In terms of ζ_0, z, and θ, (a_{ij}) is given by

$$a_{11} = -\sin\zeta_0 \sin z + \cos\zeta_0 \cos z \cos\theta$$
$$a_{12} = -\cos\zeta_0 \sin z - \sin\zeta_0 \cos z \cos\theta$$
$$a_{13} = -\cos z \sin\theta$$

$$a_{21} = \sin\zeta_0 \cos z + \cos\zeta_0 \sin z \cos\theta$$
$$a_{22} = \cos\zeta_0 \cos z - \sin\zeta_0 \sin z \cos\theta$$
$$a_{23} = -\sin z \sin\theta$$

$$a_{31} = \cos\zeta_0 \sin\theta$$
$$a_{32} = -\sin\zeta_0 \sin\theta$$
$$a_{33} = \cos\theta$$

$$\zeta_0 = 2304''.997T + 0''.302T^2 + 0''.0179T^3$$
$$z = 2304''.997T + 1''.093T^2 + 0''.0192T^3$$
$$\theta = 2004''.298T - 0''.426T^2 - 0''.0416T^3$$

with T the number of Julian centuries of 36,525 days past the epoch 1950.0.

The actual computational form of (a_{ij}) is obtained by expanding the a_{ij} in power series in ζ_0, z, θ and replacing the arguments by the above time series. The results are

$$a_{11} = 1 - 0.00029697T^2 - 0.000000013T^3$$
$$a_{12} = -a_{21} = -0.02234988T - 0.00000676T^2 + 0.00000221T^3$$
$$a_{13} = -a_{31} = -0.00971711T + 0.00000207T^2 + 0.00000096T^3$$
$$a_{22} = 1 - 0.00024976T^2 - 0.00000015T^3$$
$$a_{23} = a_{32} = -0.00010859T^2 - 0.00000003T^3$$
$$a_{33} = 1 - 0.00004721T^2 + 0.00000002T^3$$

The calling sequence has the form

(AC) = days past 0^h January 1, 1950, E.T.

CALL ROTEQ

(OP) X,,Y

X−3, X−2, X−1 contain the input vector; Y−3, Y−2, Y−1 contain the output vector; X = Y is permitted.

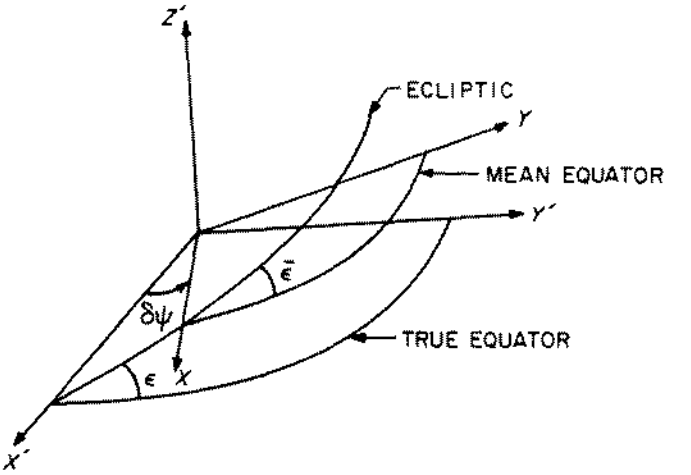
OP = PZE regards X as 1950.0 and rotates to date in Y; OP = MZE regards X as of date and rotates to 1950.0 in Y.

The matrix (a_{ij}) is saved in the COMMON locations AA, . . . , AA + 8 and recomputed only when the time has changed by 1/64 day.

The subroutine uses three cells of erasable storage starting at COMMON.

NUTATE

To describe the nutation of the Earth about its precessing mean equator, it is convenient to construct the nutation matrix N which relates the Cartesian coordinates expressed in the true equator and equinox to those in the mean equator and equinox (Sketch A-15).



Sketch A-15. Relationship between true equator and mean equator of date

$\delta\psi$ is the nutation in longitude measured from the true vernal equinox at the X' axis to the mean vernal equinox at the X axis. $\bar{\epsilon}$ is the mean obliquity, while $\epsilon = \bar{\epsilon} + \delta\epsilon$ is the true obliquity where $\delta\epsilon$ is the nutation in obliquity. Numerical expressions for the above quantities appear in the discussion of subroutine MNA following.

If N is defined in the sense

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = N \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where the primed system is the true equator and equinox and the unprimed is the mean equator and equinox, then the N_{ij} are given by

$$N_{11} = \cos\delta\psi$$
$$N_{12} = \sin\delta\psi \cos\bar{\epsilon}$$
$$N_{13} = \sin\delta\psi \sin\bar{\epsilon}$$

$$N_{21} = \sin\delta\psi \cos\epsilon$$
$$N_{22} = \cos\delta\psi \cos\epsilon \cos\bar{\epsilon} + \sin\epsilon \sin\bar{\epsilon}$$

$$N_{23} = \cos\delta\psi\cos\epsilon\sin\bar{\epsilon} - \sin\epsilon\cos\bar{\epsilon}$$

$$N_{31} = \sin\delta\psi\sin\epsilon$$

$$N_{32} = \cos\delta\psi\sin\epsilon\cos\bar{\epsilon} - \cos\epsilon\sin\bar{\epsilon}$$

$$N_{33} = \cos\delta\psi\sin\epsilon\sin\bar{\epsilon} + \cos\epsilon\cos\bar{\epsilon}$$

Since $|\delta\psi| < 10^{-4}$ and $|\delta\epsilon| < 10^{-4}$, the N_{ij} are expanded to first order in $\delta\psi$ and $\delta\epsilon$ to obtain a form which is better behaved for numerical calculation:

$$N = \begin{pmatrix} 1 & -\delta\psi\cos\bar{\epsilon} & -\delta\psi\sin\bar{\epsilon} \\ \delta\psi\cos\bar{\epsilon} & 1 & -\delta\epsilon \\ \delta\psi\sin\bar{\epsilon} & \delta\epsilon & 1 \end{pmatrix}$$

NUTATE is used as a utility routine to generate the matrix product NA, where A is obtained by calling ROTEQ; the resultant NA is used to rotate from the equator and equinox of 1950.0 to the true equator and equinox of date and is saved in the COMMON cells (NA), . . . , (NA) + 8. As N is a slowly varying matrix, it is saved and recomputed only if the time has changed by at least 0.1 day. The generation of N is effected by calling MNA which also internally stores N.

MNA, MNA1

It is the principal function of MNA to provide the rotation matrix MNA which allows vectors in the 1950.0 system to be expressed relative to the Moon's true equator and conversely.

For this purpose it is assumed that the matrix A has been formed by ROTEQ and appears in the COMMON locations AA, . . . , AA + 8. The form of the matrix N (see preceding discussion of NUTATE) depends upon the nutations $\delta\psi$ and $\delta\epsilon$. In the discussion of XYZDD to follow, M is identified as (b_{ij}).

