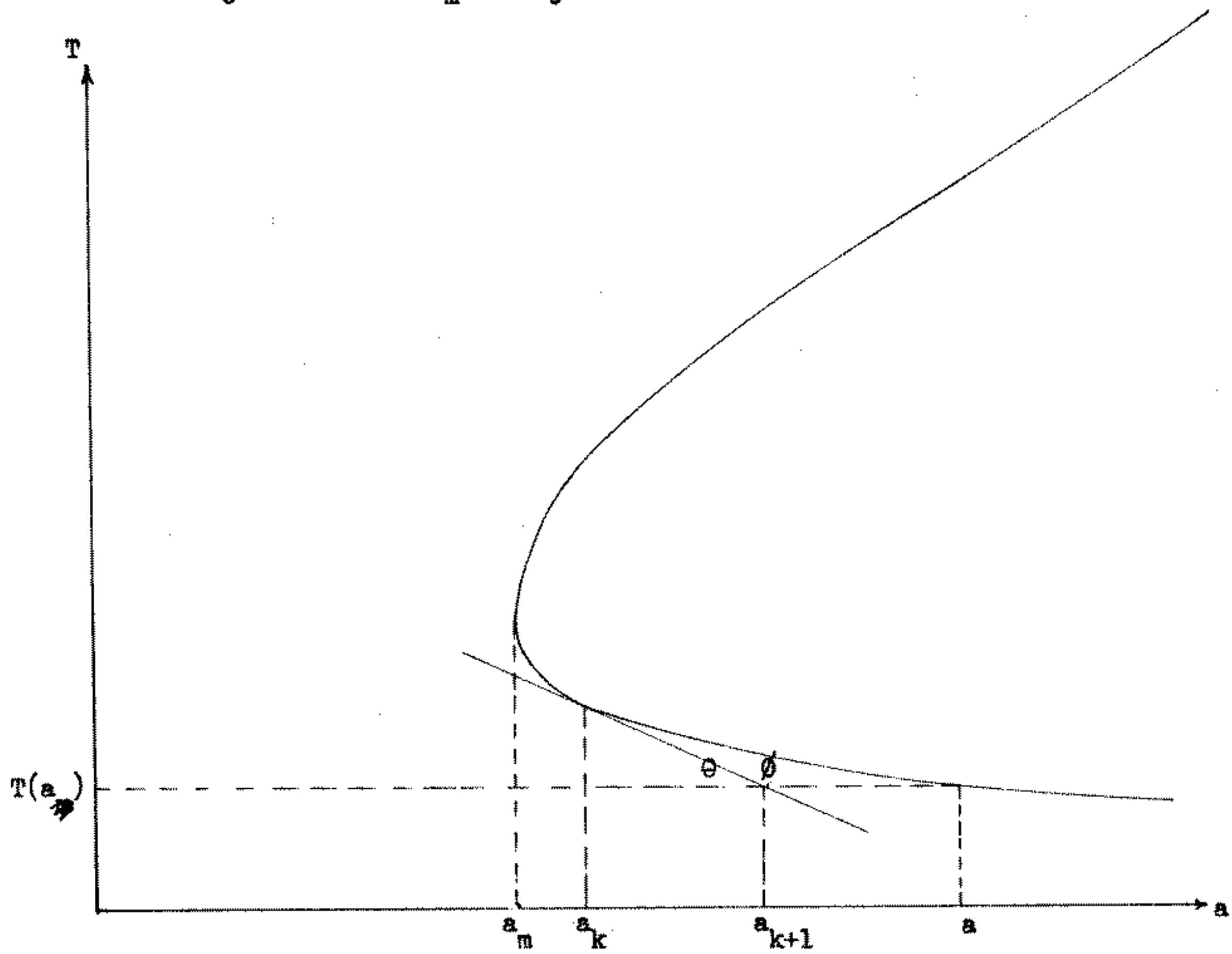
(i) $T < T(a_m)$ (ii) $T > T(a_m)$

If $T < T(a_m)$ we consider two sub-cases:

(a) $T(a_m) > T > T(r_1 + r_2)$

(b) $T(r_1 + r_2) > T$

In practice, it turns out that case (i) is more important since short flight times are desirable. Now as a increases, the kinetic energy of the particle increases, hence the sub-case (b) above will be unlikely. Consequently, we consider a method which, for case (a), will always yield a sequence $\{a_k\}$ converging to the desired value a corresponding to the prescribed flight time T. First, choose an initial value of a, say a_o, such that $T(a_m) > T(a_o) > T$.



By the figure it is evident that

$$\frac{f(a_k) - T}{a_{k+1} - a_k} = \tan\theta = \tan(\pi - \phi) = -\tan\phi = -f'(a_k)$$

Hence

$$\frac{f(a_k) - T}{f'(a_k)} = a_k - a_{k+1}$$

or

$$(11) \qquad a_{k+1} = a_k - \frac{f(a_k) - T}{f'(a_k)}$$

The sequence $\{a_k\}$ will necessarily converge to the desired value a because of the convexity of the lower half of C in the interval $a_m \leqq a \leqq r_1 + r_2$. If case (b) is true, one may still apply (11) if it is found by investigating the sign of f" in a neighborhood of a that the method yielding $\{a_k\}$ will be convergent. In a similar manner, it is easy to see that case ii presents no added difficulty and (11) may also be used to calculate the semi-major axis a by appropriate substitutions.

