The electric field and thrust per unit area are given, respectively, by

$$\left|\frac{d\phi}{dx}\right| = \left(\frac{32JHM}{\epsilon_0{}^2 e}\right)^{1/4} k \cos\psi \qquad [9]$$

$$T = (2JHM/e)^{1/2}(1 - 2k^2) \qquad [10]$$

It follows from these expressions and Eq. [4] that for a given J and H the thrust is a maximum when the following two conditions are fulfilled: 1) the velocities of the two sets of charge are equal at the exit plane of the ion motor; and 2) the electric field just outside the exit plane is zero. The first requires simply that the net accelerating voltages for the two sets of ions be proportional to their respective masses. The second requires that the net charge on the vehicle, including the charge in the ion motor, be zero. For the positive ion-electron beam, these conditions, particularly the first, are not too important because most of the energy and momentum are carried by the positive ions. For the negative ion-positive ion case, failure to maintain these conditions could reduce the thrust by as much as 29%. Thus, for example, an arrangement in which a virtual electrode is used for the final electrode will usually provide less thrust for a given J and H than one having an actual exit electrode.

The solution given by Eqs. [5] can be generalized to account for more than one species of positive or negative ion. The neglect of intrabeam collisions in the derivation of this equation is usually justified in propulsion applications (3). On the other hand, the use of a single group velocity for the electrons is generally recognized (1) to be somewhat inadequate. However, for the negative atomic ions, as for the positive atomic ions, the use of a single group velocity is appropriate.

Nomenclature

x = position along flow from potential minimum, m
ϕ = electrical potential above minimum value, v
ϵ_0 = permittivity of free space, MKS units
e = electronic charge, coul
J = current density for each ion group, amp/m²
H = total energy flux density, w/m²
$v_\pm$ = velocity of positive or negative ions, m/sec
$m_\pm$ = mass of positive or negative ions, kg
m_e = electronic mass, kg
M = $m_+ + m_-$, kg
T = net thrust per unit area, newtons/m²
ψ = angular parameter
k = amplitude parameter
λ = spatial period, m

References

1 Staff, Ramo-Wooldridge Research Lab., "Electrostatic propulsion," *Proc. Inst. Radio Engrs.* (Inst. Radio Engrs., New York, 1960), Vol. 48, pp. 477–491.

2 Gilleo, M. A. and Kash, S. W., "Propulsion by composite beams of negative and positive ions," ARS J. 31, 621–627 (1961).

3 Watson, W. K. R., "Justification of the use of the collisionless Boltzmann equation for ion beam neutralization studies, *Progress in Astronautics and Rocketry, Electrostatic Propulsion* (Academic Press Inc., New York, 1961), Vol. 5, pp. 231–235.

Out-of-Ecliptic Trajectories

RICHARD R. BURLEY[1]

NASA Lewis Research Center, Cleveland, Ohio

IN ORDER to investigate possible asymmetries in the environment of space (e.g., distributions of meteroids and solar phenomena), there is considerable scientific interest in a space probe into regions far removed from the plane of the ecliptic. The purpose of this note is to derive trajectories that yield the maximum altitude above the ecliptic for a given expenditure of propulsive energy.

The approach is the usual successive two-body approximation—that is, at a point sufficiently far from Earth (at which point the probe velocity with respect to Earth is the so-called hyperbolic velocity V_h), the probe can be considered to move in a heliocentric elliptical orbit, and the presence of Earth can be ignored. V_h is related to the booster burnout velocity V_{bo} by

$$V_{bo} = (V_h{}^2 + V_{esc}{}^2)^{1/2}$$

where V_{esc} is the parabolic escape speed at the burnout altitude. For a given value of V_h (or correspondingly, V_{bo}), the problem is then one of determining the optimum orientation of the V_h vector in space. It is assumed that the heliocentric injection point is still essentially located in the ecliptic.

