Aug 1962
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CHANGING INTERPLANETARY TRAJECTORIES

BY THE

GRAVITIONAL INFLUENCE OF A PASSING PLANET

ABSTRACT

When an interplanetary space vehicle approaches a planet on a free-Ball trajectory, the gravitional influence of the planet can radically change the trajectory about the Sun. It is possible for such vehicles to take advantage of this influence by passing the planet on a precisely calculated trajectory which will place the vehicle on an intercept trajectory with another planet. Of course, missions to several planets by one free-fall vehicle will, in general, take much more time then flights to one planet but in some cases numerical calculations have shown some remarkably short flight times for missions involving three different planets.

Since conditions favorable for interplanetary flights to particular planets do not occur often, trips to several planets by one vehicle on a free-fall trajectory is particularly attractive. The two most important prerequisites for such missions are: (1) the development of highly accurate and reliable planetary approach guidance systems, and (2) the development of spacecraft components which will give long operating life times to these vehicles.

The determination of free-fall trajectories to several planets is essentially the famous unsolved n-body problem. Thus, in order to make a detailed study of these trajectories certain simplifying assumptions must be made. This paper is based upon one fundamental assumption: At any instant one and only one gravitational field influences the vehicles motion. Under this assumption almost all of the vehicles trajectory will consist of arcs of different ellipses with the Sun at a focus, but when the vehicle comes sufficiently close to a passing planet its trajectory will be hyperbolic with respect to this planet which will

be at one of the new foci.

The paper contains results of a study of such conic trajectories performed at the Jet Propulsion Laboratory during the summer of 1961. The paper also contains numerous numerical results carried out at the Computing Facility of the University of California at Los Angeles and the computing instillation at the Jet Propulsion Laboratory. The numerical study covered simple round-trip trajectories to Venus, Mars, and Jupiter and also missions to Mars via Venus (Earth-Venus-Mars). More complex missions such as Earth-Venus-Mars-Earth were also considered. The program constructed for an IBM 7090 computer to determine these free-fall trajectories has the capability of finding solutions for missions where the vehicle rendezvouses with any number of planets.

I. INTRODUCTION

It has been discovered that conic trajectories give excellent first approximations to actual flight paths of free-fall interplanetary space vehicles. Thus, it is natural, while studying space trajectories, to assume that the vehicle moves along conic trajectories. The primary goal of this paper is to determine a conic trajectory in the vicinity of a passing planet which will enable the vehicle to pass out of its gravitational sphere of influence on a conic trajectory about the Sun which will intercept another pre-determined planet. Thus, we shall assumm that the missions begin and end at the centers of massless planets. The initial conditions are given by specifying the order in which the vehicle is to rendezvous with the given planets along with the launch date and first closest approach date. If these initial conditions are arbitrarily given then a solution may not exist or may be physically unrealizable; that is to say the resulting trajectory may take the vehicle closer to the center of a particular planet then its own radius. Hence, a numerical study of many different missions with varying initial conditions was undertaken to determine the characteristics and requirements of such missions.

No assumptions shall be made regarding the geometry of the solar system; indeed, it will not matter how eccentric the planets orbits are or how much their planes of motion differ from each other. Thus, before attacking the above problem, we shall ~~fair~~ first develop a convenient mathematical technique for handling conic trajectories in three-dimensional space. Since classical astronomy is not particularly concerned with the velocities of celestial bodies, the old method of determining orbits in space by six orbital elements, ~~ω, Ω, i, e, a and T_p~~ the argument of perihelion ω, the longitude of the ascending node Ω, the inclination i, the eccentricity e, the semi-major axis a, and the time of perihelion passage T_p, has been replaced by introducting two orthogonal constant vectors $\vec{e}$ and $\vec{h}$ which together with T_p, determines the orbit. Recasting five of the six orbital parameters in the form of these vectors enables the velocity at ~~A~~ any point on the trajectory with position vector $\vec{R}$ to be easily calculated by a simple formula.

II. CONIC TRAJECTORIES

We begin ~~ever~~ our study of conic trajectories in three-dimensional space by equating the dynamic force on a space vehicle of mass m with the force of gravitional attraction set up by the presents of a body of mass M. If Σ is any inertial frame this equation becomes

$$m \frac{d\vec{V}}{dT} = -G \frac{Mm}{R^2} \hat{R}$$

where $\vec{V}$ is the velocity of the vehicle and $\hat{R}$ is a unit vector directed from the center of the body to the vehicle separated by a distance R. We shall adhere to the convention of denoting unit vectors by placing $\wedge$ over the letter.

Since the ratio m/M is very small we may assume that the body is at rest in Σ. We shall take the center of this body as the orgin of Σ. By setting $GM = \mu$ and cancelling out m from both sides of the above equation we obtain

$$\frac{d\vec{V}}{dT} = -\frac{\mu}{R^2}\hat{R} \tag{1}$$

A. <u>The $\vec{e}$ and $\vec{h}$ Vectors of Conic Trajectories</u>

By the differentation formula for the cross product of two vectors we write

$$\frac{d}{dT}(\vec{R} \times \vec{V}) = \frac{d\vec{R}}{dT} \times \vec{V} + \vec{R} \times \frac{d\vec{V}}{dT}$$

Hence with the add of (1) we have

$$\frac{d}{dT}(\vec{R} \times \vec{V}) = 0$$

since the cross product of paralled vectors vanish. This result implies

$$\vec{R} \times \vec{V} = \int \frac{d}{dT}(\vec{R} \times \vec{V})\, dT + \vec{h} = \vec{h}$$

where $\vec{h}$ is a constant vector of integration.

$$\vec{R} \times \vec{V} = \vec{h} \tag{2}$$

From this important relation we notice that $\vec{R}$ always remains perpendicular to $\vec{h}$ and consequently the motion remains confined to a fixed plane in Σ. The angular momentum of the vehicle about the body is simply $m\vec{h}$.

We now express (2) in a slightly different form

$$\cancel{h = R \times \frac{dR}{dT}}$$

$$\vec{h} = \vec{R} \times \frac{d\vec{R}}{dT}_{\tau} = \vec{R} \times \frac{d}{dT}_{\tau}(R\,\hat{R}) = \vec{R} \times \left(\frac{dR}{dT}_{\tau} \hat{R} + R \frac{d\hat{R}}{dT} \right)$$

Thus

$$\vec{h} = R^2\,\hat{R} \times \frac{d\hat{R}}{dT}_{\tau}$$

Employing this result with (1) we find

$$\frac{d\vec{V}}{dT} \times \vec{h} = -\mu\,\hat{R} \times \left(\hat{R} \times \frac{d\hat{R}}{dT}\right)$$

and often applying the vector triple product formula of vector analysis we have

$$\frac{d\vec{V}}{dT} \times \vec{h} = -\mu \left[\left(\hat{R} \cdot \frac{d\hat{R}}{dT}\right) \hat{R} - (\hat{R} \cdot \hat{R}) \frac{d\hat{R}}{dT} \right]$$

Now since $\hat{R} \cdot \hat{R} = 1$

$$\hat{R} \cdot \frac{d\hat{R}}{dT} = \frac{1}{2} \left(\hat{R} \cdot \frac{d\hat{R}}{dT} + \frac{d\hat{R}}{dT} \cdot \hat{R}\right) = \frac{1}{2} \frac{d}{dT}(\hat{R} \cdot \hat{R}) = 0$$

Consequently

$$\frac{d\vec{V}}{dT} \times \vec{h} = \mu \frac{d\hat{R}}{dT}$$

By noting that $\vec{h}$ is a constant vector and μ is a constant scalor this equation may be written as

$$\frac{d}{dT}(\vec{V} \times \vec{h}) = \frac{d}{dT}(\mu\,\hat{R})$$

whereupon an integration leads to

$$\vec{V} \times \vec{h} = \mu \hat{R} + \vec{c}$$

where $\vec{c}$ is another constant vector of integration. By setting

$$\vec{c} = \mu \vec{e}$$

we obtain

$$\vec{V} \times \vec{h} = \mu (\hat{R} + \vec{e}) \tag{3}$$

where $\vec{e}$ is some constant vector. This vector can be expressed as

$$\vec{e} = \frac{1}{\mu} \vec{V} \times \vec{h} - \hat{R} \tag{4}$$

which follows directly from (3). Since $\vec{V} \times \vec{h}$ is a vector lying in the plane of motion the above relation implies that $\vec{e}$ lies in the plane of motion also.