We now consider the error E_{k+1} in the k+1 th iterate. $E_{k+1} = \left| a - a_{k+1} \right|$. If $\left\{ a_k \right\}$ is convergent to a then the following argument holds for either case (i) or (ii). In dealing with case (ii) one replaces f(a) by $\tilde{f}(a)$. Now by equation (11) we have

$$E_{k+1} = \left| a - a_{k+1} \right| = \left| a - a_k + \frac{f(a_k) - T}{f'(a_k)} \right|$$

But we may write

$$T = f(a) = f(a_k) + (a-a_k)\, f'(a_k) + \frac{1}{2!}(a-a_k)^2 f''(a_k) + \ldots$$

or

$$T = f(a_k) + (a-a_k)\, f'(a_k) + \tfrac{1}{2}(a-a_k)^2 f''(\zeta_k)$$

where ζ_k lies between a_k and a. Hence

$$\frac{T - f(a_k)}{f'(a_k)} = (a-a_k) + \tfrac{1}{2}(a-a_k)^2 \frac{f''(\zeta_k)}{f'(a_k)}$$

Thus

$$E_{k+1} = \left| (a-a_k) - (a-a_k) - \tfrac{1}{2}(a-a_k)^2 \frac{f''(\zeta_k)}{f'(a_k)} \right|$$

$$= \tfrac{1}{2}(a-a_k)^2 \left| \frac{f''(\zeta_k)}{f'(a_k)} \right|$$

Hence since $a_k \longrightarrow a$

(12) $$E_{k+1} \approx \tfrac{1}{2} E_k^2 \left| \frac{f''(a)}{f'(a)} \right|$$

Since $\frac{1}{2}\frac{f''(a)}{f'(a)}$ is a constant, this shows that the error in a_{k+1} is approximately proportional to the square of the error in a_k. Thus we should expect rapid convergence.

After determining the semi-major axis a with sufficient accuracy, the trajectory will be completely determined by finding the corresponding value of the eccentricity ϵ. This is obtained by making use of the dependence of ϵ on a, set up by the initial condition of requiring F to be a focus and P and Q to lie on the ellipse. It can be shown (see above reference, page 549) that if the second focus if F^* or $\tilde{F}^*$ the corresponding values of the latus rectum are given by

$$l = \left[\frac{4a}{c^2}(s - r_1)(s - r_2)\right] \sin^2 \frac{\alpha - \beta}{2}$$

$$\tilde{l} = \left[\frac{4a}{c^2}(s - r_1)(s - r_2)\right] \sin^2 \frac{\alpha - \beta}{2}$$

respectively. Making use of the relation $l = a(1 - \epsilon^2)$ and introducing x_1 and x_2 defined above, these equations can be written as

(13) $$\epsilon = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1x_2 + \sqrt{1-x_1^2}\ \sqrt{1-x_2^2}\right\}^{\frac{1}{2}}$$

(14) $$\tilde{\epsilon} = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1x_2 - \sqrt{1-x_1^2}\ \sqrt{1-x_2^2}\right\}^{\frac{1}{2}}$$

Thus if the given value of T is such that $T > T(a_m)$ then after determining a with sufficient accuracy by (11) with f(a) replaced by $\tilde{f}(a)$, the eccentricity $\tilde{\epsilon}$ of the elliptic orbit is given by (14). If $T < T(a_m)$ one uses (13) after finding a by (11).

Before considering hyperbolic trajectories, it is important to know that a solution of the above problem (having initial conditions F, $\vec{r}_1$, $\vec{r}_2$ and T prescribed) exists. Clearly if T and $\vec{r}_1$, $\vec{r}_2$ are chosen so that $\frac{c}{T}$ is sufficiently large then, since maximum $|\vec{V}| > \frac{T}{c}$ where $\vec{V}$ is the velocity of the particle, the particle may be required to have a kinetic energy such that it cannot be in a bound state. Now since this kinetic energy is given by $\mu(\frac{1}{r} - \frac{1}{2a})$, if $T < T(a_m)$ then as $a \to \infty$ the path $\widehat{PQ}$ is traversed such that $\lim_{a\to\infty} f(a) = T_o$ exists, and that $o < T_o < f(a)$ for all $a_m \leqq a < \infty$. Consequently, if the prescribed value of T is such that $T \leqq T_o$, no elliptic trajectory is possible and the solution of the above problem does not exist. This critical value T_o may be obtained by employing a device known as L'Hospital's rule. This rule for calculating limits states that

$$\lim_{t\to t_o} \frac{F(t)}{G(t)} = \lim_{t\to t_o} \frac{F'(t)}{G'(t)} \quad \text{if } F(t)\to 0 \text{ as } t\to t_o \text{ and } G(t)\to 0 \text{ as } t\to t_o.$$

If we make the change of variable $\frac{1}{t} = a^{\frac{1}{2}}$, f(a) becomes

$$\frac{1}{t^3\sqrt{\mu}}\left\{\sqrt{1 - x_2^2} + \sin^{-1} x_2 - \sqrt{1 - x_1^2} - \sin^{-1} x_1\right\}$$

where $x_1 = 1 - st^2$ and $x_2 = 1 - (s-c)t^2$. Hence as $a \to \infty$, $t \to 0 = t_o$ and we set

$$F(t) = \sqrt{1-x_2^2} + \sin^{-1} x_2 - \sqrt{1-x_1^2} - \sin^{-1} x_1$$

$$G(t) = \sqrt{\mu}\ t^3.$$

Thus
$$\lim_{a\to\infty} T = \lim_{t\to o} \frac{F'(t)}{G'(t)} = \frac{-2\,(s-c)\sqrt{\frac{1-x_2}{1+x_2}} + 2\,s\sqrt{\frac{1-x_1}{1+x_1}}}{3\sqrt{\mu}\ t}$$

By a re-application of the rule we obtain
$$\lim_{a\to\infty} T = \lim_{t\to o} \frac{1}{3\sqrt{\mu}}\left\{\frac{s-c}{1+x_2}\ \frac{dx_2}{dt}\left[\frac{\sqrt{1+x_2}}{\sqrt{1-x_2}} + \frac{\sqrt{1-x_2}}{\sqrt{1+x_2}}\right] - \frac{s}{1+x_1}\ \frac{dx_1}{dt}\left[\frac{\sqrt{1+x_1}}{\sqrt{1-x_1}} + \frac{\sqrt{1-x_1}}{\sqrt{1+x_1}}\right]\right\}$$

$$= \lim_{t\to o}\left\{\frac{-4(s-c)^2\ t}{3\sqrt{\mu}(1+x_2)\sqrt{1-x^2_2}} + \frac{4s^2\ t}{3\sqrt{\mu}(1+x_1)\sqrt{1-x^2_1}}\right\}$$