The numerical expressions for the necessary quantities appear below:

$\delta\epsilon = \Delta\epsilon + d\epsilon$, where $\Delta\epsilon$ denotes the long-period and $d\epsilon$ the short-period terms for the nutation in obliquity. In a similar manner the nutation in longitude $\delta\psi$ is given with long-period and short-period terms $\Delta\psi$ and $d\psi$.

$$\begin{aligned}\Delta\epsilon = {} & 25\overset{\circ}{.}5844\times10^{-4}\cos\Omega - 0\overset{\circ}{.}2511\times10^{-4}\cos2\Omega \\ & + 1\overset{\circ}{.}5336\times10^{-4}\cos2L + 0\overset{\circ}{.}0666\times10^{-4}\cos(3L-\Gamma') \\ & - 0\overset{\circ}{.}0258\times10^{-4}\cos(L+\Gamma') \\ & - 0\overset{\circ}{.}0183\times10^{-4}\cos(2L-\Omega) \\ & - 0\overset{\circ}{.}0067\times10^{-4}\cos(2\Gamma'-\Omega)\end{aligned}$$

$$\begin{aligned}d\epsilon = {} & 0\overset{\circ}{.}2456\times10^{-4}\cos2\mathbb{C} + 0\overset{\circ}{.}0508\times10^{-4}\cos(2\mathbb{C}-\Omega) \\ & + 0\overset{\circ}{.}0369\times10^{-4}\cos(3\mathbb{C}-\Gamma') \\ & - 0\overset{\circ}{.}0139\times10^{-4}\cos(\mathbb{C}+\Gamma') \\ & - 0\overset{\circ}{.}0086\times10^{-4}\cos(\mathbb{C}-\Gamma'+\Omega) \\ & + 0\overset{\circ}{.}0083\times10^{-4}\cos(\mathbb{C}-\Gamma'-\Omega) \\ & + 0\overset{\circ}{.}0061\times10^{-4}\cos(3\mathbb{C}+\Gamma'-2L) \\ & + 0\overset{\circ}{.}0064\times10^{-4}\cos(3\mathbb{C}-\Gamma'-\Omega)\end{aligned}$$

$$\begin{aligned}\Delta\psi = {} & -(47\overset{\circ}{.}8927 + 0\overset{\circ}{.}0482\,T)\times10^{-4}\sin\Omega \\ & + 0\overset{\circ}{.}5800\times10^{-4}\sin2\Omega - 3\overset{\circ}{.}5361\times10^{-4}\sin2L \\ & - 0\overset{\circ}{.}1378\times10^{-4}\sin(3L-\Gamma') \\ & + 0\overset{\circ}{.}0594\times10^{-4}\sin(L+\Gamma') \\ & + 0\overset{\circ}{.}0344\times10^{-4}\sin(2L-\Omega) \\ & + 0\overset{\circ}{.}0125\times10^{-4}\sin(2\Gamma'-\Omega) \\ & + 0\overset{\circ}{.}3500\times10^{-4}\sin(L-\Gamma') \\ & + 0\overset{\circ}{.}0125\times10^{-4}\sin(2L-2\Gamma')\end{aligned}$$

$$\begin{aligned}d\psi = {} & -0\overset{\circ}{.}5658\times10^{-4}\sin2\mathbb{C} \\ & - 0\overset{\circ}{.}0950\times10^{-4}\sin(2\mathbb{C}-\Omega) \\ & - 0\overset{\circ}{.}0725\times10^{-4}\sin(3\mathbb{C}-\Gamma') \\ & + 0\overset{\circ}{.}0317\times10^{-4}\sin(\mathbb{C}+\Gamma') \\ & + 0\overset{\circ}{.}0161\times10^{-4}\sin(\mathbb{C}-\Gamma'+\Omega) \\ & + 0\overset{\circ}{.}0158\times10^{-4}\sin(\mathbb{C}-\Gamma'-\Omega) \\ & - 0\overset{\circ}{.}0144\times10^{-4}\sin(3\mathbb{C}+\Gamma'-2L) \\ & - 0\overset{\circ}{.}0122\times10^{-4}\sin(3\mathbb{C}-\Gamma'-\Omega) \\ & + 0\overset{\circ}{.}1875\times10^{-4}\sin(\mathbb{C}-\Gamma') \\ & + 0\overset{\circ}{.}0078\times10^{-4}\sin(2\mathbb{C}-2\Gamma') \\ & + 0\overset{\circ}{.}0414\times10^{-4}\sin(\mathbb{C}+\Gamma'-2L) \\ & + 0\overset{\circ}{.}0167\times10^{-4}\sin(2\mathbb{C}-2L) \\ & - 0\overset{\circ}{.}0089\times10^{-4}\sin(4\mathbb{C}-2L)\end{aligned}$$

$$\begin{aligned}\Omega = {} & 12\overset{\circ}{.}1127902 - 0\overset{\circ}{.}0529539222\,d + 20\overset{\circ}{.}795\times10^{-4}\,T \\ & + 20\overset{\circ}{.}81\times10^{-4}\,T^2 + 0\overset{\circ}{.}02\times10^{-4}\,T^3\end{aligned}$$

$$\begin{aligned}\mathbb{C} = {} & 64\overset{\circ}{.}37545167 + 13\overset{\circ}{.}1763965268\,d - 11\overset{\circ}{.}31575\times10^{-4}\,T \\ & - 11\overset{\circ}{.}3015\times10^{-4}\,T^2 + 0\overset{\circ}{.}019\times10^{-4}\,T^3\end{aligned}$$

$$\begin{aligned}\Gamma' = {} & 208\overset{\circ}{.}8439877 + 0\overset{\circ}{.}1114040803\,d - 0\overset{\circ}{.}010334\,T \\ & - 0\overset{\circ}{.}010343\,T^2 - 0\overset{\circ}{.}12\times10^{-4}\,T^3\end{aligned}$$

$$\begin{aligned}L = {} & 280\overset{\circ}{.}08121009 + 0\overset{\circ}{.}9856473354\,d + 3\overset{\circ}{.}03\times10^{-4}\,T \\ & + 3\overset{\circ}{.}03\times10^{-4}\,T^2\end{aligned}$$

$$\begin{aligned}\Gamma = {} & 282\overset{\circ}{.}08053028 + 0\overset{\circ}{.}470684\times10^{-4}\,d + 4\overset{\circ}{.}5525\times10^{-4}\,T \\ & + 4\overset{\circ}{.}575\times10^{-4}\,T^2 + 0\overset{\circ}{.}03\times10^{-4}\,T^3\end{aligned}$$

T is the number of Julian centuries of 36,525 days past the epoch 0^h January 1, 1950, E.T., while d is the number of days past the same epoch. The program uses d in double precision. The mean obliquity is calculated from

$$\bar{\epsilon} = 23°.4457587 - 0°.01309404\,T - 0°.0088 \times 10^{-4}\,T^2 + 0°.0050 \times 10^{-4}\,T^3$$

The quantity $\delta\alpha = \delta\psi \cos \bar{\epsilon}$ is computed and stored in the COMMON cell NUTRA for the GHA routine to use as the nutation in right ascension for calculation of the true value of the Greenwich hour angle of the vernal equinox.

The librations are given by

$$\begin{aligned}
\sigma \sin I = &- 0°.0302777 \sin g + 0°.0102777 \sin (g + 2\omega) \\
&- 0°.00305555 \sin (2g + 2\omega) \\
\tau = &- 0°.003333 \sin g + 0°.0163888 \sin g' \\
&+ 0°.005 \sin 2\omega \\
\rho = &- 0°.0297222 \cos g + 0°.0102777 \cos (g + 2\omega) \\
&- 0°.00305555 \cos (2g + 2\omega) \\
I = &\ 1°.535
\end{aligned}$$

The following expressions have been programmed for g, g', and ω:

$$\begin{aligned}
g &= 215°.54013 + 13°.064992\,d \\
g' &= 358°.009067 + 0°.9856005\,d \\
\omega &= 196°.745632 + 0°.1643586\,d
\end{aligned}$$

Evidently $g = \mathbb{C} - \Gamma'$, the mean anomaly of the Moon; $g' = L - \Gamma$, the mean anomaly of the Sun; and $\omega = \Gamma' - \Omega$, the argument of the perigee of the Moon. All quantities relate to mean motions of the Sun and the Moon.

$$\begin{aligned}
\cos i = &\ \cos (\Omega + \sigma + \delta\psi) \sin \epsilon \sin (I + \rho) \\
&+ \cos \epsilon \cos (I + \rho), \quad 0 < i < 90° \\
\sin \Omega' = &- \sin (\Omega + \sigma + \delta\psi) \sin (I + \rho) \csc i, \\
&- 90° < \Omega' < 90° \\
\sin \Delta = &- \sin (\Omega + \sigma + \delta\psi) \sin \epsilon \csc i \\
\cos \Delta = &- \sin (\Omega + \sigma + \delta\psi) \sin \Omega' \cos \epsilon \\
&- \cos (\Omega + \sigma + \delta\psi) \cos \Omega', \quad 0 \leqq \Delta < 360°
\end{aligned}$$

$$\begin{aligned}
\Lambda &= \Delta + (\mathbb{C} + \tau) - (\Omega + \sigma) \\
\epsilon &= \bar{\epsilon} + \delta\epsilon
\end{aligned}$$

The calling sequence to MNA is

(AC) = fractional day past 0^h of epoch T in E.T.