Let θ be the angle measured from $\vec{e}$ in the positive direction (i.e., counterclockwise) to $\vec{R}$. Hence in view of (2) and (3) we have

$$h^2 = \vec{h} \cdot \vec{h} = \vec{h} \cdot \vec{R} \times \vec{V} = \vec{R} \cdot \vec{V} \times \vec{h} = \vec{R} \cdot \mu (\hat{R} + \bar{e})$$

Thus

$$\frac{h^2}{\mu} = R + R\,e \cos \theta = R\,(1 + e \cos \theta)$$

Consequently we obtain

$$R = \frac{\frac{h^2}{\mu}}{1 + e \cos \theta} \tag{5}$$

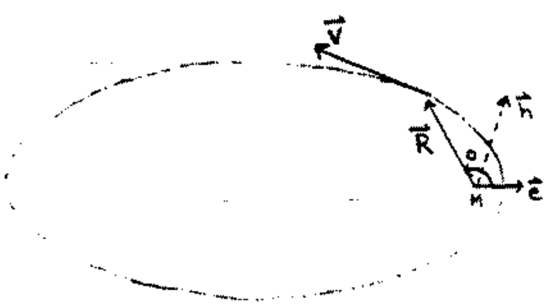
This is the general equation of a conic with eccentricity e and semi-latus rectum

$$\ell_1 = \frac{h^2}{\mu} \tag{6}$$

$$R = \frac{l}{1 + e \cos \theta} \tag{7}$$

Thus we obtain the well-known fact that the trajectory is a conic section.

Since (7) implies that R is smallest when $\theta = 0$, the direction of $\vec{e}$ is along the direction of perihelion.



B. <u>The Calculation of $\vec{e}$ and $\vec{h}$ Vectors</u>

If two position vectors on a conic trajectory are known along with its semi-major axis a and eccentricity e the $\vec{e}$ and $\vec{h}$ vectors can be calculated. The $\vec{h}$ vector can easily be obtained from

$$\vec{h} = \pm \frac{\vec{R}_1 \times \vec{R}_2}{\left|\vec{R}_1 \times \vec{R}_2\right|} \sqrt{a \, \mu \, \left|1-e^2\right|} \tag{8}$$

where the choice of signs depends upon $\measuredangle \; \vec{R}_1, \vec{R}_2$ and the direction of motion. The calculation of the $\vec{e}$ vector can be carried out by the following formula:

$$\vec{e} = \alpha \, \vec{R}_1 + \beta \, \vec{R}_2 \tag{9}$$

where

$$\alpha = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} \qquad \beta = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} \tag{10}$$

with

$$\left\{ \begin{array}{ll} b_i = \dfrac{1}{R_i} + e^2 - 1 & \\ a_{ii} = 1 & \\ a_{ij} = \vec{R}_i \cdot \vec{R}_j + 1 - R_i & i \neq j \end{array} \right\} \tag{11}$$

and

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{12}$$

The derivation of (8) is omitted since it is immediately obvious but (9) along with (10), (11) and (12) is more involved. These formulas can be established by first noting that since the $\vec{e}$ vector lies in the plane of motion, two scalors α and β exist such that (9) holds. If we denote the vehicles velocity vectors at $\vec{R}_1$ and $\vec{R}_2$ by $\vec{V}_1$ and $\vec{V}_2$ respectively and dot each side of (9) by $\frac{1}{\mu} \vec{V}_1 \times \vec{h}$ and $\frac{1}{\mu} \vec{V}_2 \times \vec{h}$ we obtain the following equations

$$\frac{1}{\mu} \vec{V}_1 \times \vec{h}\cdot\vec{e} = \frac{1}{\mu} \vec{V}_1 \times \vec{h}\cdot\vec{R}_1\alpha + \frac{1}{\mu} \vec{V}_1 \times\vec{h}\cdot\vec{R}_2 \beta$$

$$\frac{1}{\mu} \vec{V}_2 \times \vec{h}\cdot\vec{e} = \frac{1}{\mu} \vec{V}_2 \times \vec{h}\cdot\vec{R}_1\alpha + \frac{1}{\mu} \vec{V}_2 \times \vec{h}\cdot\vec{R}_2 \beta$$

By employing (3) and noting that (5) and (6) imply

$$\hat{R} \cdot \vec{e} = \frac{1}{R} - 1$$

the formulas (10), (11) and (12) can easily be seen to follow.

C. Velocity Vector as a Function of $\vec{e}$, $\vec{h}$ and $\vec{R}$

We now come to a very useful formula which expresses the vehicles velocity vector in terms of the e and $\bar{h}$ vectors and its unit position vector $\hat{R}$. The derivation follows by making use of the vector tripal product formula

$$\vec{h} \times (\vec{V} \times \vec{h}) = (\vec{h}\cdot\vec{h})\, \vec{V} - (\vec{h}\cdot\vec{V})\, \vec{h}$$

Thus since $\vec{V}$ is perpendicular to $\vec{h}$ we have

$$h^2 \vec{V} = \vec{h} \times (\vec{V} \times \vec{h})$$

and hence by employing (3) we obtain

$$\vec{V} = \frac{\mu}{h^2} \vec{h} \times (\hat{R} + \vec{e}) \tag{13}$$

As an immediate application of (13) we now derive the well-known energy equation. With the aid (13) together with (6) we may write

$$V^2 = \frac{1}{1} (\vec{V} \cdot \vec{h} \times \hat{R} + \vec{V} \cdot \vec{h} \times \vec{e})$$

By the box product formulas this becomes

$$V^2 = \frac{1}{Rl} \left[\vec{h} \cdot \vec{R} \times \vec{V} + R (\vec{e} \cdot \vec{V} \times \vec{h}) \right]$$

After making use of (2) and (3) this can be written as

$$V^2 = \frac{1}{R\lambda} \left[h^2 + \mu (R e \cos \theta + R e^2) \right]$$

Consequently with the aid of (5) this is expressed as

$$V^2 = \frac{1}{R\lambda} \left[2h^2 + \mu R (e^2 - 1) \right]$$

and by (6) we obtain the famous energy equation

$$v^2 = \mu \left(\frac{2}{R} \mp \frac{1}{a} \right) \tag{14}$$

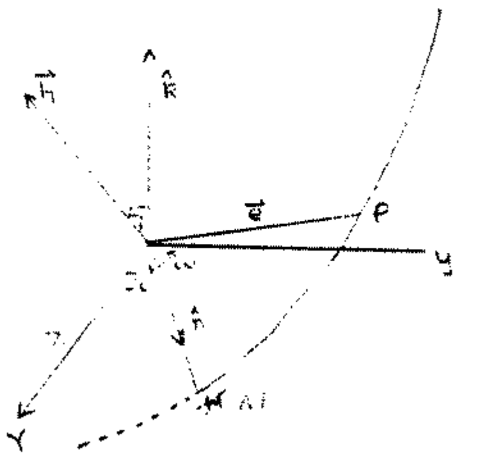
where the negative or positive sign is chosen according to whether the trajectory is elliptic or hyperbolic.

the E and H

D. Relations between $\vec{e}$, $\vec{h}$ Vectors and Classical Orbital Elements

The vectors $\vec{e}$ and $\vec{h}$ along with the time of perhelion passage T_p completely determines the conic trajectory. These vectors represent six constant scalors five of which are independent. To an astronomer, who is primarily interested in knowing where to point his telescope, the old method of defining an orbit by giving its classical orbital elements Ω, i, ω, a, e, T_p is very convenient. But in Astronautics velocity vectors play an impartant roll. The determination of velocity vectors of celestial bodies having orbits defined by osculating elements usually requires slow and cumbersome numerical differentation of position vectors. By defining the trajectories in terms of osculating $\vec{e}$ and $\vec{h}$ vectors, velocity vectors can be immediately calculated by (13).