Now $\lim_{t\to o} x_1 = \lim_{t\to o} x_2 = 1$ hence

$$\lim_{a\to\infty} T = \frac{-2(s-c)^2}{3\sqrt{\mu}}\cdot\lim_{t\to o}\frac{t}{\sqrt{1-x^2_2}} + \frac{2s^2}{3\sqrt{\mu}}\cdot\lim \frac{t}{\sqrt{1-x^2_1}}$$

Let $L_1 = \lim_{t\to o}\dfrac{t}{\sqrt{1-x^2_1}}$. Then

$$L_1 = \lim_{t\to o}\frac{1}{\frac{1}{2}(1-x^2_1)^{-\frac{1}{2}}\,(-2x_1)\,\frac{dx_1}{dt}} = \lim_{t\to o}\frac{\sqrt{1-x^2_1}}{2x_1 st}$$

$$= \frac{1}{2s}\lim_{t\to o}\frac{\sqrt{1-x^2_1}}{t} = \frac{1}{2sL_1}$$

Thus we obtain $L_1 = \dfrac{1}{\sqrt{2s}}$. In a similar manner we find letting

$$L_2 = \lim \frac{t}{\sqrt{1-x^2_2}}$$

$$L_2 = \frac{1}{\sqrt{2(s-c)}}$$

Consequently we obtain
$$T_o = \lim_{a\to\infty} T = \frac{2}{3}\cdot\frac{s^2}{\sqrt{\mu}}\cdot\frac{1}{\sqrt{2s}} - \frac{2}{3}\cdot\frac{(s-c)^2}{\sqrt{\mu}}\cdot\frac{1}{\sqrt{2(s-c)}}$$

$$\therefore\ (15)\quad T_o = \frac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right)$$

Hence if the prescribed T is such that $T \leqq \dfrac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right)$ an elliptical trajectory will be impossible.

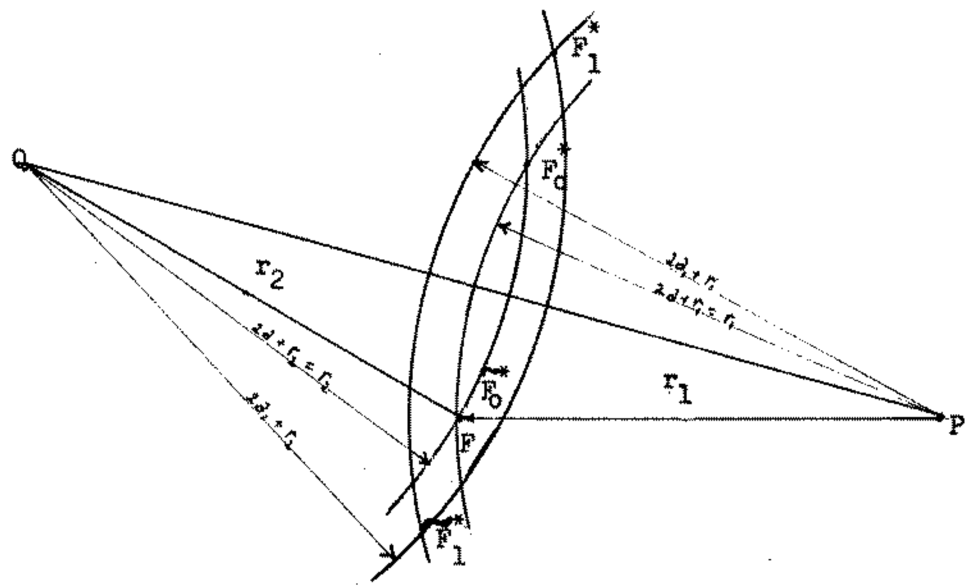
We turn now to the case when the prescribed value of T is such that an elliptic trajectory is impossible. That is to say when $T \leqq T_o$. In this case, we must consider hyperbolic trajectories. The trajectory will be parabolic if $T = T_o$. It will be shown that when the semi-major axis a of a hyperbolic trajectory (with vacant focus F*) increases indefinitely the corresponding time of flight approaches T_o, and the path becomes parabolic. We proceed as before specifying P and Q to lie on the path with $\vec{r}_1 = \overrightarrow{OP}$, $\vec{r}_2 = \overrightarrow{OQ}$ with O the center of an attractive body of mass M which, of course, is at a focus F of the hyperbolic path. Since the field is attractive, P and Q must both lie on the concave branch of the hyperbola with F its nearest focus. Hence if F* is a second focus

$$\overline{PF}^* - r_1 = \overline{QF}^* - r_2 = 2a,$$

according to the definition of a hyperbola which we take to be the locus of points the difference of whose distances from two fixed points (foci) is constant. Thus

$$\overline{PF}^* = 2a + r_1 \qquad \overline{QF}^* = 2a + r_2$$

Hence the vacant foci can be described as the intersection of families of circles about P and Q with radii $2a + r_1$ and $2a + r_2$, respectively.



These circles will intersect in two points $(F_1^*, \tilde{F}_1^*)$. Unlike the elliptic case, the minimum value of $a = 0$. In this case one vacant focus $\tilde{F}_0^*$ coincides with F. The other $\tilde{F}_0^*$ is such that $\overline{PQ}$ bisects $\overline{FF_0^*}$ and hence the path $\widehat{PQ}$ is $\overline{PQ}$ and corresponds to an infinite velocity. The flight time T in this case must, of course, be 0. The path corresponding to $\tilde{F}_0^* = F$ is $\overline{PF}$ to $\overline{FQ}$. These cases, of course, are unrealized. Hence there exists two possible hyperbolic paths having the same semi-major axis a corresponding to the vacant foci F_1^* or $\tilde{F}_1^*$. We observe from the figure that the path with vacant focus at F_1^* has greater eccentricity ϵ than the eccentricity $\tilde{\epsilon}$ of the path with vacant focus at $\tilde{F}^*$. $\epsilon > \tilde{\epsilon}$