(MQ) = integer days past 0^h January 1, 1950 of the epoch T

```
CALL MNA
PZE 1,,A
PZE 1,,B
```

The cells A, A + 1, A + 2 contain the 1950.0 position vector $\mathbf{R} = (X, Y, Z)$, while the output vector $\mathbf{r} = (x, y, z)$ in the Moon-fixed coordinate system is placed in the locations B, B + 1, B + 2. The coordinate transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = MNA \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

The inverse transformation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = (MNA)' \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is indicated by

```
CALL MNA
PZE 1,,A
PZE 0,,B
```

A, A + 1, A + 2 contain **r** and the output **R** is placed in B, B + 1, B + 2.

If MNA1 is called instead of MNA, the matrices are not recomputed unless time has changed by 0.01 day.

The subroutines use four cells of erasable storage starting at COMMON.

MNAMD, MNAMD1

As it is necessary to form the Moon-fixed velocity, the subroutine MNAMD has been provided to accomplish this task. As in the preceding discussion of MNA, the formulas for transforming positions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = MNA \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

for the transformation from 1950.0 position to Moon-fixed position and inversely,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = (MNA)' \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for the position transformation in the other direction.

To obtain velocity transformations, the above formulas are differentiated and the approximation is made that

$$\dot{N} = \dot{A} = 0$$

Thus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = MNA \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} + \dot{M}NA \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

and for the inverse transformation

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = (MNA)' \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + (\dot{M}NA)' \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In computing $\dot{M}$ the rates for the slowly varying angles Ω' and i are taken to be zero.

$$\dot{M} = (\dot{M}_{ij})$$

where

$$\dot{M}_{11} = (-\sin\Lambda\cos\Omega' - \cos\Lambda\sin\Omega'\cos i)\,\dot{\Lambda}$$
$$\dot{M}_{12} = (-\sin\Lambda\sin\Omega' + \cos\Lambda\cos\Omega'\cos i)\,\dot{\Lambda}$$
$$\dot{M}_{13} = (\cos\Lambda\sin i)\,\dot{\Lambda}$$
$$\dot{M}_{21} = (-\cos\Lambda\cos\Omega' + \sin\Lambda\sin\Omega'\cos i)\,\dot{\Lambda}$$
$$\dot{M}_{22} = (-\cos\Lambda\sin\Omega' - \sin\Lambda\cos\Omega'\cos i)\,\dot{\Lambda}$$
$$\dot{M}_{23} = (-\sin\Lambda\sin i)\,\dot{\Lambda}$$
$$\dot{M}_{31} = 0$$
$$\dot{M}_{32} = 0$$
$$\dot{M}_{33} = 0$$

From the formula

$$\Lambda = \Delta + (\mathfrak{C} + \tau) - (\Omega + \sigma)$$

obtain

$$\dot{\Lambda} = \dot{\Delta} + \dot{\mathfrak{C}} + \dot{\tau} - \dot{\Omega} - \dot{\sigma}$$

The adopted numerical expressions for the rates are

$$\dot{\Delta} = \frac{-\cos(\Omega + \sigma + \delta\psi)\sin\epsilon\,(\dot{\Omega} + \dot{\sigma})}{\sin i\cos\Delta}$$

$$\dot{\mathfrak{C}} = 0.266170762 \times 10^{-6} - 0.12499171 \times 10^{-13}\,T \text{ rad/sec}$$

$$\dot{\Omega} = -0.1069698435 \times 10^{-7} + 0.23015329 \times 10^{-13} T \text{ rad/sec}$$

$$\dot{\tau} = -0.1535272946 \times 10^{-9}\cos g + 0.569494067 \times 10^{-10}\cos g' + 0.579473484 \times 10^{-11}\cos 2\omega \text{ rad/sec}$$

$$\dot{\sigma} = -0.520642191 \times 10^{-7}\cos g + 0.1811774451 \times 10^{-7}\cos(g + 2\omega) - 0.1064057858 \times 10^{-7}\cos(2\omega + 2g) \text{ rad/sec}$$

The calling sequence to MNAMD has the form

(AC) = fractional day past 0^h of epoch T under consideration

(MQ) = integer days past 0^h January 1, 1950, E.T. to T in E.T.

```
CALL MNAMD
PZE 1,,A
PZE 1,,B
PZE 1,,C
```

The 1950.0 position vector $\mathbf{R} = (X, Y, Z)$ is input to cells A, A + 1, A + 2, while the 1950.0 velocity vector $\mathbf{V} = (\dot{X}, \dot{Y}, \dot{Z})$ occupies locations B, B + 1, B + 2. The output vector $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ is placed in C, C + 1, C + 2.

If the inverse transformation is desired, the calling sequence is modified to read

```
CALL MNAMD
PZE 1,,A
PZE 1,,B
PZE 0,,C
```

The Moon-fixed position vector $\mathbf{r} = (x, y, z)$ occupies cells A, A + 1, A + 2 as the Moon-fixed velocity vector $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ uses B, B + 1, B + 2 for input. The 1950.0 velocity vector $\mathbf{V} = (\dot{X}, \dot{Y}, \dot{Z})$ is the output and is placed in locations C, C + 1, C + 2.

The alternate entry MNAMD1 differs from the entry MNAMD in that the matrices M and $\dot{M}$ are recomputed only if time has changed by 0.01 day.

The subroutines use four words of erasable storage starting at COMMON.

3. Ephemeris

INTR, INTR1

The subroutine INTR assumes a high-density ephemeris tape on A8 with 20-day records of 596 words in the following format:

T_0	integer days past 0^h January 1, 1950, E.T. (floating point)
$X_☾(T_j)$, $\delta^2X_☾(T_j)$, $\delta^4X_☾(T_j)$ $Y_☾(T_j)$, $\delta^2Y_☾(T_j)$, $\delta^4Y_☾(T_j)$ $Z_☾(T_j)$, $\delta^2Z_☾(T_j)$, $\delta^4Z_☾(T_j)$ $X_☉(T_j)$, $\delta^2X_☉(T_j)$, $\delta^4X_☉(T_j)$ $Y_☉(T_j)$, $\delta^2Y_☉(T_j)$, $\delta^4Y_☉(T_j)$ $Z_☉(T_j)$, $\delta^2Z_☉(T_j)$, $\delta^4Z_☉(T_j)$	geocentric block $j = 0,\ldots,20$ time interval is 1 day 378 words
$X_♀(T_j)$, $\delta^2X_♀(T_j)$, $\delta^4X_♀(T_j)$ $Y_♀(T_j)$, $\delta^2Y_♀(T_j)$, $\delta^4Y_♀(T_j)$ $Z_♀(T_j)$, $\delta^2Z_♀(T_j)$, $\delta^4Z_♀(T_j)$ $X_♂(T_j)$, $\delta^2X_♂(T_j)$, $\delta^4X_♂(T_j)$ $Y_♂(T_j)$, $\delta^2Y_♂(T_j)$, $\delta^4Y_♂(T_j)$ $Z_♂(T_j)$, $\delta^2Z_♂(T_j)$, $\delta^4Z_♂(T_j)$ (Nine words per time point representing what was the Earth-Moon barycenter used in an older version) $X_♃(T_j)$, $\delta^2X_♃(T_j)$, $\delta^4X_♃(T_j)$ $Y_♃(T_j)$, $\delta^2Y_♃(T_j)$, $\delta^4Y_♃(T_j)$ $Z_♃(T_j)$, $\delta^2Z_♃(T_j)$, $\delta^4Z_♃(T_j)$	heliocentric block $j = 0, 4, 8, 12, 16, 20$ time interval is 4 days 216 words

The last word of the record is the check sum for the previous 595 words.