A typical heliocentric orbit is shown in Fig. 1. The altitude of any point on this orbit is determined by the inclination α between the ecliptic and orbital planes, the radius r, and the travel angle θ. The altitude can be expressed in terms of these parameters as

$$h = r \sin\theta \sin\alpha$$

The radius r of any point in the orbit is a function of three parameters: the heliocentric velocity at the injection point $V_{v/s}$, the path angle β of the vehicle at the point where the probe enters the orbit, and the travel angle θ. The heliocentric velocity vector can be related to the hyperbolic velocity vector as illustrated in Fig. 2. The direction of the hyperbolic velocity vector is specified in terms of η and λ, which are functions of the launch conditions. At the point where the probe enters the heliocentric orbit (i.e., at the ecliptic), its distance from the sun r_e is assumed to be 1 a.u. Also, the orbital velocity of Earth V_e is assumed to be perpendicular to this radius and equal to 18.5 mps. Adding this velocity vectorially to V_h gives $V_{v/s}$. Then, by using the equations of conservation of energy and angular momentum, the altitude can be expressed as a function of V_h, η, λ, θ:

$$H = \frac{(A)^{1/2}\bar{V}_h \sin\eta \sin\theta}{1 + (AB + 1)^{1/2}\cos(\theta_0 + \theta)}$$

where

$$A = \bar{V}_h{}^2(1 - \cos^2\eta \sin^2\lambda) + 2\bar{V}_h{}^2 \cos\eta \cos\lambda + 1$$

$$B = \bar{V}_h{}^2 + 2\bar{V}_h \cos\eta \cos\lambda - 1$$

$$\theta_0 = \cos^{-1}\left[\frac{A - 1}{(AB + 1)^{1/2}}\right]$$

$$\bar{V}_h = V_h/V_e$$

$$H = h/r_e$$

Thus the problem becomes one of finding the values of η, λ, and θ which will maximize H as a function of $\bar{V}_h$. These values can be found by differentiating H with respect to the three angles and equating the resulting expressions to zero, yielding

$$\cos\theta = 1 - A$$

$$\cos\lambda = 1/(2^{1/2} - \bar{V}_h)$$

The relationship between $\bar{V}_h$ and η was not tractable to an explicit solution. Therefore, the altitude was determined first as a function of η by substituting the expressions for θ and λ along with several values of η into the altitude equation. Then a curve of altitude as a function of η was plotted for each value of $\bar{V}_h$. From this curve, the maximum altitude could be determined as a function of $\bar{V}_h$.

This relationship is shown in Fig. 3. Also shown is the maximum altitude using a circular orbit whose radius is equal to 1 a.u. This is an orbit that would be obtained by setting

Received February 16, 1962.

[1] Research Engineer, Mission Analysis Branch. Member ARS.