The classical orbital elements can easilybbe calculated for a trajectory described by $\vec{e}$, $\vec{h}$ and T_p by referring to the following figure where Σ is taken to be an ecliptic coordinate system.



We take H to denote the point where the vehicle rises above the ecliptic which is the x y plane and $\hat{n}$ as a unit vector directed twoard $\overset{M}{\text{H}}$. The point of perihelion is denoted by P. The x axis is directed toward the vernal equinox for some epoch.

Now the time of perihelion passage T_p is already given and $|\vec{e}| = e$. Thus, only four of the six orbital elements remain to be calculated. From (6) the semi-major axis a can be obtained by

$$a = \frac{h^2}{\mu \, |1-e^2|}$$

The inclination i of the trajectory can be calculated from the relation

$$\cos i = \frac{h_3}{h}$$

where we write $\vec{h} = (h_1, h_2, h_3)$ and $\vec{e} = (e_1, e_2, e_3)$. The unit vector $\hat{n}$ is given by

$$\hat{n} = (\cos \Omega, \sin \Omega, 0)$$

Consequently since

$$\hat{k} \times \vec{h} = h \sin i \ \hat{n}$$

The longitude of the ascending node can be obtained by

$$\sin \Omega = \frac{h_1}{h \sin i}$$

The argument of perihelion ω may be calculated from

$$\cos \omega = \frac{\hat{n} \cdot \vec{e}}{e}$$

Similarly one easily finds that if a trajectory is defined by the classical orbital elements, the $\vec{e}$ and $\vec{h}$ vectors can be calculated by the following formulas:

$$e_1 = e(\cos \Omega \cos \omega - \cos i \sin \Omega \sin \phi$$

$$e_2 = e(\sin \Omega \cos \omega + \cos i \cos \Omega \sin \omega)$$

$$e_3 = e \sin i \sin \omega$$

$$h_1 = h \sin i \sin \Omega$$

$$h_2 = -h \sin i \cos \Omega$$

$$h_3 = h \cos i$$

where $$h = \sqrt{\mu a \ |1-e^2|}$$

~~E. Equations Relating Time with Position~~

~~Let π denote the plane of motion.~~

E. Lambert's Theorem

We now come to a fundamental theorem of Celestial mechanics known as Lamberts theorem. This profound result will play an important roll in the determination of interplanetary trajectories. The theorem states that the time required for a body to move from a point P with position vector $\vec{R}_1$ to a point Q with position vector $\vec{R}_2$ depends only on $R_1 + R_2$ and the distance c between P and Q.

To prove Lambert's theorem we shall employ the Hamilton - Jacobi theory of analytical mechanics. This formulation of mechanics is very beautiful and contains many important ideas which were carried over to Quantum mechanics. On the practical side however, little use of the theory can be found. Thus to those who remember the elegant mathematical structure of the Hamilton - Jacobi theory the following application should be warmly received.

We recall that the solution of a problem in analytical mechanics described by the generalized coordinates $q_1, q_2, \ldots, q_m$ can be obtained by simple differentiations of Hamilton's principal function $S\,(q_1 \ldots q_n,\ t,\ \alpha_1 \ldots \alpha_n)$. The arguments $\alpha_1 \ldots \alpha_n$ of S are
and constants depending of the initial conditions. The solution is obtained by solving
slowing the system

$$\beta_i = \frac{\partial S}{\partial \alpha_i} \qquad i = 1, 2, \ldots n$$

of n equations for the generalized coordinates q_i q_i Σ_i

$$q_i = q_i\,(\alpha_1 \ldots \alpha_n,\ \beta_1 \ldots \beta_n,\ t)$$

where β_i are additional constants depending on the initial conditions. The principal function is related to the generalized momenta P_i by P_i

$$\dot{q}_i = P_i = \frac{\partial S}{\partial q_i} \qquad i = 1, \ldots, n$$

P_i

consequently if

$$H(q_1, \ldots, q_n, p_1, \ldots p_n, t)$$

principal

is the Hamiltonian of the system, the original function is obtained by ~~binding~~ finding the solution of the Hamilton - Jacobi equation

$$H\left(q_1, \ldots, q_n, \frac{\partial s}{\partial q_1}, \ldots, \frac{\partial s}{\partial q_n}; \tau\right) + \frac{\partial s}{\partial \tau} = 0 \qquad \text{15 } (\cancel{21})$$

Now in our problem the system is conservative with energy E. Hence the Hamiltonian is time independent. The problem of finding S from 15 (~~22~~) for conservative systems can be simplified by introducing a function W $(q_1 \ldots q_n, \alpha_1, \ldots \alpha_n)$ by assuming

$$S = W - \alpha_1 T \qquad \text{16 } (\cancel{22})$$

which is immediately suggested by 15 (~~22~~). When this is substituted into ~~(22)~~ we obtain ~~commonly~~ the Hamilton-Jacobi equation

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1 + E \qquad (17)$$

if W is a solution of 17 (~~24~~), S can be obtained from 16 (~~23~~).

We now apply the above theory to ~~move~~ prove Lambert's theorem. Define generalized coordinates q_1 and q_2 by

$$q_1 = \frac{1}{2}(R_2 - c) \qquad\qquad q_2 = \frac{1}{2}(R_2 + c)$$

Omitting the calculations, the Hamiltonian function becomes

$$\frac{1}{2}\left\{\frac{q_1^2 - \frac{R_1^2}{4}}{q_1^2 - q_2^2}\, p_1^2 - \frac{q_2^2 - \frac{R_1^2}{4}}{q_1^2 - q_2^2}\, p_2^2\right\} - \frac{\mu}{q_1 + q_2}$$

Hamilton's characteristic function W can then be obtained by

$$\frac{1}{2}\left\{\frac{\left(q_1-\frac{R_1}{2}\right)\left(q_1+\frac{R_1}{2}\right)}{\left(q_1-q_2\right)\left(q_1+q_2\right)}\left(\frac{\partial W}{\partial q_1}\right)^2-\frac{\left(q_2-\frac{R_1}{2}\right)\left(q_2+\frac{R_1}{2}\right)}{\left(q_1-q_2\right)\left(q_1+q_2\right)}\left(\frac{\partial W}{\partial q_2}\right)^2\right\}=E.$$

it is easy to see that this equation will be satisfied if

$$\left(\frac{\partial W}{\partial q_1}\right)^2=2\left(\frac{\mu}{q_1+\frac{R_1}{2}}+E\right)$$

$$\left(\frac{\partial W}{\partial q_2}\right)^2=2\left(\frac{\mu}{q_2+\frac{R_1}{2}}+E\right)$$