The time required to traverse the path $\widehat{PQ}$ when the vacant focus is at $\tilde{F}^*$ or F^* was expressed byBöttin as

$$T = \sqrt{\frac{a^3}{\mu}} \left[(\sinh \alpha - \alpha) - (\sinh \beta - \beta) \right]$$

$$\tilde{T} = \sqrt{\frac{a^2}{\mu}} \left[(\sinh \alpha - \alpha) + (\sinh \beta - \beta) \right]$$

where $\sinh \frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$, $\sinh \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$, corresponding to paths having vacant focus at F^* or $\tilde{F}^*$, respectively. Thus since $\sqrt{\frac{s}{2a}} \geqq 0$ $\sqrt{\frac{s-c}{2a}} \geqq 0$, $\alpha, \beta, \geqq 0$. Also since in this case $\sinh \alpha \geqq \alpha$, $\sinh \beta \geqq \beta$, it is clear that $T \leqq \tilde{T}$, which we expect by observing the figure. Employing the identities, $\sinh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x - 1)}$, $\cosh^2 x - \sinh^2 x = 1$ and $\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$ for $x > 1$, the above expressions can be written as

$$T = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2 \right]$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 + \sqrt{y_2^2 - 1} - \cosh^{-1} y_2 \right]$$

where $y_1 = 1 + \frac{s}{a}$, $y_2 = 1 + \frac{s-c}{a}$. Let the right-hand sides of these equations be denoted by h(a) and $\tilde{h}(a)$, respectively. Thus

$$(16) \quad h(a) = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2 \right] = T$$

(17) $$\tilde{h}(a) = \sqrt{\frac{a^3}{\mu}}\left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 + \sqrt{y_2^2 - 1} - \cosh^{-1} y_2\right] = \tilde{T}$$

Omitting the details, we find

(18) $$\frac{dh(a)}{da} = h'(a) = \frac{3}{2}\cdot\frac{h(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

(19) $$\frac{d\tilde{h}(a)}{da} = \tilde{h}'(a) = \frac{3}{2}\cdot\frac{\tilde{h}(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{-(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

We now consider the limits of the equations as $a \to o$ and $a \to \infty$. In doing this we shall make use of the expansion for $\cosh^{-1} x$,

$$\cosh^{-1} x = \log 2x - \frac{1}{2}\cdot\frac{1}{2x^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4x^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{6x^2} - \ldots \quad (x > 1)$$

$$\lim_{a \to o} h(a) = \lim\left\{a\sqrt{a}\cdot\sqrt{\left(1+\frac{s}{a}\right)^2 - 1}\right\} + \lim\left\{a\sqrt{a}\ (\cosh^{-1} y_2 - \cosh^{-1} y_1)\right\}$$

$$- \lim\left\{a\sqrt{a}\cdot\sqrt{\left(1+\frac{s-c}{a}\right)^2 - 1}\right\}$$

$$= \lim\left\{\sqrt{a}\cdot\sqrt{(a+s)^2 - a^2}\right\} + \lim\left\{a\sqrt{a}\left(\log\frac{y_2}{y_1}\right.\right.$$

$$\left.\left. + \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}\left(\frac{1}{y_1^4} - \frac{1}{y_2^4}\right) + \ldots\right)\right\}$$

$$- \lim\left\{\sqrt{a}\cdot\sqrt{(a+s-c)^2 - a^2}\right\}$$

$$= 0 + \lim\left\{a\sqrt{a}\ \log\frac{y_2}{y_1}\right\} - 0 \quad \text{since}$$

$$y_1 = 1 + \frac{s}{a} \to \infty \text{ and } y_2 = 1 + \frac{s-c}{a} \to \infty \text{ as } a \to o$$

$$\lim_{a \to o}\frac{y_2}{y_1} = \lim_{a \to o}\frac{1+\frac{s-c}{a}}{1+\frac{s}{a}} = \lim\frac{a+s-c}{a+s} = \frac{s-c}{s}.$$ Hence

$$\lim_{a \to o} h(a) = 0 \quad \text{as expected.}$$

From the above results we may write

$$\lim_{a \to o}\tilde{h}(a) = -\lim\left\{a\sqrt{a}\ (\log 2y_1 + \log 2y_2)\right\}$$

$$= -\lim\left\{a\sqrt{a}\ \log 4y_1 y_2\right\}$$

$$= -\lim\left\{a\sqrt{a}\ \log y_1 y_2\right\}$$

Now $\lim y_1 y_2 = \lim\left\{\left(1+\frac{s}{a}\right)\left(1+\frac{s-c}{a}\right)\right\} = \lim \frac{s(s-c)}{a^2}$. Hence

$$\lim_{a\to o} \tilde{h}(a) = -\lim\left\{a\sqrt{a}\ \log\frac{s(s-c)}{a^2}\right\} = \lim\left\{a\sqrt{a}\ \log a^2\right\}$$

$$= 2\lim\left\{\sqrt{a}\ \log a^a\right\}$$ But $a^a \to 1$ as $a \to o$. Consequently

$\lim_{a\to o} \tilde{h}(a) = 0$ as expected.

We now find $\lim_{a\to o} h'(a)$

$$\lim_{a\to o} h'(a) = \frac{3}{2}\lim_{a\to o}\frac{h(a)}{a} + \lim\left\{\frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}\right\}$$

$$= \frac{3}{2}\left\{\lim\left[\sqrt{\frac{a}{\mu}}\left[\sqrt{y_1^2-1} - \cosh^{-1}y_1 - \sqrt{y_2^2-1} + \cosh^{-1}y_2\right]\right]\right\}$$

$$+ \lim\left\{\frac{1}{\sqrt{a\mu}}\left[(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right]\right\}$$

$$= \frac{3}{2}\left[\lim\sqrt{\frac{a}{\mu}}\left(\frac{\sqrt{(a+s)^2-a^2}}{a} - \frac{\sqrt{(a+s-c)^2-a^2}}{a}\right)\right]$$

$$+ \lim\frac{1}{\sqrt{a\mu}}\cdot(s-c-s)$$ since

$\lim_{a\to o}\sqrt{a}\ (\cosh^{-1}y_2 - \cosh^{-1}y_1) = 0$ as shown above and $\frac{y_2-1}{y_2+1}\to 1$,