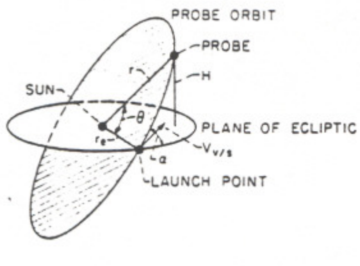

Fig. 1 Geometric relations for out-of-ecliptic trajectories

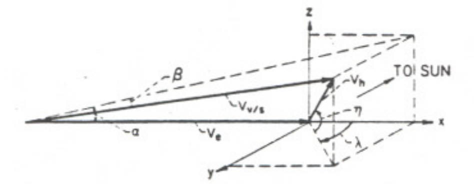

Fig. 2 Velocity relations at orbit injection

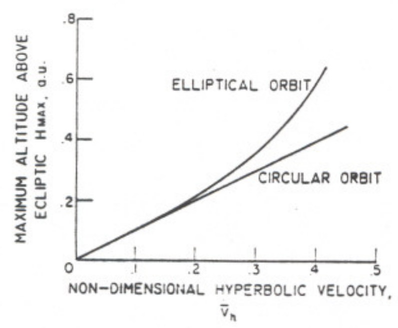

Fig. 3 Maximum altitude for elliptical and circular orbits

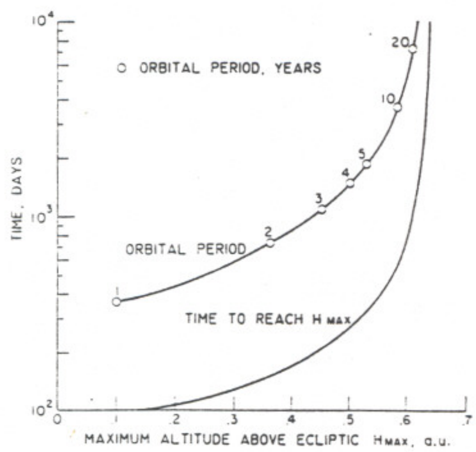

Fig. 4 Effect of maximum altitude on probe travel time

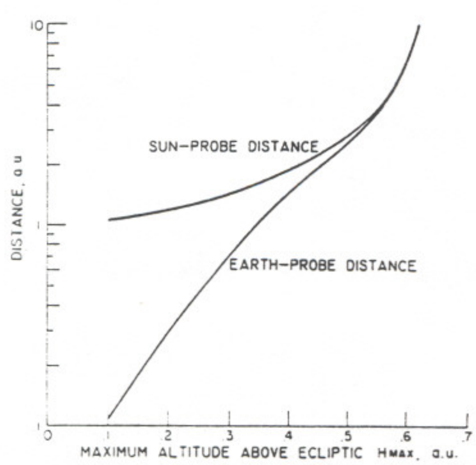

Fig. 5 Variation of probe distance from sun and Earth with maximum altitude (Earth's orbit assumed circular)

$\lambda = 0$ and $V_{s/v} = V_e$, and it is the type of out-of-ecliptic shot usually considered in the literature. It is seen that, when $\bar{V}_h$ is larger than about 0.2, significantly greater distances out of the ecliptic can be reached by traveling along the elliptical orbit. Conversely, if a given distance is desired, less $\bar{V}_h$ is required using the ellipse.

Having determined the maximum altitude out of the ecliptic, it is also of interest to know the period of the elliptical orbit as a function of maximum H. This is shown in Fig. 4. If the probe is to return and pass close to Earth after reaching maximum H, its period must be an integer number of years. Therefore, only particular values of H can be reached in an optimum fashion. It is seen that if it is required to reach large values of H, the period becomes excessive. In contrast, when traveling along the circular orbit, the period is always 1 yr (and, in fact, an Earth rendezvous will always occur 6 months after launch).

Because of the large period, an alternative method might be to send back information from the probe at the time it reaches H_{max}, without waiting for it to return to the vicinity of Earth. Fig. 4 shows that this time is quite reasonable. The disadvantage is that communication distances become large. This is shown in Fig. 5, along with the corresponding distance of the probe from the sun.

It is shown in this note that an optimum trajectory yields significantly greater distances above the ecliptic than the circular-orbit type. Although in reaching these distances the period becomes excessive, the time for the probe to reach H_{max} is quite remarkable. The presented data are limited to $\bar{V}_h < 0.414$, so that the heliocentric orbit is always elliptical. For larger values of $\bar{V}_h$ it is possible to have hyperbolic heliocentric orbits whose H is any desired value. This leads to a balancing of $\bar{V}_h$, time, and communication distance which is beyond the scope of this note.

Influence of Meteoroid Hazards on Selection of Spacecraft Propellants

RICHARD J. WEBER[1]

NASA Lewis Research Center, Cleveland, Ohio

FOR any given mission, it is ordinarily desirable to employ propellants with the highest specific impulse because this results in the smallest propellant-to-gross weight fraction. However, as has been pointed out by others, e.g., Refs. 1 and 2, it is also desirable to employ propellants with high density since this tends to reduce the tank weight for any given propellant fraction.

It is obvious that high propellant density will be most advantageous in situations where the tank weight is a large fraction of the gross weight. One possible factor that may create this situation in space vehicles is the need for protection against meteoroid damage. This note considers, in a very elementary fashion, the weight of a hypothetical space stage whose tank thickness is determined on this basis alone. Required wall thickness to give the rather high zero-puncture probability of 0.999 were obtained from Ref. 3. A single stage having a characteristic velocity increment of 11,000 fps is considered; this value is approximately equivalent to escape from Earth orbit or to a lunar landing or takeoff. A single spherical tank was assumed, since it has the minimum surface-to-volume ratio. Tank weight was varied directly as the surface area. The weight of engines, thrust structure, etc., was taken as 3% of the gross weight. The payload was 12,500 lb.

Received April 23, 1962.

[1] Assistant Chief, Mission Analysis Branch. Member ARS.