These equations imply

$$W=\pm\sqrt{2}\int_{q_1=\frac{1}{2}R_1}^{q_1=\frac{1}{2}\left(R_2-c\right)}\sqrt{\frac{\mu}{q_1+\frac{R_1}{2}}+E}\;dq_1+f\left(q_2\right)$$

or

$$W=\pm\sqrt{2}\int_{q_2=\frac{1}{2}R_1}^{q_2=\frac{1}{2}\left(R_2+c\right)}\sqrt{\frac{\mu}{q_2+\frac{R_1}{2}}+E}\;dq_2+g\left(q_1\right)$$

Thus we obtain

$$W=\pm\sqrt{2}\int_{\frac{1}{2}r_1}^{\frac{1}{2}\left(R_2-c\right)}\sqrt{\frac{\mu}{q_1+\frac{R_1}{2}}+E}\;dq_1\;\pm\sqrt{2}\int_{\frac{1}{2}R_1}^{\frac{1}{2}\left(R_1+c\right)}\sqrt{\frac{\mu}{q_2+\frac{R_1}{2}}+E}\;dq_2$$

Since W represents the length of a geodesic in a two-dimensional Riemannian manifold where the motion takes place along geodesocs, W must be positive for arbitrary values of R_1, R_2 and c. With this restriction W is given by

$$W = \sqrt{2}\int_{\frac{1}{2}(R_2-c)}^{\frac{1}{2}R_1}\sqrt{\frac{\mu}{q_1+\frac{R_1}{2}} + E}\; dq_1 + \sqrt{2}\int_{\frac{1}{2}R_1}^{\frac{1}{2}(R_2+c)}\sqrt{\frac{\mu}{q_2+\frac{R_1}{2}} + E}\; dq_2$$

which can be combined to yield

$$W = \sqrt{2}\int_{\frac{1}{2}(R_2-c)}^{\frac{1}{2}(R_2+c)}\sqrt{\frac{\mu}{\xi+\frac{R_1}{2}} + E}\; d\xi$$

η

If a new variable of integration ζ is introduced by letting

$$\zeta = \xi + \frac{R_1}{2}$$

we obtain

$$W = \sqrt{2}\int_{\frac{1}{2}(R_1+R_2-c)}^{\frac{1}{2}(R_1+R_2+c)}\sqrt{\frac{\mu}{\zeta} + E}\; d\xi \qquad (24)$$

The time variable T which appears in (23) is the time interval between the initial time T_1 and any time T_2 later. Thus since

$$\beta_1 = T_1 = \frac{\partial s}{\partial e} \quad E$$

(23) implies

$$T = \frac{\partial W}{\partial E}$$

The beautiful simplicity of the proof now becomes evident. In view of (18) we write

$$T = 2\int_{\frac{1}{2}(R_1+R_2-c)}^{\frac{1}{2}(R_1+R_2+c)} \frac{\partial}{\partial E}\sqrt{\frac{\mu}{\zeta}+E}^{\,\frac{3}{2}}\, d\zeta$$

which becomes

$$T = \frac{1}{\sqrt{2}}\int_{s-c}^{s} \frac{\zeta\, d\zeta}{\sqrt{E\zeta^2+\mu\zeta}} \qquad \text{(19)}$$

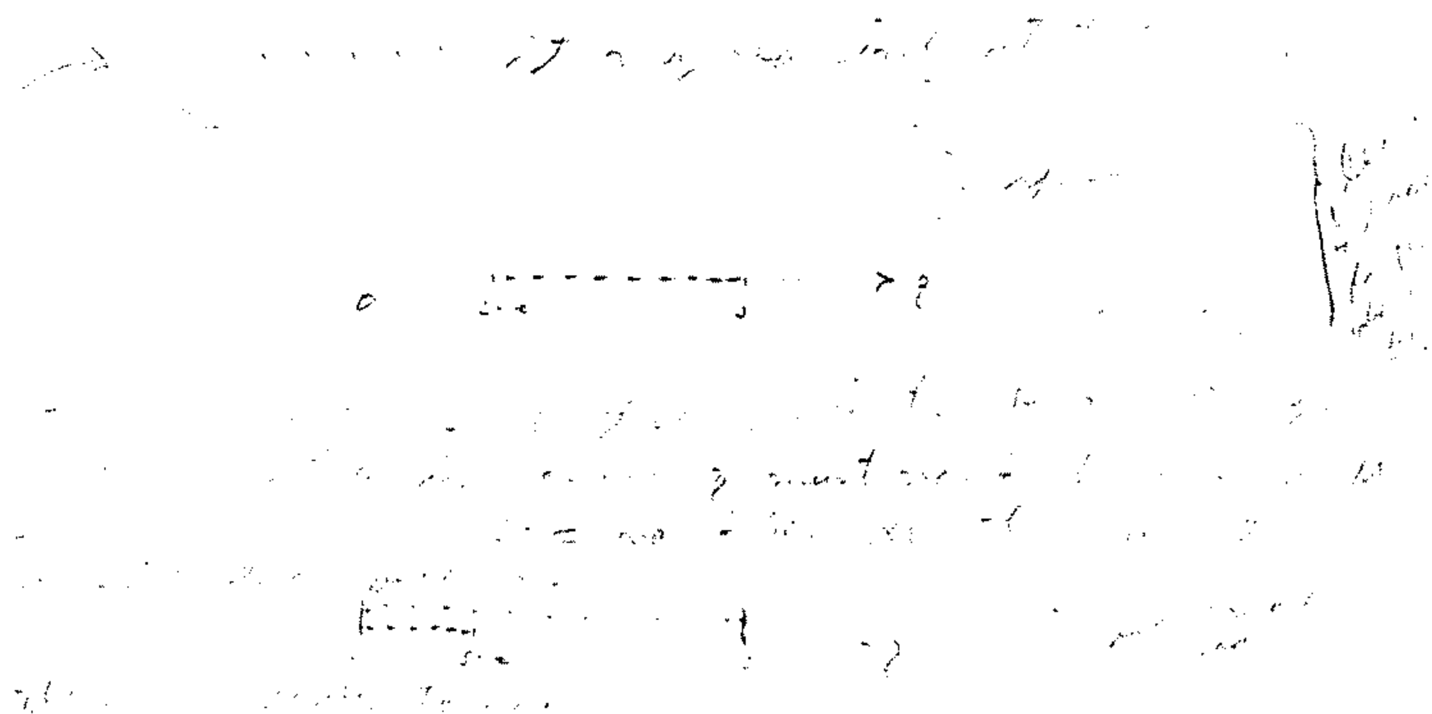
where s is the semi-perimeter of the triangle FPQ where F denotes the principal focus

$$S = \frac{1}{2}(R_1 + R_2 + c)$$

Now the energy E equals the kinetic energy $\frac{1}{2}V^2$ plus the potential energy $-\frac{\mu}{R}$. Thus in view of the energy equation (14)

$$E = \mp \frac{\mu}{2a}$$

where the negative sign corresponds to elliptic orbits and the positive sign with hyperbolic orbits. For parobolic orbits E = 0. In this case the orbit will take on the following general shape.