$\frac{y_1-1}{y_2+1}\to 1$ as $a\to o$

$$\therefore\quad \lim_{a\to o} h'(a) = \lim\left[\frac{\frac{3}{2}}{\sqrt{a\mu}}\ (s-(s-c)) + \frac{1}{\sqrt{a\mu}}\ (s-c-s)\right]$$

$$= \frac{1}{2}\lim_{a\to o}\frac{c}{\sqrt{a\mu}} = +\infty$$

Since $\lim_{a\to o}\tilde{h}(a) = 0$ and $\tilde{h}(a) \leqq \tilde{h}(a)$, $\lim_{a\to o} h'(a) = +\infty$ implies

$$\lim_{a\to o}\tilde{h}'(a) = +\infty$$

We now compute $\lim_{a\to\infty} h(a)$. Let $\frac{1}{t} = a^{\frac{1}{2}}$. Then

$$\lim_{a\to\infty} h(a) = \lim_{t\to o}\left\{\frac{1}{\sqrt{\mu}}\ t^3\left[\sqrt{y_1^2-1} - \cosh^{-1}y_1 - \sqrt{y_2^2-1} + \cosh^{-1}y_2\right]\right\}$$

Employing L'Hospital's rule with

$$F(t) = \sqrt{y_1^2-1} - \cosh^{-1}y_1 - \sqrt{y_2^2-1} + \cosh^{-1}y_2$$

$$G(t) = \mu \sqrt{t^3}$$

$$\lim_{a \to \infty} h(a) = \lim_{t \to o} \frac{\frac{dF}{dt}}{\frac{dG}{dt}} = \lim \frac{\frac{1}{2}(y^2{}_1-1)^{-\frac{1}{2}} \cdot 2y_1 \frac{dy_1}{dt} - \frac{1}{\sqrt{y^2{}_1-1}}\frac{dy_1}{dt} - \frac{1}{2}(y^2{}_2-1)^{-\frac{1}{2}} \cdot 2y_2 \frac{dy_2}{dt}}{3\sqrt{\mu}\ t^2}$$

$$+ \frac{1}{\sqrt{y^2{}_2-1}} \frac{dy_2}{dt}$$

$$= \lim \left(2s\sqrt{\frac{y_1-1}{y_1+1}} - 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}}\right) \frac{1}{3\sqrt{\mu}\ t}$$

$$= \lim \frac{\frac{d}{dt}\left\{ 2s\sqrt{\frac{y_1-1}{y_1+1}} - 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}} \right\}}{\frac{d}{dt}(3\sqrt{\mu}\ t)}$$

$$= \lim_{t \to o} \frac{\frac{s}{y_1+1}\frac{dy_1}{dt}\left\{\sqrt{\frac{y_1+1}{y_1-1}} - \sqrt{\frac{y_1-1}{y_1+1}}\right\} - \frac{s-c}{y_2+1}\frac{dy_2}{dt}\left\{\sqrt{\frac{y_2+1}{y_2-1}} - \sqrt{\frac{y_2-1}{y_2+1}}\right\}}{3\sqrt{\mu}}$$

$$= \lim_{t \to o} \frac{1}{3\sqrt{\mu}} \left\{ \frac{2s^2 t}{y_1+1}\left(\frac{2}{\sqrt{y^2{}_1-1}}\right) - \frac{2(s-c)^2 t}{y_2+1}\left(\frac{2}{\sqrt{y^2{}_2-1}}\right) \right\}$$

$$= \frac{1}{3\sqrt{\mu}} \left\{ 2s^2 \lim_{t \to o} \frac{t}{\sqrt{y^2{}_1-1}} - 2(s-c)^2 \lim_{t \to o} \frac{t}{\sqrt{y^2{}_2-1}} \right\}$$

$$= \frac{1}{3\sqrt{\mu}} \left\{ 2s^2 L_1 - 2(s-c)^2 L_2 \right\}$$

where $L_1 = \lim_{a \to o} \frac{t}{\sqrt{y^2{}_1-1}}$, $L_2 = \lim_{a \to o} \frac{t}{\sqrt{y^2{}_2-1}}$

$$\therefore\ L_1 = \lim \frac{1}{\frac{1}{2}(y^2{}_1-1)^{-\frac{1}{2}} 2y_1 \frac{dy_1}{dt}}$$

$$= \lim \frac{\sqrt{y^2{}_1-1}}{y_1 \cdot 2st} = \frac{1}{2s} \lim \frac{\sqrt{y^2{}_1-1}}{t} = \frac{1}{2s} \cdot \frac{1}{L_1}$$

$$\therefore\ L_1 = \frac{1}{\sqrt{2s}}$$

In a similar manner we find $L_2 = \frac{1}{\sqrt{2(s-c)}}$. Hence we obtain

$$\lim_{a\to\infty} h(a) = \frac{1}{3\sqrt{\mu}}\left\{\frac{2s^2}{\sqrt{2s}} - \frac{2(s-c)^2}{\sqrt{2(s-c)}}\right\}$$

$$= \frac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right) \qquad \text{or}$$

$$\lim_{a\to\infty} h(a) = T_o$$

We now calculate $\lim_{a\to\infty} \tilde{h}(a)$. Let us choose $\frac{1}{t} = a^{\frac{1}{2}}$ so that as $a \to \infty$, $t \to o$ and apply L'Hopital's rule with

$$\tilde{F} = \sqrt{y_1^2-1} - \cosh^{-1} y_1 + \sqrt{y_2^2-1} - \cosh^{-1} y_2$$

$$\tilde{G} = \sqrt{\mu}\; t^3$$

Thus

$$\lim_{a\to\infty} \tilde{h}(a) = \lim_{t\to o} \frac{\tilde{F}}{\tilde{G}} = \lim_{t\to o} \frac{\tilde{F}'}{\tilde{G}'}$$

$$= \lim_{t\to o}\left\{\tfrac{1}{2}(y_1^2-1)2y_1 \frac{dy_1}{dt} - \frac{1}{\sqrt{y_1^2-1}}\frac{dy_1}{dt} + \tfrac{1}{2}(y_2^2-1)^{-\frac{1}{2}} 2y_2 \frac{dy_2}{dt} - \frac{1}{\sqrt{y_2^2-1}}\frac{dy_2}{dt}\right\}\frac{1}{3\sqrt{\mu}\, t^2}$$