From record to record the time must be incremented by 20 days. In addition, the time on the first record T_F and the time on the last record T_L are subroutine parameters which give the base point of the ephemeris and also a check for time out of the range of the ephemeris. The symbolic locations are TFIRST and TLAST for T_F and T_L respectively.

The lunar coordinates are assumed to use the Earth radius as a unit of length, while all other coordinates are expressed in terms of the Astronomical Unit. As the program runs in km, conversion factors are provided at SCALE1 for the Earth radius and SCALE2 for the Astronomical Unit. The rectangular coordinates are assumed to be expressed in the mean equator and equinox of the epoch 1950.0 E.T., the beginning of the Besselian year.

As the argument of the tables is E.T. (Ephemeris Time) and the program uses U.T. (Universal Time), the subroutine E.T. is used to form the double-precision ephemeris time in sec E.T. = U.T. + ΔT, where the constant ΔT appears at GRAV−2 and thus may be input via INP1 in the symbolic mode.

Beginning at GRAV, a list of gravitational coefficients for the bodies appears in the units km³/sec². As a function of the central body, certain sets of these coefficients are provided for the subroutine BODY in the COMMON list KB0, . . . , KB6. The following illustrates the transfers:

Central body	Effective noncentral bodies
Earth	Moon, Sun; Jupiter if $R_{⊕♃} \geqq 10^6$ km
Moon	Earth, Sun
Sun	Earth, Moon, Venus, Mars, Jupiter
Venus	Earth, Moon, Sun, Mars, Jupiter
Mars	Earth, Moon, Venus, Sun, Jupiter
Jupiter	Earth, Moon, Venus, Mars, Sun

The entry INTR takes as argument the double-precision seconds past 0^h January 1, 1950, U.T., stored in T, T + 1, makes the conversion to E.T., and interpolates as a function of the central body on the required coordinates for the bodies listed above. There are two conditions under which actual interpolation takes place:

1. Central body has changed
2. Time has changed

If neither (1) nor (2) is satisfied, then INTR gives an immediate return.

In contrast with INTR, the entry INTR1 always interpolates; in addition, this entry obtains the positions of all the bodies in terms of the central body instead of the selective list used with BODY. The positions appear in the COMMON bank XN. Additionally, INTR1 numerically differentiates the positions to obtain the velocities which are deposited in the bank XN. in COMMON.

Positioning of the ephemeris tape on A8 is accomplished by the following scheme:

1. If $T < T_F$ or $T \geqq T_L$, an error point is given and ABORT is called.
2. If $T \geqq T_N + 20$, where T_N is the time on the record currently in core, the tape is searched in a forward direction until the correct record is found. If the tape has not been previously read, a dummy T_N causes a forward search.
3. If $T_N \leqq T < T_N + 20$, interpolation proceeds.
4. If $T < T_N$, the correct number m of backspaces is calculated.
   a. If $m \leqq 15$, the tape is backspaced m times and proceeds to do a forward search.

b. If $m > 15$, the tape is backspaced 1 file and a forward search is undertaken.

After the correct record has been found, it is read into core and both check-summed and redundancy-tested. Reading of the desired record is attempted a maximum of 10 times, after which an error comment is printed and ABORT is called. In the forward search the above two tests are not made.

The following Everett's formula is used for the interpolation:

$$y(t) = \left\{uy_0 + ty_1\right\} + \left\{\frac{u(u^2-1)}{3!}\delta^2 y_0 + \frac{t(t^2-1)}{3!}\delta^2 y_1\right\} + \left\{\frac{u(u^2-1)(u^2-4)}{5!}\delta^4 y_0 + \frac{t(t^2-1)(t^2-4)}{5!}\delta^4 y_1\right\}$$

where

$$y_0 = y(T_j)$$
$$y_1 = y(T_j + h)$$
h = ephemeris interval
$$t = \frac{T - T_j}{h}$$
$$u = 1 - t$$
$$T_j \leqq T < T + h$$

To obtain a formula for the velocity, the above Everett's form is differentiated and scaled:

$$\dot{y}(T) = \frac{1}{h}\frac{dy(t)}{dt} = \frac{1}{h}\left\{-y_0 + y_1\right\} + \frac{1}{h}\left\{-\frac{3u^2-1}{3!}\delta^2 y_0 + \frac{3t^2-1}{3!}\delta^2 y_1\right\} + \frac{1}{h}\left\{-\frac{5u^4-15u^2+4}{5!}\delta^4 y_0 + \frac{5t^4-15t+4}{5!}\delta^4 y_1\right\}$$

The ephemeris tape currently used has the following modified differences for the Moon:

$$\delta_m^2 y = \delta^2 y - 0.01312\,\delta^6 y + 0.0043\,\delta^8 y$$
$$\delta_m^4 y = \delta^4 y - 0.27827\,\delta^6 y + 0.0685\,\delta^8 y$$

Thus δ_m^2 and δ_m^4 are used in the Everett's formula instead of δ^2 and δ^4 to provide for the influence of the higher differences.

The following constants are used:

E.R. = 6378.165
A.U. = 0.149599×10^9
} km

$$\left.\begin{aligned}
\mu_\oplus &= 0.3986032 \times 10^6 \\
\mu_\mathbb{C} &= 0.4900759 \times 10^4 \\
\mu_\odot &= 0.132715445 \times 10^{12} \\
\mu_\mathbb{Q} &= 0.3247695 \times 10^6 \\
\mu_\sigma &= 0.4297780 \times 10^5 \\
\mu_\mathbb{4} &= 0.1267106 \times 10^9
\end{aligned}\right\} \text{ km}^3/\text{sec}^2$$

The subroutine uses 20 cells of erasable storage starting at COMMON.

4. Encke Method Calculations

ENCKE, ORTHO

The subroutine ENCKE has been provided to perform the calculation of the Encke contribution to the acceleration

$$\frac{\mu}{R_0^3}\left(\mathbf{R}F(Q) - \boldsymbol{\rho}\right)$$

where $\mathbf{R} = \mathbf{R}_0 + \boldsymbol{\rho}$. The solution $\mathbf{R}_0$ for the position in the two-body orbit is provided by KEPLER, and is saved from step to step so that a new $\mathbf{R}_0$ is calculated only when the time has changed; thus KEPLER is called normally once per integration step while using the Adams-Moulton predictor-corrector.

$F(Q) = 1 - (1 + 2Q)^{-3/2}$ is calculated from the series expansion

$$F(Q) = Q\sum_{j=0}^{6} a_j Q^j$$

where

$$a_0 = 3$$
$$a_1 = -7.5$$
$$a_2 = 17.5$$
$$a_3 = -39.375$$
$$a_4 = 86.625$$
$$a_5 = -187.6875$$
$$a_6 = 402.1875$$

and

$$Q = \frac{\boldsymbol{\rho}\cdot\left(\mathbf{R}_0 + \frac{\boldsymbol{\rho}}{2}\right)}{R_0^2}$$

The expansion gives accurate results for $|Q| \leqq 0.03$; if the limit is exceeded, an error print will be given and the trajectory will be terminated. Normally, Q grows slowly enough so that rectification may be performed at the end

of the integration step; however, for wild trajectories the error procedure has been observed to occur.