From the above figure two possible ~~cases~~ are evident. In the derivation of (26) it was tacitly assumed that the variable of integration ζ does not assume the value of zero as it goes from s - c to s, although it may vanish at the end points. This corresponds to the points P and Q. Now if the point M is passed such that the body passes from P to Q, S - c vanishes at the Point M. Thus corresponding to these two core for parobolic trajectories (26) yields

$$T = \frac{1}{2}\mu \int_{s-c}^{s} \zeta^{\frac{1}{2}} \, d\zeta$$

$\rightarrow \sqrt{\zeta}\, d\zeta$

and

$$\tilde{T} = \frac{1}{\sqrt{2\mu}} \int_0^{s-c} \zeta^{\frac{1}{2}} d\zeta + \frac{1}{\sqrt{2\mu}} \int_0^{s} \zeta^{\frac{1}{2}} d\zeta$$

from which we obtain

$$T = \frac{\sqrt{2}}{3\sqrt{\mu}} \left[S^{\frac{3}{2}} - (s-c)^{\frac{3}{2}} \right]$$ 20 ~~(26)~~

$$\tilde{T} = \frac{\sqrt{2}}{3\sqrt{\mu}} \left[S^{\frac{3}{2}} + (s-c)^{\frac{3}{2}} \right]$$ 21 ~~(27)~~

It is easy to see that the same cases are true for hyperbolic trajectories.

We now come to the important case of elligtic trajectories. The determination of two time intervals T and $\tilde{T}$ analogous to those obtained for parobolic and hyperbolic trajectories hinge upon an inequality between the semi-perimeter s of the triangle FPQ and the semi-major axis a. This inequality is

$$s \leq 2a$$

The proof follows from an elementary theorem from plane geometry. Let F^* denote the vacant focus. Then since

$$\overline{FP} + \overline{F^*P} = 2a$$

$$\overline{FQ} + \overline{F^*Q} = 2a$$

we may write

$$2s = (\overline{FP} + \overline{F^*P}) + (\overline{FQ} + \overline{F^*Q}) + \overline{PQ} - \overline{F^*P} - \overline{F^*A} \quad Q$$

WHENCE

$$2s + (\overline{F^*P} + \overline{F^*Q} - \overline{PQ}) = 4a$$

and since the sum of any two sides of a plane triangle is greater than or equal to the third side, the inequality follows.

By refering to the above figure we observe that if Q coincides with M, $S = 2a$.
Thus ~~by (26)~~ the interval of integration yielding $\widetilde{T}$ is

consequently

$$T = \frac{1}{\sqrt{2}} \int_{s-c}^{s} \frac{\zeta d\zeta}{\sqrt{E\zeta^2 + \mu\zeta}}$$

$$T = \frac{1}{\sqrt{2}} \int_{s-c}^{2a} \frac{\zeta d\zeta}{\sqrt{E\zeta^2 + \mu\zeta}} + \frac{1}{\sqrt{2}} \int_{s}^{2a} \frac{\zeta d\zeta}{\sqrt{E\zeta^2 + \mu\zeta}}$$

Setting $E = -\frac{\mu}{2a}$, the integration yields

$$T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1-\gamma_2^2} + \sin^{-1}\gamma_2 - \sqrt{1-\gamma_1^2} - \sin^{-1}\gamma_1 \right\} \quad (24)$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1-\gamma_2^2} + \sin^{-1}\gamma_2 + \sqrt{1-\gamma_1^2} + \sin^{-1}\gamma_1 \right\} \quad (25)$$

where

$$\gamma_1 = 1 - \frac{s}{a} \qquad \gamma_2 = 1 - \frac{s-c}{a}$$

If $\vec{R}_1$ and $\vec{R}_2$ are the position vectors corresponding to perihelion and aphelion respectively, $R_1 = a(1-e)$ and $R_2 = a(1+e)$. Thus we find that in this case $s = 2a$. Consequently

$$\gamma_1 = -1 \qquad \gamma_2 = 1$$

Hence by (24) we easily find that the period P of an elliptic trajectory is given by

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

By subtracting (24) or (25) from P the flight times T and $\tilde{T}$ corresponding to the above cases where $\overline{R_1 R_2}$ does not and does intersect $\overline{FF^*}$ can be determined when $180° \le \measuredangle \vec{R}_1, \vec{R}_2 \le 360°$. This yields

$$T = \sqrt{\frac{a^3}{\mu}} \left\{ 2\pi - \sqrt{1-\gamma_2^2} - \sin^{-1}\gamma_2 + \sqrt{1-\gamma_1^2} + \sin^{-1}\gamma_1 \right\} \qquad (26)$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \pi - \sqrt{1-\gamma_2^2} - \sin^{-1}\gamma_2 - \sqrt{1-\gamma_1^2} - \sin^{-1}\gamma_1 \right\} \qquad (27)$$

If $360° \le \measuredangle \vec{R}_1 \vec{R}_2 \le 540°$ the case of short flight time is obtained by adding P to (24). In this case we have

$$T = \sqrt{\frac{a^3}{\mu}} \left\{ 2\pi + \sqrt{1-\gamma_2^2} + \sin^{-1}\gamma_2 - \sqrt{1-\gamma_1^2} - \sin^{-1}\gamma_1 \right\} \qquad (28)$$

Following the above notation, let $\bar{R}_1$ and $\bar{R}_2$ denote the position vectors of P and Q (or $\tilde{Q}$). Then by the definition of conic sections one can easily show with the aid of elementary trigonometry that the values of the eccentricity corresponding to (22) and (23) for hyperbolic trajectories are

$$e = \left\{ 1 + \frac{2}{c^2}(s-R_1)(s-R_2)\left(\rho_1 \rho_2 + \sqrt{\rho_1^2-1}\sqrt{\rho_2^2-1} - 1\right) \right\}^{\frac{1}{2}} \qquad (29)$$

and

$$\tilde{e} = \left\{ 1 + \frac{2}{c^2}(s-R_1)(s-R_2)\left(\beta_1\beta_2 - \sqrt{\beta_1^2-1}\,\sqrt{\beta_2^2-1} - 1\right) \right\}^{\frac{1}{2}} \qquad (30)$$

respectively. For elliptical trajectories the eccentricity corresponding to (24), (27) and (29) is given by

$$e = \left\{ 1 - \frac{2}{c^2}(s-R_1)(s-R_2)\left(1 - \gamma_1\gamma_2 + \sqrt{1-\gamma_1^2}\,\sqrt{1-\gamma_2^2}\right) \right\}^{\frac{1}{2}} \qquad (31)$$

corresponding to the case where $\overline{R_1R_2}$ intersects $\overline{FF^*}$ the eccentricity is given by

$$\tilde{e} = \left\{ 1 - \frac{2}{c^2}(s-R_1)(s-R_2)\left(1 - \gamma_1\gamma_2 - \sqrt{1-\gamma_1^2}\,\sqrt{1-\gamma_2^2}\right) \right\}^{\frac{1}{2}} \qquad (32)$$

$$e = \left\{ 1 + \frac{2}{c^2} \; (s-R_1)\,(s-R_2)\,\left(\rho_1\, \rho_2 - \sqrt{\rho_1^2 - 1}\,\sqrt{\rho_2^2 - 1} - 1\right) \right\}^{\frac{1}{2}}$$

where ρ_1 and ρ_2 are defined as in ([illegible]) and ([illegible]). For eliptical trajectories

$$e = \left\{ 1 - \frac{2}{c^2} (s-R_1)\,(s-R_2)\,\left(1-\rho_1\, \rho_2 + \sqrt{1-\rho_1^2}\,\sqrt{1-\rho_2^2}\right) \right\}^{\frac{1}{2}}$$

$$\tilde{e} = \left\{ 1 - \frac{2}{c^2} (s-R_1)\,(s-R_2)\,\left(1-\rho_1\, \rho_2 - \sqrt{1-\rho_1^2}\,\sqrt{1-\rho_2^2}\right) \right\}^{\frac{1}{2}}$$

Where in this case ρ_1 and ρ_2 are those quantities appearing in (31) - ([illegible]).