$$\lim_{a\to\infty} \tilde{h}(a) = \lim_{t\to o}\left\{\frac{\frac{dy_1}{dt}}{\sqrt{y_1^2-1}}(y_1 - 1) + \frac{\frac{dy_2}{dt}}{\sqrt{y_2^2-1}}(y_2 - 1)\right\}\frac{1}{3\sqrt{\mu}\, t^2}$$

$$= \lim_{t\to o}\left\{2s\sqrt{\frac{y_1-1}{y_1+1}} + 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}}\right\}\frac{1}{3\sqrt{\mu}\, t}$$

$$= \lim_{t\to o}\left\{s\left[\frac{\sqrt{y_1+1}\,(y_1-1)^{-\frac{1}{2}}\frac{dy_1}{dt} - \sqrt{y_1-1}\,(y_1+1)^{-\frac{1}{2}}\frac{dy_1}{dt}}{y_1+1}\right] + (s-c)\left[\frac{\sqrt{y_2+1}\,(y_2-1)^{-\frac{1}{2}}\frac{dy_2}{dt} - \sqrt{y_2-1}\,(y_2+1)^{-\frac{1}{2}}\frac{dy_2}{dt}}{y_2+1}\right]\right\}\frac{1}{3\sqrt{\mu}}$$

$$= \lim_{t\to o}\left\{\frac{s\frac{dy_1}{dt}}{y_1+1}\left(\sqrt{\frac{y_1+1}{y_1-1}} - \sqrt{\frac{y_1-1}{y_1+1}}\right) + \frac{(s-c)\frac{dy_2}{dt}}{y_2+1}\left(\sqrt{\frac{y_2+1}{y_2-1}} - \sqrt{\frac{y_2-1}{y_2+1}}\right)\right\}\frac{1}{3\sqrt{\mu}}$$

$$= \lim_{t\to o}\left\{\frac{2s^2 t}{y_1+1}\left(\frac{2}{\sqrt{y_1^2-1}}\right) + \frac{2(s-c)^2 t}{y_2+1}\left(\frac{2}{\sqrt{y_2^2-1}}\right)\right\}\frac{1}{3\sqrt{\mu}}$$

Now we know $\lim_{t\to o} y_2 = \lim_{t\to o} y_1 = 1$. So, recalling

$$L_1 = \lim_{t\to o} \frac{t}{\sqrt{y_1^2-1}} \quad \text{and} \quad L_2 = \lim_{t\to o} \frac{t}{\sqrt{y_2^2-1}} \quad \text{we obtain}$$

$$\lim_{a\to\infty} \widetilde{h}(a) = \frac{1}{3\sqrt{\mu}} \left\{ 2s^2 L_1 + 2(s-c)^2 L_2 \right\}$$

$$= \frac{2}{3\sqrt{2\mu}} \left(\sqrt{s^3} + \sqrt{(s-c)^3} \right)$$

Let us define $\widetilde{T}_o = \frac{2}{3\sqrt{2\mu}} \left(\sqrt{s^3} + \sqrt{(s-c)^3} \right)$ so that

$$\lim_{a\to\infty} \widetilde{h}(a) = \widetilde{T}_o > T_o$$

Hence we conclude that if a prescribed value of T is such that $T > T_o$ but $T < \widetilde{T}_o$, i.e.,

$$T_o < T < \widetilde{T}_o$$

two trajectories are possible; an elliptical trajectory and a hyperbolic trajectory. Now clearly as the distance between P and Q approaches zero, (i.e., as $c \to o$), one would expect all flight times to correspondingly approach zero. That this is <u>not</u> true can be seen by observing the expression for $\widetilde{T}_o$. We notice that as $c \to o$ $\widetilde{T}_o \to \frac{4}{3\sqrt{2\mu}} \sqrt{s^3}$. But since $s = \frac{1}{2}(r_1 + r_2 + c)$, $s \to \frac{1}{2}(r_1 + r_1) = r_1$. Hence $\widetilde{T}_o \to \frac{4}{3\sqrt{2\mu}} \sqrt{r_1^3}$. This should not be too surprising for, by the above figure, we notice that the hyperbolic path with F^* as vacant focus always passes around F so that when $c \to o$ the path approaches the path from P to F and F back to P. This can also be demonstrated analytically by using the expression for the kinetic energy of our unit mass particle: $\frac{1}{2} V^2 = \frac{\mu}{r}$ yielding $V = \sqrt{\frac{2\mu}{r}}$. Now the time T required for the particle to go from P to F is

$$T = \int_o^{r_1} \frac{ds}{v}$$

$$= \int_o^{r_1} \frac{ds}{\sqrt{\frac{2\mu}{r}}} = \frac{1}{\sqrt{2\mu}} \int_o^{r_1} r^{\frac{1}{2}}\, dr$$

$$= \frac{1}{\sqrt{2\mu}} \frac{2}{3} r^{\frac{1}{2}} \Big|_o^{r_1}$$

$$= \frac{2}{3\sqrt{2\mu}} r^3$$

Thus the time to make the round trip is

$$\frac{4}{3}\,\frac{\sqrt{r^3}}{\sqrt{2\mu}} = \lim \widetilde{T}_o \quad \text{as} \quad c \to o.$$

Notice that $f(a_m) = T(a_m) = f(\frac{s}{2}) = \frac{1}{2}\sqrt{\frac{s^3}{2\mu}}\left\{\sqrt{1-\left(1-\frac{s-c}{\frac{s}{2}}\right)^2}\right.$

$$\left. + \sin^{-1}\left(1-\frac{s-c}{\frac{s}{2}}\right) - \sqrt{1-\left(1-\frac{s}{\frac{s}{2}}\right)^2} - \sin^{-1}\left(1-\frac{2s}{s}\right)\right\}$$

$$T(a_m) = f(a_m) = \frac{1}{2}\sqrt{\frac{s^3}{2\mu}}\left\{\frac{\pi}{2} + \frac{2\sqrt{c}}{s}\sqrt{s-c} + \sin^{-1}\left(-1+\frac{2c}{s}\right)\right\}$$