ENCKE deposits the true position $\mathbf{R}$ in the COMMON cells QX, QX + 1, QX + 2 and the acceleration term $\mu/R_0^3\,(\mathbf{R}\,F(Q) - \rho)$ in the cells CX.., CX.. + 1, CX.. + 2.

At the osculation epoch T_0 the subroutine ORTHO provides the Encke scheme with initial conditions:

$$\rho\,(T_0) = \mathbf{R}(T_0) - \mathbf{R}_0(T_0)$$
$$\dot{\rho}\,(T_0) = \mathbf{V}(T_0) - \mathbf{V}_0(T_0)$$

$\rho\,(T_0)$ is placed in the COMMON cells CX, CX + 1, CX + 2, while $\dot{\rho}\,(T_0)$ is placed in storage locations CX., CX. + 1, CX. + 2.

CONIC

The subroutine CONIC supplies the Encke machinery with the necessary orbital elements given an epoch T_0 and the Cartesian position and velocity vectors $\mathbf{R}_0$ and $\mathbf{V}_0$. Under certain circumstances described below the derived elements are nonosculating.

The computation starts with the formation of the angular momentum c_1 given by

$$c_1\,\mathbf{W} = \mathbf{R}_0 \times \mathbf{V}_0$$

If $c_1 > (0.99\epsilon)R_0V_0$, where $\epsilon = 0.5 \times 10^{-3}$, the orbit is considered nonrectilinear and the subroutine proceeds in the normal case. However, for $c_1 \leqq (0.99\epsilon)R_0V_0$, $\mathbf{V}_0$ is replaced by $\mathbf{V}_0^*$ given by

$$\mathbf{V}_0^* = V_0\left[\sqrt{1-\epsilon^2}\,\operatorname{sgn}\,(\mathbf{R}_0\cdot\mathbf{V}_0)\,\frac{\mathbf{R}_0}{R_0} + \epsilon\mathbf{M}\right]$$

where

$$\mathbf{M} = \begin{cases} \dfrac{1}{\sqrt{X_0^2 + Y_0^2}}(Y_0, -X_0, 0) & \text{if } X_0^2 + Y_0^2 \neq 0, \\ (1,0,0) & \text{otherwise} \end{cases}$$

and the routine cycles back to recompute the angular momentum. Observe that $c_1^* = \epsilon R_0V_0$ so that $\mathbf{V}_0^*$ is acceptable; of course, $V_0^* = V_0$ and $c_3^* = c_3$.

Next come the elements

$$p = \frac{c_1^2}{\mu}\text{, the semilatus rectum}$$

$$c_3 = V_0^2 - \frac{2\mu}{R_0}\text{, the "energy" or } \textit{vis viva} \text{ integral}$$

$$1 - \epsilon^2 = -\frac{c_1^2 c_3}{\mu^2}$$

At this point the eccentricity ϵ is computed and tested:

$$\epsilon = \begin{cases} \sqrt{1-(1-\epsilon^2)} & \text{if radicand} > 0, \\ 0 & \text{otherwise} \end{cases}$$

If the computed ϵ is smaller than 0.01, then a circular orbit is assumed and the remaining elements are made consistent with the assumption of $\epsilon = 0$. There follows in quick succession

$$q = \frac{p}{1+\epsilon}\text{, the closest approach distance}$$

$$\lambda = \frac{1-\epsilon^2}{(1+\epsilon)^2}\text{, the pericenter parameter}$$

$$g = \frac{c_1}{2q^2}\text{, the mean motion for the pericenter method}$$

$$a = \frac{\mu}{|c_3|}\text{, the semimajor or transverse axis}$$

$$n = \frac{\sqrt{|c_3|}}{a}\text{, the mean motion}$$

$$b = a\sqrt{|1-\epsilon^2|}\text{, the semiminor or conjugate axis}$$

and finally,

$$1 - \epsilon = \frac{1-\epsilon^2}{1+\epsilon}$$

It remains to calculate $\mathbf{P}$ and $\mathbf{Q}$ along with $\Delta T = T_0 - T_p$ and to make the two sets agree sufficiently so that the Encke starting values will not be too large.

If $\epsilon = 0$,

$$\mathbf{P} = \frac{\mathbf{R}_0}{R_0} \text{ and } \mathbf{Q} = \frac{\mathbf{W}\times\mathbf{R}_0}{|\mathbf{W}\times\mathbf{R}_0|} \text{ with } \Delta T = 0$$

Otherwise, the vectors are constructed:

$$\epsilon R_0^2\,\mathbf{P} = (\epsilon R_0\cos v_0)\,\mathbf{R}_0 - (\epsilon R_0 \sin v_0)\,\mathbf{W}\times\mathbf{R}_0$$
$$\epsilon R_0^2\,\mathbf{Q} = (\epsilon R_0\sin v_0)\,\mathbf{R}_0 + (\epsilon R_0 \cos v_0)\,\mathbf{W}\times\mathbf{R}_0$$

Divide by R_0, and normalize the resultant vectors to obtain $\mathbf{P}$ and $\mathbf{Q}$. The expressions involving the true anomaly at epoch are calculated from

$$\epsilon R_0\cos v_0 = p - R_0$$

$$\epsilon R_0 \sin v_0 = \frac{c_1}{\mu}\mathbf{R}_0\cdot\mathbf{V}_0$$

To obtain ΔT, the applicability of the pericenter method for $|\lambda| < 0.45$ is tested. w_0 is formed according to

$$w_0 = \begin{cases} \dfrac{\sin v_0}{1+\cos v_0} & \text{if } \cos v_0 \geqq 0 \\ \dfrac{1-\cos v_0}{\sin v_0} & \text{otherwise} \end{cases}$$

and tested. If $|w_0| \leqq w_{max}$, then

$$g\Delta T = \sum_{j=0}^{6} a_j(\lambda)\, w_0^{2j}$$

where the coefficients $a_j(\lambda)$ along with w_{max} are given in the discussion of the subroutine PERI.

Whenever $|w_0| > w_{max}$, the eccentric anomaly and Kepler's equation are resorted to. The scheme is divided into two cases according to the value of ε:

(1) $\varepsilon < 1$, elliptic case

The following expressions are constructed for the eccentric anomaly:

$$\varepsilon \cos E_0 = 1 - \frac{R_0}{a}$$

$$\varepsilon \sin E_0 = \frac{\mathbf{R}_0 \cdot \mathbf{V}_0}{a\sqrt{|c_3|}}$$

from which E_0 is determined.

If $|\varepsilon \sin E_0| \leqq |\varepsilon \cos E_0|$, then the auxiliary variable E^* is constructed:

$$E^* = \sin^{-1} \frac{\mathbf{R}_0 \cdot \mathbf{V}_0}{a\varepsilon\sqrt{|c_3|}}, \quad -\frac{\pi}{2} < E^* < \frac{\pi}{2}$$

Then E_0 is given by

$$E_0 = \begin{cases} E^* & \text{if } 1 - \dfrac{R_0}{a} > 0 \\ \pi\, \mathrm{sgn}\,(E^*) - E^* & \text{otherwise} \end{cases}$$

On the other hand, if $|\varepsilon \sin E_0| > |\varepsilon \cos E_0|$, then

$$E^* = \cos^{-1} \frac{1}{\varepsilon}\left(1 - \frac{R_0}{a}\right), \quad 0 < E^* < \pi,$$

with

$$E_0 = \mathrm{sgn}\,(\mathbf{R}_0 \cdot \mathbf{V})\, E^*$$

Finally, ΔT is calculated from

$$n\Delta T = E_0 - \varepsilon \sin E_0 = E_0 - \frac{\mathbf{R}_0 \cdot \mathbf{V}_0}{a\sqrt{|c_3|}}$$

(2) $\varepsilon > 1$, hyperbolic case

The eccentric anomaly F_0 is found from the relation

$$\varepsilon \sinh F_0 = \frac{\mathbf{R}_0 \cdot \mathbf{V}_0}{a\sqrt{c_3}} = \varepsilon\, \alpha$$

$$F_0 = \mathrm{sgn}\,(\mathbf{R}_0 \cdot \mathbf{V}_0)\, \ln\left(|\alpha| + \sqrt{1 + \alpha^2}\right)$$

Then ΔT is obtained from Kepler's equation:

$$n\Delta T = \varepsilon \sinh F_0 - F_0 = \frac{\mathbf{R}_0 \cdot \mathbf{V}_0}{a\sqrt{c_3}} - F_0$$

QUADKP

The subroutine QUADKP was written to provide an iterative solution to Kepler's equation for the elliptic and hyperbolic cases using a second-order gradient method. However, only the machinery for the latter case has been utilized in the main program for the Encke solution.