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Interplanetary conic trajectories of free fall space vehicles in the foreseeable future will be elliptical. Thus if such a vehicle is to move along an elliptical path leaving apoint P at a time T_1 and arriving at a point Q at a time T_2 the semi - major axis of the trajectory may be calculated by one of the formulas (24) ~~and~~ (28). Hence it is important to have a general idea of the properties of these functions. Moreover, in view of the energy equation (14), it is particularly important to know how the functions compare with each other.

Consider the set of all pairs of position vectors $\vec{R}_1$ and $\vec{R}_2$ of points P and Q such that $R_1 + R_2$ and c remain invariant. Corresponding to each such pair of points P and Q let us pass all possible elliptic paths obtained by ~~all possible ellip~~ varying the semi-major axis a ~~ad~~ and eccentricity e~~, which are associated with each of the formulas ( ) - ( ).~~ The graphs of T vs. ~~A~~ a of each of the five functions can then be plotted for identical values of $R_1 + R_2$ and c.

August 14, 1962

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In the above figure T_{A1} and T_{A2} are asymptotic values which can be shown to be

$$T_{A1} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left\{\sqrt{s^3} - \sqrt{(s-c)^3}\right\}$$

$$T_{A2} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left\{\sqrt{s^3} + \sqrt{(s-c)^3}\right\}$$

When a = a (minimum) = $\frac{s}{2}$, the graph of (24) joins (25) at time T_{M1} and the graph of (26) joins (27) at time T_{M2}. By subtituting $a = \frac{s}{2}$ into (24) or (25) we ~~bind~~ find

$$T_{M1} = \sqrt{\frac{s^3}{2\mu}}\left\{\sqrt{\frac{c}{s}\left(1 - \frac{c}{s}\right)} + \frac{1}{2}\sin^{-1}\left(\frac{2c}{s} - 1\right) + \frac{\pi}{4}\right\}$$

and substituting $a = \frac{s}{2}$ into (26) or (27) yields

$$T_{M2} = \sqrt{\frac{s^3}{2\mu}}\left\{\frac{3\pi}{4} - \sqrt{\frac{c}{s}\left(1 - \frac{c}{s}\right)} - \frac{1}{2}\sin^{-1}\sqrt{\frac{2c}{s} - 1}\right\}$$

With respect to Σ, a vehicle on the elliptical path from a point P to a point Q corresponding to the minimum value of the semi - major axis a will have a minimum energy . Since $\vec{V} = \vec{V}_p + \vec{V}'$ where we assume that the point P corresponds to the position of some planet, this property of minimum energy trajecteries will also be ~~fual~~ true for launch energies with respect to Σ' centered at P only if $\vec{V}_p$ is parallel to $\vec{V}'$. Thus ~~in particular since the earth orbit is extremely eccentric and since most planets move about the sun in almost the same plane, launch energy with respect to the earthwill move a minimum value if $a \approx \frac{s}{2}$~~ in practice where launch energies will be relatively low $\vec{V}'$ will be nearly parallel to $\vec{V}_p$. Hence trajectories having semi-major axis equal to $\frac{s}{2}$ will correspond approximately to trajectories ~~yielding~~ having minimum launch ener

III. Using the Gravitational Influence of a Passing Planet.

The effort taking place in the development of space vehicles designed for the exploration of the solar systems is rapidly gaining momentum. Recent advances

August 22, 1962

in many fields such as metallurgy, chemistry and electronics are being applied to actual hardwárd as soon as they become available. With the arrival of new sophisticated long life interplanetary space craft many new complex deep space operations will be ~~passeble~~ possible. Such vehicles equipted with an advanced planetary approach guidance ~~equipment~~ system could accurately control its entry into the vicinity of a passing planet. If the mission does not require the vehicle to land or to orbit the planet the small guidance package along with the planets gravitational influence gives the vehicle the potential of radically changing its trajectory about the sun.

We now consider the problem of finding a conic approximation of the trajectory of a free fall vehicle in the vicinity of a passing planet such that its influence will enable the vehicle to rendezvous with another planet. Let Σ denote any arbitrary inertial frame with the [illegible] Let Σ' be a parallel [illegible] translation of Σ with new origin located at the center of ~~x~~ a planet influencing the motion of the vehicle. Let τ denote the region of gravitational influence about the planet. It can be shown that τ can be taken as a spherical region with center at the planets center and radius ρ^* given by

$$\rho^* = \left(\frac{m}{M}\right)^{\frac{2}{5}} R$$

where R is the distance between the sun of mass M and the planet of mass m. The problem is formally stated as follows:

Suppose a free fall interplanetary space vehicle leaves the planet P_1 at time T_1 and makes a closest approach to the planet P_2 at time T_2. The influence of P_2 then causes the vehicle to rendezvous with a third planet P_3 (P_3 may or may not be P_1, indeed it may be another space vehicle orbiting the sun). The planets P_1, P_2 and P_3 along with T_1 and T_2 are given. The elliptical transfer trajectory from P_1 to P_2, the hyperbolic trajectory in τ, and the elliptical transfer trajectory from P_2 to P_3 at the time T_3 are to be determined.

August 22, 1962

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The following notation shall be employed throughout this section:

(a) $\widehat{P_1 P_2}$ = the elliptical transfer trajectory from P_1 to P_2

(b) $\widehat{P_2 P_3}$ = the elliptical transfer trajectory from P_2 to P_3

(c) $\vec{R}_i\ (T)$ = position vector of P_i with respect to Σ at time T ($i = 1,2,3$)

(d) $\vec{r}\ (T)$ = position vector of vehicle with respect to Σ at time T

(e) $\vec{\rho}\ (T)$ = position vector of vehicle with respect to Σ' at time T

(f) $\vec{V}_{\rho i}\ (T)$ = velocity vector of ρ_i with respect to Σ at time T ($i = 1,2,3$)

(g) $\vec{V}_{\rho 2}$ = velocity vector of ρ_2 with respect to Σ at time of ~~closest approach T_2~~

(h) $\vec{V}\ (T)$ = velocity vector of vehicle with respect to Σ at time T

(i) $\vec{V}'\ (T)$ = velocity vector of vehicle with respect to Σ' at time T

(j) T_1^*, T_2^* = time at which vehicle enters and leaves τ respectively

(k) a_1, l_1; a_3, l_3 = semi - major axis and semi - latus rectum of $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$ respectively.

(l) $\vec{e}_1$, $\vec{h}_1$; $\vec{e}_3$, $\vec{h}_3$ = E and H vectors of $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$ respectively

August 22, 1962

(m) $a_2, \vec{e}_2, \vec{h}_2$ = semi - major axis and E and H vectors of the hyperbolic trajectory in τ with respect to Σ' (with respect to Σ, the trajectory in τ is not a conic and hence these quantities have no meaning)

(n) R = radius of P_2

(o) d = distance of closest approach to the surface of P_2

(p) $\mu_2 = m_2 G$ where m_2 is the mass of P_2 and G is the gravitational constant

For definiteness we shall assume that $\measuredangle \vec{R}_1(T_1), \vec{R}_2(T_2)$ and $\measuredangle \vec{R}_2(T_2), \vec{R}_3(T_3)$ are not greater than 540° so that one of the formulas (24) - (28) will always be applicable.