Hence

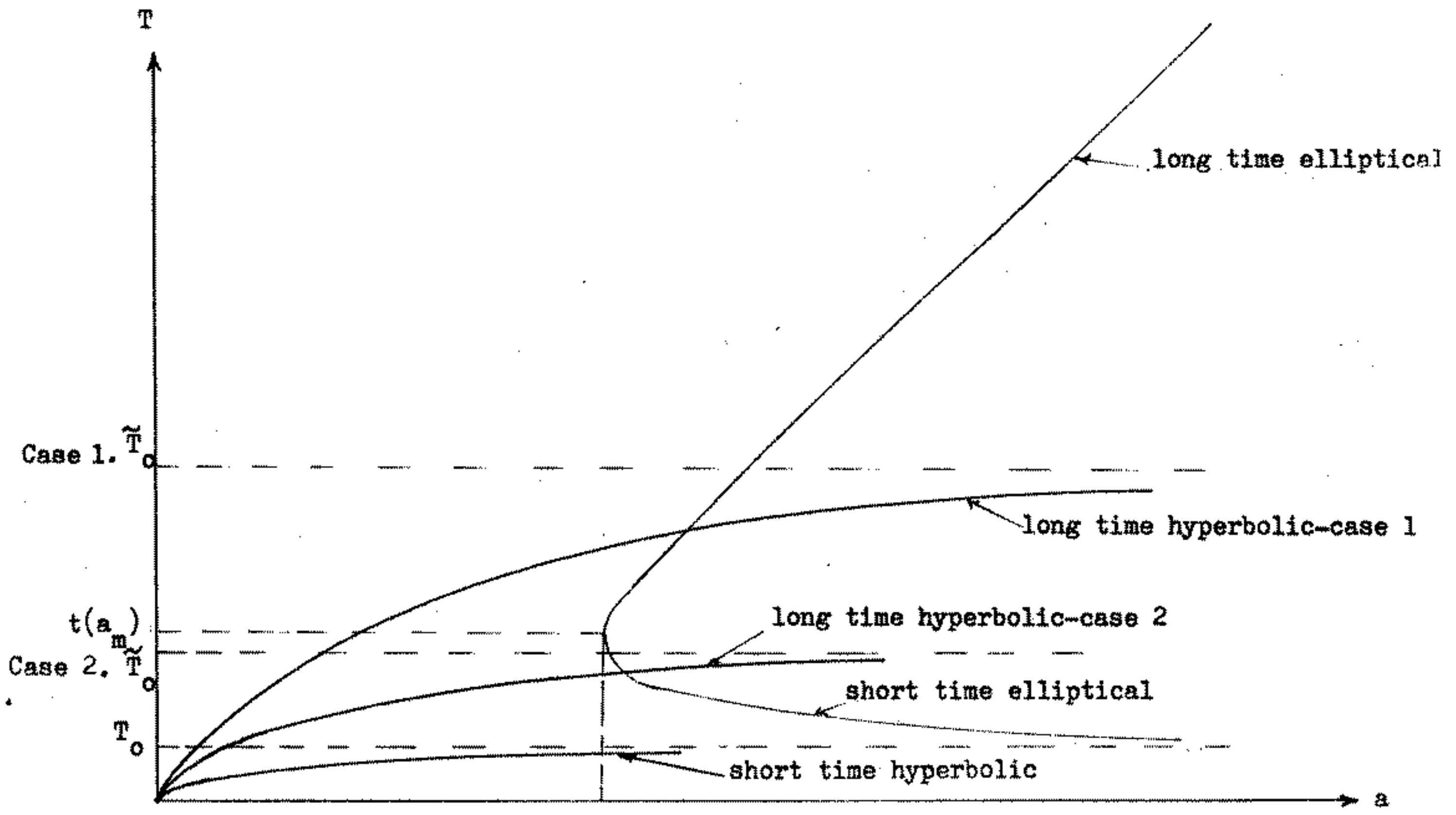
$$T(a_m)\Big|_{c=o} = 0 \quad \text{and} \quad T(a_m)_{c=s} = \frac{\pi}{2}\sqrt{\frac{s^3}{2\mu}}.$$

Now $\widetilde{T}_o\Big|_{c=o} = \frac{4}{3\sqrt{2\mu}}\sqrt{s^3}$ and $\widetilde{T}_o\Big|_{c=s} = \frac{2}{3\sqrt{2\mu}}\sqrt{s^3}$. Thus we may have $T(a_m) > \widetilde{T}_o$ when c is near max. or $T(a_m) < \widetilde{T}_o$ when c is small.

With this information and information concerning f', f", h', and limits as a approaches limiting values, we can construct general shapes of the graph of T vs. a.

"Graph of T vs. a"

Case 1: $T(a_m) < \widetilde{T}_o$ Case 2: $\widetilde{T}_o < T(a_m)$



The first step in making a detailed study of possible conic trajectories associated with prescribed initial values $\vec{r_1}$, $\vec{r_2}$, T should be determining whether case 1, $f(a_m) < \widetilde{T}_o$, or case 2, $\widetilde{T}_o < f(a_m)$, is true so that a general graph may be obtained.

To complete our analysis of hyperbolic trajectories we write an iteration method for obtaining a corresponding to $T < \widetilde{T}_o$

$$a_{k+1} = a_k - \frac{h(a_k) - T}{h'(a_k)}$$

Since $T_o < \widetilde{T}_o$, a second hyperbolic trajectory is possible if $T < T_o$

$$a_{k+1} = a_k - \frac{\widetilde{h}(a_k) - T}{\widetilde{h}'(a_k)}$$

Thus if $T_o < T < \widetilde{T}_o$ two different conic trajectories exist; an elliptical trajectory and a hyperbolic trajectory. For the hyperbolic paths Battin shows that

$$l = \left[\frac{4a}{c^2}(s-r_1)(s-r_2)\sinh^2 \tfrac{1}{2}(\alpha+\beta)\right]$$

$$\widetilde{l} = \left[\frac{4a}{c^2}(s-r_1)(s-r_2)\sinh^2 \tfrac{1}{2}(\alpha-\beta)\right]$$

corresponding to paths with vacant foci F^* and $\widetilde{F}^*$, respectively. Since $l = a(\epsilon^2 - 1)$, these equations yield

$$\epsilon = \left\{1 + \frac{2}{c^2}(s-r_1)(s-r_2)(y_1y_2 + \sqrt{y_1^2-1}\sqrt{y_2^2-1} - 1\right\}^{\frac{1}{2}}$$

$$\widetilde{\epsilon} = \left\{1 + \frac{2}{c^2}(s-r_1)(s-r_2)(y_1y_2 - \sqrt{y_1^2-1}\sqrt{y_2^2-1} - 1\right\}^{\frac{1}{2}}$$

Summary of results for alternative method of determining possible conic paths associated with prescribed values of $\vec{r_1}$, $\vec{r_2}$, M_1 and T where $\vec{r_1} = \vec{FP}$, $r_2 = \vec{FQ}$, M is the mass of the single gravitating body at the focus F and T is the flight time from P to Q.