Let Kepler's equation be represented in the hyperbolic case by

$$f(F) = \varepsilon \sinh F - F - M, \quad M = n\,(T - T_p)$$

Then for the approximate solution F_j the Taylor series expansion through second-order terms may be used to obtain a new estimate F_{j+1} of the root $f(F) = 0$.

$$0 = f(F_j + \delta F_j) = f(F) \approx f(F_j) + \delta F_j f'(F_j) + \frac{\delta F_j^2}{2} f''(F_j)$$

Solving directly for the roots of the quadratic,

$$\delta F_j = F_{j+1} - F_j = \frac{2f(F_j)}{-f'(F_j) - \sqrt{f'^2(F_j) - 2f(F_j)f''(F_j)}}$$

where the minus sign is taken before the radical to insure that $\delta F_j \to 0$ as $f(F_j) \to 0$; for a wild guess, however, the radicand may become negative, in which case the radical is replaced by zero. With a good initial approximation the latter case arises only infrequently.

A similar result may be obtained for the elliptic case, namely

$$f(E) = E - \varepsilon \sin E - M$$

Convergence in either the hyperbolic or elliptic case is evidently given by

$$F_{j+1} - F = O\left((F_j - F)^3\right)$$

The program is made complex by the attention necessarily paid to obtaining a good initial approximation for starting the higher order iteration scheme and the behavior of Kepler's equation when ε is near 1 and M is small.

The initial approximation is obtained as a function of ε and M:

(1) $\varepsilon > 1.1, \quad F_{-1} = \dfrac{M}{\varepsilon - 1}$

$$F_0 = \begin{cases} \operatorname{sgn}(M) \min\left\{1, \ln \dfrac{2|M|}{\varepsilon}\right\} & \text{if } |F_{-1}| > 1 \\ F_{-1} - \dfrac{\dfrac{F_{-1}^3}{3!} + \dfrac{F_{-1}^5}{5!}}{\dfrac{\varepsilon - 1}{\varepsilon} + \dfrac{F_{-1}^2}{2!} + \dfrac{F_{-1}^4}{4!}} & \text{if } |F_{-1}| \leqq 1 \end{cases}$$

(1 iteration by Newton's method)

(2) $1 \leqq \varepsilon \leqq 1.1$

$$F_0 = \begin{cases} \operatorname{sgn}(M) \min\left\{1, \ln \dfrac{2|M|}{\varepsilon}\right\} & \text{if } |M| > 1 \\ (6M)^{1/3} & \text{otherwise} \end{cases}$$

(3) $0.9 < \varepsilon \leqq 1$

$$E_0 = (6M)^{1/3}$$

(4) $0 < \varepsilon \leqq 0.9, \quad E_{-1} = \dfrac{M}{1 - \varepsilon}$

$$E_0 = \begin{cases} 2.5 \times \operatorname{sgn}(M) & \text{if } |E_{-1}| > 3 \\ E_{-1} - \dfrac{\dfrac{E_{-1}^3}{3!} - \dfrac{E_{-1}^5}{5!}}{\dfrac{1 - \varepsilon}{\varepsilon} + \dfrac{E_{-1}^2}{2!} - \dfrac{E_{-1}^4}{4!}} & \text{if } |E_{-1}| \leqq 3 \end{cases}$$

(1 iteration by Newton's method)

(5) $\varepsilon = 0$

$$E_0 = M = E$$

In the iteration scheme, a modification occurs in the numerical evaluation of f, f', and f'' for the hyperbolic case.

(1) For $|F_j| < 1.98$

$$\varepsilon \sinh F_j - F_j = (\sinh F_j - F_j) + (\varepsilon - 1) \sinh F_j$$

$$\sinh F_j - F_j = \frac{F_j^3}{3!} \sum_{i=0}^{5} \frac{(3!)^{i+1}}{(2i+3)!} \left(\frac{F_j^2}{3!}\right)^i$$

$$\sinh F_j = F_j + (\sinh F_j - F_j)$$

(2) For $|F_j| < 2.28$

$$\cosh F_j - 1 = \frac{F_j^2}{2!} \sum_{i=0}^{6} \frac{(2!)^{i+1}}{(2i+2)!} \left(\frac{F_j^2}{2!}\right)^i$$

$$\varepsilon - \cosh F_j = (\varepsilon - 1) - (\cosh F_j - 1)$$

The power series developments are employed since the quantity $1 - \varepsilon$ is regarded as an independent orbital element and ε is made consistent with this choice. For small values of F_j, underflow is avoided by choosing fewer terms in the expansions.

Additionally, if $M = 0$, F is set to zero and no iteration is performed. The calling sequence has the form

```
(AC)  = M
(MQ)  = 1 - ε
CALL  QUADKP
DEC   ε
PZE   A
(ERROR RETURN)
```

For the elliptic case, it is assumed that M has been normalized so that $|M| \leqq \pi$.

The location A contains a positive number for the elliptic case and a negative number for the hyperbolic case to choose the correct form of Kepler's equation for the rectilinear orbit.

As a convergence criterion, $\epsilon = 5 \times 10^{-8}$ has been chosen to apply such that the normal return is given with (AC) $= F$ or E whenever $|f(F_j)| < \epsilon |M|$ and $F = F_j$ for the hyperbolic case, or $|f(E_j)| < \epsilon |M|$ and $E = E_j$ for the elliptic case. However, if the process fails to converge to within ϵ in $N = 50$ iterations, the following comment is printed:

QUADRATIC METHOD FAILED

M ε F_j (or E_j) $f(F_j)$ [or $f(E_j)$]

and the error return is given with (AC) $= F_j$ or E_j. Generally, convergence is obtained in five or fewer iterations. After the first time QUADKP has been called in a phase, subsequent initial approximations are obtained by using the solution at the previous time point.

The subroutine uses eleven words of erasable storage starting at COMMON.

KEPLER, PERI, SPEED

KEPLER is the subroutine which provides the solution to the two-body problem at the epoch T. The necessary elements are assumed to have been provided at the osculation epoch by ORTHO and CONIC. Different methods of solution are chosen according to the following criteria:

1. The pericenter method is used whenever
   a. $|\lambda| < 0.45$ and
   b. $|w_0| \leqq g\,\Delta T_{\max}$
2. QUADKP is used for $\epsilon > 1$ and if item 1 is not satisfied.
3. If the conditions in item 1 are not met and if $\epsilon < 1$, then the methods described below for the ellipse are used.

As the orbital elements furnished by CONIC are nonosculating if true osculating elements give nearly rectilinear results, the case for $\epsilon = 1$ and $c_3 \neq 0$ need not be treated.