A. The Fundamental Equation

$T_1^+ < t < T_2^-$

It follows from the above notations that

$R(t)$ t

$$\vec{\Gamma}(T) = \vec{R}_2(T) + \vec{\rho}(T)$$

whence by differentiation leads to

t

$$\vec{V}(T) = \vec{V}_2(T) + \vec{V}'(T) .$$

~~where $T_1 < T < T_2$~~ Since half of the total time that the vehicle spends in τ is very small compared to the period of P_2 about the sun we may write

t

$$\vec{V}(T) = \vec{V}_2 + \vec{V}'(T)$$

and consequently

[redacted] $\vec{V}(T_i^*) = \vec{V}_2 + \vec{V}'(T_i^*)$ $(i = 1,2)$ (33)

Since $V^2 = \vec{V} \cdot \vec{V}$ these equations yield

August 22, 1962

$$V^2(T_1^*) = V_{\infty 2}^2 + 2\vec{V}_{\infty 2}\cdot\vec{V}'(T_1^*) + V'^2(T_1^*) \qquad (34)$$

By invoking ~~involving~~ the energy equation (14) for hyperbolic trajectories we write

$$V'^2(T_1^*) = \mu_2\left(\frac{2}{\rho(T_1^*)} + \frac{1}{a_2}\right)$$

The radius of τ at T_1^* which is $\rho(T_1^*)$ is almost identical with the radius of τ at T_2^* which is $\rho(T_2^*)$. Thus the above equation implies that the vehicles energy with respect to Σ' as it enters τ is the same as its energy as it leaves τ.

$$V'^2(T_1^*) = V'^2(T_2^*) \qquad 35$$

Upon substituting this result into the difference of the equations given by (34) we find

$$V^2(T_2^*) - V^2(T_1^*) = 2\vec{V}_{\infty 2}\cdot\left(\vec{V}'(T_2^*) - \vec{V}'(T_1^*)\right) \qquad (36)$$

Taking the difference of the two equations given by (33) we have

$$\vec{V}'(T_2^*) - \vec{V}'(T_1^*) = \vec{V}(T_2^*) - \vec{V}(T_1^*)$$

and substituting this result into (36) we obtain an important equation by which all three parts of the total trajectory can be determined.

$$V^2(T_2^*) - V^2(T_1^*) = 2\vec{V}_{\infty 2}\cdot\left(\vec{V}(T_2^*) - \vec{V}(T_1^*)\right) \qquad (37)$$

It should be born in mind that this equation in essence says nothing more than (35). Its value lies in its form where the quantities are given with respect to Σ and not Σ'.

August 23, 1962

B. The Determination of the Elliptical Orbits Associated with the Transfer Trajectories.

By the orbits associated with the transfer trajectories we mean the two closed elliptical orbits about the sun where $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$ are sections. The elliptical trajectory $\widehat{P_1 P_2}$ begins at the center of P_1 with position vector $\vec{R}(T_1$
$\vec{R}_1\ (T_1)$ and ends at a point on the surface of τ at T_1' with position vector $\vec{T}\ (T_1')$.
The elliptical trajectory $\widehat{P_2 P_3}$ begins at a point on the surface of τ at T_2' with position vector $\vec{T}\ \overset{R}{(T_2')}$ and ends at the center of P_3 with position vector $\vec{R}_3\ (T_3)$.
Consider the following figure drawn with respect to Σ.

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August 23, 1962

In the above figure the short solid line represents a small portion of P_2's orbit about the sun when the vehicle is nearby. The points D, E and F are the planets positions at T_1^*, T_2 and T_2^* respectively. The longer solid line represents a small portion of the vehicles trajectory near P_2. The point A is the position of the vehicle at the time T_1^* as it enters τ, B is its position at T_2 when it is closest to P_2 and the point C is the position of the vehicle at time T_2^* as it leaves the moving region τ. The trajectory of the vehicle bounded by A and C is not conic since the figure is drawn with respect to Σ. When viewed from Σ' this portion of the trajectory is hyperbolic. The vehicle's ellipticle trajectories outside τ appear as straight line segments because of the scale of the figure. The sun is very far away and therefore the vectors $\vec{R}_1 (T_1^*)$, $\vec{R}_2 (T_2)$ and $\vec{R}_3 (T_2^*)$ appear as parallel vectors. The dotted lines are continuations of $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$. The points B' and B" correspond to the positions of the vehicle moving on the orbits of $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$ at the time T_2 as ~~if~~ if P_2 did not exist. The figure clearly displays some very important facts.

It is easy to see that the position vectors of B' and B" are almost identical with $\vec{R}_2 (T_2)$. Thus by employing Lambert's Theorem by using the appropriate formula from (24) - (28) with $T = T_2 - T_1$, $\vec{R}_1 = \vec{R}_1 (T_1)$ and $\vec{R}_2 = \vec{R}_2 (T_2)$, the semi - major axis a_1 of $\widehat{P_1 P_2}$ can be calculated. Then by using either (31) or (33) depending upon which formula ~~of (24)-(28)~~ was used to calculate a_1, the eccentricity e_1 can be found. Consequently since $l_1 = a_1 (1 - e_1^2)$ the vectors $\vec{e}_1$ and $\vec{h}_1$ corresponding to $P_1 P_2$ can be calculated by (8) - (14), ~~corresponds~~ similarly by setting $T = T_3 - T_2$, $\vec{R}_1 = \vec{R}_2 (T_2)$ and $\vec{R}_2 = \vec{R}_3 (T_3)$ an application of Lambert's Theorem yields $a_3 = a_3 (T_3)$. Since T_3 is unknown a_3 is

new paragraph

written as $a_3\ (T_3)$ meaning that a_3 is a function of T_3. Following the above prodedure the functions $e_3\ (T_3)$, $l_3\ (T_3)$, $e_3\ (T_3)$ and $\overrightarrow{h_3}\ (T_3)$ can in theory be obtained. In practice these functions are not actually determined since high speed digital computers makes it possible to give T_3 an actual trial numerical value. Thus $a_3\ (T_3)$, $e_3\ (T_3)$, $l_3\ (T_3)$, $\overrightarrow{e_3}\ (T_3)$ and $\overrightarrow{h_3}\ (T_3)$ all take on actual numerical values corresponding to the trial value given to T_3. The actual value of T_3 can be obtained by noticing a second important fact suggested from the figure.

It is clearly evident that the vehicles velocity vector at A and C are almost identical with the hypothetical velocity vectors at B' and B". Consequently in view of the first observation these velocities can easily be obtained ~~using (—)~~ by (13)

$$\overrightarrow{V}\,(T_1^*) = \overrightarrow{V_1^*} = \frac{1}{l_1}\ \overrightarrow{h_1} \times \left(\hat{R}_2\,(T_2) + \overrightarrow{e_1}\right)$$

$$\overrightarrow{V}\,(T_2^*) = \overrightarrow{V_2^*}(T_3) = \frac{1}{l_3\,(T_3)}\ \overrightarrow{h_3}\,(T_3) \times \left(\hat{R}_2\,(T_2) + \overrightarrow{e_3}\,(T_3)\right)$$

Upon substituting these values into (37) we obtain the equation

$$V_2^2\,(T_3) - V_1^2 = 2\,\overrightarrow{V}_{p2} \cdot (\overrightarrow{V_2}\,(T_3) - \overrightarrow{V_1}) \qquad (38)$$

from which the value of T_3 can be calculated. In practice each trial value of T_3 yields trial values of a_3, e_3, l_3, $\overrightarrow{e_3}$, $\overrightarrow{h_3}$ and hence by (13) a trial value of $\overrightarrow{V_2}$. If the trial value of T_3 yielding the trial value of $\overrightarrow{V_2}$ does not satisfy (38) a new trial value of T_3 is considered. Thus a systematic search for T_3 can be instituted which will yield a solution of (38). The corresponding trial values of a_3, e_3, l_3, $\overrightarrow{e_3}$ and $\overrightarrow{h_3}$ then become the actual values for $\widehat{P_2\,P_3}$. Thus since $\overrightarrow{e_1}$,

August 23, 1962

$\vec{h}_1$ and $\vec{e}_3$, $\vec{h}_3$ are known the elliptical orbits associated with $\widehat{P_1 P_2}$ and $\widehat{P_2 P_3}$ are completely determined. We emphasize at this point that even though $\vec{V}\,(T_1^*)$ and $\vec{V}\,(T_2^*)$ are also known T_1^* and T_2^* remain to be calculated.