(i) Calculate $f(a_m)$ and $\widetilde{T}_o$ to determine general graph of T vs. a by: Case 1, $f(a_m) < \widetilde{T}_o$, Case 2, $\widetilde{T}_o < f(a_m)$ (see graph of T vs. a).

(ii) Calculate T_o to determine whether an elliptical path is possible (if $T < T_o$ an elliptical path is impossible.)

(iii) Determine whether an elliptical path and a hyperbolic path are <u>both</u> possible (i.e., if $T_o < T < \widetilde{T}_o$).

(iv) Determine the functions yielding T:

a) if $f(a_m) \leqq T$ use $\tilde{f}(a)$ for elliptic path

b) if $T_o < T \leqq f(a_m)$ use $f(a)$ for elliptic path

c) if $T_o < T \leqq T_o$ then hyperbolic path also exists with T given by $\tilde{h}(a)$

d) if $T < T_o$ only hyperbolic paths exist, $T = h(a)$ for short hyperbolic flight times, $T = h(a)$ for long hyperbolic flight times

(v) Determine a with sufficient accuracy by

$$a_{k+1} = a_k - \frac{F(a) - T}{\frac{dF}{da}} \qquad \left\{a_k\right\} \rightarrow a$$

where $F(a)$ is the function yielding T

error in k'th iterate $= E_k = \left| a - a_k \right|$ has the relation

$$E_{k+1} \approx \tfrac{1}{2} E^2_k \left|\frac{F''(a)}{F'(a)}\right|$$

showing rapid convergence to solution $T(a) = T$

(vi) Determine the eccentricity after obtaining good approximation of a by:

$$\epsilon = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 + \sqrt{1-x^2_1}\sqrt{1-x^2_2})\right\}^{\frac{1}{2}}$$

$$\tilde{\epsilon} = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 - \sqrt{1-x^2_1}\sqrt{1-x^2_2})\right\}^{\frac{1}{2}}$$

for an elliptical path when T is given by $T = f(a)$, $T = \tilde{f}(a)$, respectively.

$$\epsilon = \left\{1 + \frac{2}{c^2}(s - r_1)(s - r_2)(y_1 y_2 + \sqrt{y_1^2-1}\sqrt{y^2_2-1} -1)\right\}^{\frac{1}{2}}$$

$$\tilde{\epsilon} = \left\{1 + \frac{2}{c^2}(s - r_1)(s - r_2)(y_1 y_2 - \sqrt{y^2_1-1}\sqrt{y^2_2-1} -1)\right\}^{\frac{1}{2}}$$

for hyperbolic paths when the prescribed time T is given by $T = h(a)$, and $T = \tilde{h}(a)$, respectively.

(vii) Formulas for above expressions:

$$c = \text{distance from P to Q} = \overline{PQ}$$

$$= \sqrt{r^2_1 + r^2_2 - 2 r_1 r_2 \cos\theta} \qquad \theta = \angle PFQ$$

$$= \sqrt{\vec{r}^2_1 + \vec{r}^2_2 - 2\,\vec{r}_1 \cdot \vec{r}_2}$$

$$s = \frac{r_1 + r_2 + c}{2} \qquad a_m = \frac{s}{2}$$

$$x_1 = 1 - \frac{s}{a} \qquad x_2 = 1 - \frac{s-c}{a}$$

$$y_1 = 1 + \frac{s}{a} \qquad y_2 = 1 + \frac{s-c}{a}$$

$$f(a) = \sqrt{\frac{a^3}{\mu}}\left\{\sqrt{1-x^2{}_2} + \sin^{-1}x_2 - \sqrt{1-x^2{}_1} - \sin^{-1}x_1\right\}$$

$$\tilde{f}(a) = \sqrt{\frac{a^3}{\mu}}\left\{\pi + \sqrt{1-x^2{}_2} + \sin^{-1}x_2 + \sqrt{1-x^2{}_1} + \sin^{-1}x_1\right\}$$

$$\frac{df(a)}{da} = f'(a) = \frac{3}{2}\frac{f(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

$$\frac{d\tilde{f}}{da} = \tilde{f}'(a) = \frac{3}{2}\frac{\tilde{f}(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} + s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

$$h(a) = \sqrt{\frac{a^3}{\mu}}\left\{\sqrt{y^2{}_1-1} - \cosh^{-1}y_1 - \sqrt{y^2{}_2-1} + \cosh^{-1}y_2\right\}$$

$$\tilde{h}(a) = \sqrt{\frac{a^3}{\mu}}\left\{\sqrt{y^2{}_1-1} - \cosh^{-1}y_1 + \sqrt{y^2{}_2-1} - \cosh^{-1}y_2\right\}$$

$$\frac{dh}{da} = h'(a) = \frac{3}{2}\frac{h(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

$$\frac{d\tilde{h}}{da} = \tilde{h}'(a) = \frac{3}{2}\frac{\tilde{h}(a)}{a} - \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} + s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

μ = GM where G is the universal gravitational constant and M is the mass of the body about which the conic trajectory takes place.

It is found convenient to use a year as unit of time and A.U. as unit of distance.

$$T(a_m) = \sqrt{\frac{s^3}{2\mu}}\left\{\sqrt{\frac{c}{s}\left(1-\frac{c}{s}\right)} + \frac{1}{2}\sin^{-1}\left(\frac{2c}{s}-1\right) + \frac{\pi}{4}\right\}$$

$$T_0 = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left\{\sqrt{s^3} - \sqrt{(s-c)^3}\right\}$$

MM:ls