Pericenter Method

The first problem to be solved in the pericenter method is the determination of w from the formula

$$w_0 = g\,(T - T_p) = \int_0^w \frac{1 + u^2}{(1 + \lambda u^2)^2}\,du$$

where $g = c_1/2q^2$. The quadrature is approximated by the expansion

$$w \sum_{j=0}^{6} a_j w^{2j}$$

where the coefficients are a function of λ:

$$a_0 = 1$$

$$a_j = (-1)^j \frac{1}{2j+1}\left[(j+1)\,\lambda^j - j\lambda^{j-1}\right]$$

$$w_{\max} = \max\left\{\left(\frac{\epsilon}{a_7}\right)^{1/14}, \quad \left(\frac{\epsilon}{3a_7}\right)^{1/12}\right\},$$

where $\epsilon = 0.5 \times 10^{-8}$.

$$g\,\Delta T_{\max} = w_{\max} \sum_{j=0}^{6} a_j\, w_{\max}^{2j}$$

w_1, the initial approximation, is given by

$$w_1 = \begin{cases} (3w_0)^{1/3} & \text{if } |w_0| > 3 \\ 1/2\, w_0 & \text{if } 1 < |w_0| \leqq 3 \\ w_0 & \text{if } 0 \leqq |w_0| \leqq 1 \end{cases}$$

Iteration proceeds by Newton's method:

$$w_{j+1} = w_j - \frac{f(w_j)}{f'(w_j)}$$

where

$$f(w) = w \sum_{j=0}^{6} a_j w^{2j} - w_0$$

and

$$f'(w) = \frac{1 + w^2}{(1 + \lambda w^2)^2}$$

Convergence is usually obtained in a maximum of four iterations for cases which have arisen in practice, assuming a criterion of $|f(w_j)| < 5 \times 10^{-8}\,|w_0|$.

When convergence has been obtained the coordinates may be calculated from the formulas

$$\mathbf{R} = \frac{1 - w^2}{1 + \lambda w^2}\,q\,\mathbf{P} + \frac{2w}{1 + \lambda w^2}\,q\,\mathbf{Q}$$

$$\mathbf{V} = -\frac{(1 + \lambda)\,w\,c_1}{q\,(1 + w^2)}\,\mathbf{P} + \frac{(1 - \lambda w^2)\,c_1}{q\,(1 + w^2)}\,\mathbf{Q}$$

Hyperbolic Case

If method 2 is to be used, the subroutine QUADKP is called and returns with F, the solution to Kepler's equation. The coordinates are then calculated from the expressions

$$\mathbf{R} = a\,(\epsilon - \cosh F)\,\mathbf{P} + a\sqrt{\epsilon^2 - 1}\,\sinh F\,\mathbf{Q}$$

$$\mathbf{V} = -\frac{a\sqrt{c_3}\,\sinh F}{R}\,\mathbf{P} + \frac{a\sqrt{c_3}\,\sqrt{\epsilon^2 - 1}\,\cosh F}{R}\,\mathbf{Q}$$

Elliptic Case

Begin by defining the auxiliary quantity M^* by

$$M^* \equiv M = n(T - T_p) \pmod{2\pi} \quad -\pi < M^* \leqq \pi$$

and

$$h_2 = -\left\{\operatorname{sgn}(M^*)\right\}\frac{\epsilon}{2}$$

Then for $|M^*| \leqq \pi/6 - \epsilon/2$, take

$$E_2 = M$$

$$E_1 = M - h_2$$

$$E_0 = M + h_2$$

and for $|M^*| > \pi/6 - \epsilon/2$ the approximations are

$$E_2 = M - h_2$$

$$E_1 = M - 2h_2$$

$$E_0 = M$$

Now if $|M^*| \geqq 0.25$, iteration is performed using Newton's method with E_2 as the initial approximation:

$$E_{j+1} = E_j - \frac{f(E_j)}{f'(E_j)}$$

where $f(E) = E - \epsilon \sin E - M$. Convergence for Newton's method is evidently given by

$$E_{j+1} - E = O((E_j - E)^2)$$

where $f(E) = 0$.

Whenever $|M^*| < 0.25$ Muller's method[2] is used:

$$E_{j+1} = E_j + h_{j+1}$$

$$h_{j+1} = \lambda_{j+1} h_j$$

$$\lambda_{j+1} = -\frac{2\delta_j f(E_j)}{g_j + \{\mathrm{sgn}(g_j)\}\sqrt{|a_j|}}$$

$$a_j = g_j^2 - 4\lambda_j \delta_j f(E_j) \times [\lambda_j f(E_{j-2}) - \delta_j f(E_{j-1}) + f(E_j)]$$

$$g_j = \lambda_j^2 f(E_{j-2}) - \delta_j^2 f(E_{j-1}) + (\lambda_j + \delta_j) f(E_j)$$

$$\delta_j = 1 + \lambda_j$$

Initially, of course,

$$\lambda_2 = \frac{E_2 - E_1}{E_1 - E_0} = -\frac{1}{2}$$

The convergence rate is given by

$$E_{j+1} - E = O(\epsilon_M^3)$$

where $\epsilon_M = \max(|E_j - E|, |E_{j-1} - E|, |E_{j-2} - E|)$ and $f(E) = 0$. The method owes much of its usefulness to the obtaining of E_{j+1} by interpolation, which makes it relatively insensitive to $f'(E) \approx 0$.

For both the Newton and Muller methods, convergence is defined by

$$|f(E_j)| \leqq \epsilon |M|$$

where $\epsilon = 5 \times 10^{-8}$ or until E_{j+1} and E_j agree to 25 bits. In practice, for the values of M and ϵ encountered, three or four iterations are usually sufficient for convergence.

Whenever $0.5 < \epsilon < 1$, the following series expansions are resorted to:

(1) For $|E_j| < 1.98$

$$E_j - \epsilon \sin E_j = (E_j - \sin E_j) + (1 - \epsilon) \sin E_j$$

$$E_j - \sin E_j = \frac{E_j^3}{3!} \sum_{i=0}^{5} \frac{(3!)^{i+1}}{(2i+3)!} \left(-\frac{E_j^2}{3!}\right)^i$$

(2) For $|E_j| < 2.28$

$$1 - \cos E_j = \frac{E_j^2}{2!} \sum_{i=0}^{6} \frac{(2!)^{i+1}}{(2i+2)!} \left(-\frac{E_j^2}{2!}\right)^i$$

$$\cos E_j - \epsilon = (1 - \epsilon) - (1 - \cos E_j)$$

Here ϵ is made consistent with the choice of $1 - \epsilon$ as an independent orbital element.

The coordinates are obtained after convergence by use of the formulas

$$\mathbf{R} = a(\cos E - \epsilon)\,\mathbf{P} + a\sqrt{1 - \epsilon^2} \sin E\,\mathbf{Q}$$

$$\mathbf{V} = -\frac{a\sqrt{|c_3|} \sin E}{R}\,\mathbf{P} + \frac{a\sqrt{|c_3|}\sqrt{1 - \epsilon^2} \cos E}{R}\,\mathbf{Q}$$

As usually only the position is required for the two-body orbit in the Encke scheme, KEPLER is additionally used to calculate **R**. Whenever **V** is needed, the subroutine SPEED is called upon, which makes use of the previous solution E, F, or w. **R** is placed in the COMMON cells QX0, QX0 + 1, QX0 + 2 by KEPLER, while SPEED places **V** in the cells QX0., QX0. + 1, QX0. + 2 and also calculates the true velocity $\mathbf{V} + \dot{\rho}$ which is placed in the cells QX., QX. + 1, QX. + 2.

After the first use of KEPLER in a phase, further time points use as initial approximation the solution at the preceding time point.

5. Perturbations

HARMN, HARMN1

The oblate potential of the Earth is assumed to contain the second, third, and fourth spherical harmonics:

$$U_\oplus = \frac{\mu_\oplus}{R}\left\{\frac{Ja_\oplus^2}{3R^2}(1 - 3\sin^2\phi) + \frac{Ha_\oplus^3}{5R^3}(3 - 5\sin^2\phi)\sin\phi + \frac{Da_\oplus^4}{35R^4}(3 - 30\sin^2\phi + 35\sin^4\phi)\right\}$$

where $\mu_\oplus$ is the gravitational coefficient of the Earth and $a_\oplus$ is the equatorial radius of the Earth.