C. The Determination of the Hyperbolic Trajectory

We now consider that part of the vehicles trajectory in τ. The seemingly difficult task of finding this trajectory turns out to be surprisingly easy.

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The above ~~This~~ figure is drawn with respect to Σ'. Hence the vehicles trajectory in τ is hyperbolic. The points A, B and C correspond to the points A, B and C of figure 1. From ([illegible]) we calculate the hyperbolic excess velocity vectors at A and C.

$$\vec{V}'\,(T_1^*) = \cancel{\vec{v}_1} = \vec{V}_1^{*\prime} - \vec{V}_2$$

$$\vec{V}'\,(T_2^*) = \cancel{\vec{v}_2} = \vec{V}_2^{*\prime} - \vec{V}_2$$

from which $V_1^{*\prime 2}$ and $V_2^{*\prime 2}$ are claculated. In view of (31) we calculate the average of these quantities

$$\bar{V}'^2 = \frac{1}{2}\left(V_1^{*\prime 2} + V_2^{*\prime 2}\right).$$

Now with respect to Σ', the points A and C may be taken at infinity. Thus by (14) the semi - major axis a_2 of the hyperbolic path can be easily obtained from

$$a_2 = \frac{\mu_2}{\bar{V}'^2} \qquad (39)$$

If b_2 denotes the length of the semi - minor axis of the hyperbolic trajectory

$$\tan \phi = \frac{b_2}{a_2}$$

where ϕ is one half of the angle between the asymptotes. Thus since the eccentricity e_2 is related to a_2 and b_2 by

$$e_2 = \sqrt{1 + \left(\frac{b_2}{a_2}\right)^2}$$

we obtain

$$\cos \phi = \frac{1}{e_2} \qquad (40)$$

Thus by observing the above figure we find

$$\vec{V}_1^{*\prime} \cdot \vec{V}_2^{*\prime} = V_1^{*\prime} V_2^{*\prime} \cos 2 \left(\frac{\pi}{2} - \beta\right)$$

which is expressible as

$$\vec{V}_1' \cdot \vec{V}_2' = V_1' V_2' \left(1 - 2 \cos^2 \beta\right)$$

It then follows that the eccentricity of the hyperbolic path can be calculated by

$$e_2 = \left\{ \frac{2 \vec{V}_1^{*\prime} \cdot \vec{V}_2^{*\prime}}{\underbrace{V_1^{*\prime} V_2^{*\prime} - \vec{V}_1^{*\prime} \cdot \vec{V}_2^{*\prime}}} \right\}^{\frac{1}{2}} \qquad \frac{2 V_1^{*\prime} V_2^{*\prime}}{V_1^{*\prime} V_2^{*\prime} - \vec{V}_1^{*\prime} \cdot \vec{V}_2^{*\prime}} \qquad (41)$$

The distance of closest approach which the vehicle makes with the surface of P_2 can now be obtained by

$$d = a_2 \left(e_2 - 1 \right) - R \qquad (42)$$

~~since the distance RP_2 is $a_2 (e_2 - 1)$~~. If this quantity turns out to be negative the trajectory is obviously physically unrealizable. The value of T_3 is then disgarded and the search continues until a new T_3 is found which yields a solution of (37) and also a positive value for d. If the search for T_3 proceeds by taking increasing thial trial values of T_3 the solution will give the shortest possible time for T_3.

After T_3 has been determined which will give a positive value to d, the magnitude of the H vector can be calculated by

$$h_2 = \sqrt{a_2 \left(e_2^2 - 1 \right) \mu_2} \qquad (43)$$

From figure 1), one sees that the E and H vectors may be calculated by

$$\vec{e}_2 = \frac{\vec{V}_1' - \vec{V}_2'}{\left|\vec{V}_1' - \vec{V}_2'\right|} \, e_2 \qquad (44)$$

$$\vec{h}_2 = \frac{\vec{V}_1' \times \vec{V}_2'}{\left|\vec{V}_1' \times \vec{V}_2'\right|} \, h_2 \qquad (45)$$

The position vectors of the points A and C of figures ( ) and ( ) with respect to Σ' can now be calculated by employing (4). We take $\rho\,(T_1^*) = \rho\,(T_2^*) = \left(\frac{m}{M}\right)^{\frac{2}{5}} \; R_2\,(T_2)$ since the radii of the spheres of influence at times T_1^* and T_2^* are almost identical with the radius at the time T_2.

$$\vec{\rho}\,(T_i^*) = \left(\frac{1}{\mu 2}\,\vec{V}_i' \times \vec{h}_2 - e_2\right)\left(\frac{m}{M}\right)^{\frac{2}{5}} R_2\,(T_2) \qquad (i = 1,2) \qquad (46$$

~~The~~ one half of the amount of time ~~ΔT~~ $\frac{1}{2}\Delta T$ which the vehicle spends in ~~T~~ may be calculated by (2?) with

$$R_1 = a_2\,(e_2 - 1)$$

$$R_2 = \left(\frac{m}{M}\right)^{\frac{2}{5}} R_2\,(T_2)$$

$$c = \sqrt{a_2^2\,\left(e_2 - 1\right)^2 + \left(\frac{m}{M}\right)^{\frac{4}{5}} R_2^2\,(T_2) + 2a_2\left(1 - \frac{1}{e_2}\right)\left(\frac{m}{M}\right)^{\frac{2}{5}} R_2\,(T_2)}$$

which follows from (40)

Setting $E = \frac{\mu}{2a}$, (26) yields

$$T = \frac{1}{\sqrt{a\mu}} \int_{s-c}^{s} \frac{\zeta \, d\zeta}{\sqrt{\left(\frac{\zeta}{a} + 1\right)^2 - 1}}$$

$$\tilde{T} = \frac{1}{\sqrt{a\mu}} \int_{0}^{s-c} \frac{\zeta \, d\zeta}{\sqrt{\left(\frac{\zeta}{a} + 1\right)^2 - 1}} + \frac{1}{\sqrt{a\mu}} \int_{0}^{s} \frac{\zeta \, d\zeta}{\sqrt{\left(\frac{\zeta}{a} + 1\right)^2 - 1}}$$

By defining $\beta = \frac{\zeta}{a} + 1$ these integrals can be written as

$$T = \sqrt{\frac{a^3}{\mu}} \int_{\rho_1}^{\rho_2} \frac{(\beta - 1)\, d\beta}{\sqrt{\beta^2 - 1}}$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \int_{1}^{\rho_1} \frac{(\beta - 1)\, d\beta}{\sqrt{\beta^2 - 1}} + \sqrt{\frac{a^3}{\mu}} \int_{1}^{\rho_2} \frac{(\beta - 1)\, d\beta}{\sqrt{\beta^2 - 1}}$$

where $\rho_1 = \frac{s-c}{a} + 1$ and $\rho_2 = \frac{s}{a} + 1$. Carrying out the integration leads to

$$T = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{\rho_2^2 - 1} - \cosh^{-1} \rho_2 - \sqrt{\rho_1^2 - 1} + \cosh^{-1} \rho_1 \right] \qquad (28)$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{\rho_2^2 - 1} - \cosh^{-1} \rho_2 + \sqrt{\rho_1^2 - 1} - \cosh^{-1} \rho_1 \right] \qquad (29)$$