$\mathbf{R} = (X, Y, Z)$ is the position vector from the Earth's center of mass expressed in the mean equator and equi-

[2] David E. Muller, "A Method for Solving Algebraic Equations Using an Automatic Computer," *Mathematical Tables and Other Aids to Computation*, 1956, pp. 208-215.

nox of 1950.0. To obtain ϕ, the geocentric latitude, $\mathbf{r} = (x, y, z)$, the position vector expressed in the true equator and equinox of date, must be obtained. NUTATE provides the necessary rotation matrix A:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Thus $\sin \phi = z/R$.

To obtain the perturbing acceleration, $\nabla U_\oplus$ is formed:

$$\nabla U_\oplus = \left(\frac{\partial U_\oplus}{\partial u_1}, \frac{\partial U_\oplus}{\partial u_2}, \frac{\partial U_\oplus}{\partial u_3} \right)$$

where $u_1 = X$, $u_2 = Y$, and $u_3 = Z$.

$$\frac{\partial U_\oplus}{\partial u_j} = -\frac{J\mu_\oplus}{R^2}\frac{a_\oplus^2}{R^2}\left\{\left(1 - \frac{5z^2}{R^2}\right)\frac{u_j}{R} + 2\frac{z}{R}a_{3j}\right\}$$

$$-\frac{H\mu_\oplus}{R^2}\frac{a_\oplus^3}{R^3}\left\{\left(3 - 7\frac{z^2}{R^2}\right)\frac{z}{R}\frac{u_j}{R} + \left(-\frac{3}{5} + \frac{3z^2}{R^2}\right)a_{3j}\right\}$$

$$-\frac{D\mu_\oplus}{R^2}\frac{a_\oplus^4}{R^4}\left\{\left(\frac{3}{7} - 6\frac{z^2}{R^2} + 9\frac{z^4}{R^4}\right)\frac{u_j}{R} + \left(\frac{12}{7} - 4\frac{z^2}{R^2}\right)\frac{z}{R}a_{3j}\right\}$$

where $j = 1, 2, 3$.

The calling sequence for the setup entry is

```
CALL    HARMN
PZE     X,,B
PZE     K,,ZT
PZE     R
```

X, X + 1, X + 2 contain the vector $\mathbf{R} = (X, Y, Z)$; B, B + 1, B + 2 will contain $-\nabla U_\oplus$, the negative of the perturbing acceleration; K contains $\mu_\oplus$, the Earth's gravity coefficient; ZT contains z, the distance above the true equator of the Earth; and R contains R, the distance to the center of the Earth.

HARMN1 is the execution entry which assumes the above storage layout. In addition, provisions have been made to omit the calculation of the various harmonics as a function of the geocentric range. The internal parameters are listed in the following:

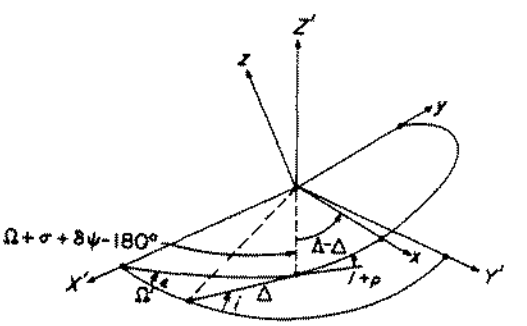
Location	Quantity	Nominal value	Explanation
HARMN + 2	J	1.62345×10^{-3}	Coefficient for second harmonic
+ 3	H	-0.575×10^{-3}	Coefficient for third harmonic
+ 4	D	0.7875×10^{-5}	Coefficient for fourth harmonic
+ 5	$a_\oplus$	6378.165 km	Earth radius
+ 6	R_2	500,000 km	$R > R_2$ suppress second harmonic
+ 7	R_3	200,000 km	$R > R_3$, suppress third harmonic
+ 10	R_4	100,000 km	$R > R_4$, suppress fourth harmonic

As HARMN is contained in the symbol table for INP1, the above parameters may be input in the symbolic mode of INP1.

The subroutine uses 15 cells of erasable storage starting at COMMON.

XYZDD, XYZDD1

For purposes of computing the oblate potential, the Moon is assumed to have a triaxial ellipsoidal figure. The moments of inertia A, B, and C are taken about the principal axes of the ellipsoid x, y, and z originated at the Moon's center of mass.



Sketch A-16. Geometry of the true equator of the Moon

In Sketch A-16, the X', Y', Z' frame is the Earth's true equator and equinox; the $x-y$ plane lies in Moon's true equator with z completing the right-hand system by lying along the Moon's spin axis. i is the inclination of the Moon's true equator to the Earth's true equator; Ω' is the right ascension of the ascending node of the Moon's true equator; Λ is the anomaly from the node to the x axis; Δ is the anomaly from the node to the ascending node of the Moon's true equator on the ecliptic; ϵ is the true obliquity of the ecliptic; $\delta\psi$ is the nutation in longitude; Ω is the mean longitude of the descending node of the Moon's mean equator on the ecliptic; $\mathbb{C}$ is the mean longitude of the Moon; I is the inclination of the Moon's mean equator to the ecliptic; σ is the libration in the node; τ is the libration in the mean longitude; and ρ is the libration in the inclination. The anomalies are related by $\Lambda - \Delta = (\mathbb{C} + \tau) - (\Omega + \sigma)$. Expressions for the above quantities appear in the discussion of subroutine MNA.

The two rectangular systems are related through Λ, Ω', and i by the rotation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

where

$$b_{11} = \cos\Lambda\cos\Omega' - \sin\Lambda\sin\Omega'\cos i$$

$$b_{12} = \cos\Lambda\sin\Omega' + \sin\Lambda\cos\Omega'\cos i$$

$$b_{13} = \sin\Lambda\sin i$$

$$b_{21} = -\sin\Lambda\cos\Omega' - \cos\Lambda\sin\Omega'\cos i$$

$$b_{22} = -\sin\Lambda\sin\Omega' + \cos\Lambda\cos\Omega'\cos i$$

$$b_{23} = \cos\Lambda\sin i$$

$$b_{31} = \sin\Omega'\sin i$$

$$b_{32} = -\cos\Omega'\sin i$$

$$b_{33} = \cos i$$

Combining the above rotation with the one to rotate 1950.0 coordinates to true of-date, as described in NUTATE, derives the additional relation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where X, Y, and Z are the 1950.0 coordinates.

The following form of the potential function which accounts for a second harmonic has been adopted:

$$U_{\mathbb{C}} = \frac{G}{R}\,\frac{(A+B+C-3I)}{2R^2}$$

$$G = \frac{\mu_{\mathbb{C}}}{m_{\mathbb{C}}} = k^2, \text{ the universal gravitational constant}$$

$$I = A\left(\frac{x}{R}\right)^2 + B\left(\frac{y}{R}\right)^2 + C\left(\frac{z}{R}\right)^2$$

To obtain an expression for the perturbing acceleration

$$\nabla U_{\mathbb{C}} = \left(\frac{\partial U_{\mathbb{C}}}{\partial u_1}, \frac{\partial U_{\mathbb{C}}}{\partial u_2}, \frac{\partial U_{\mathbb{C}}}{\partial u_3}\right)$$

is formed, where $u_1 = X$, $u_2 = Y$, and $u_3 = Z$.

$$\frac{\partial U_{\mathbb{C}}}{\partial u_j} = \frac{G}{R^2}\left\{\left[-\frac{3}{2}\,\frac{A+B+C}{R^2} + \frac{15}{2}\,\frac{I}{R^2}\right]\frac{u_j}{R} - \frac{3}{R^3}\left[Am_{1j}x + Bm_{2j}y + Cm_{3j}z\right]\right\}$$

where $j = 1, 2, 3$. In current use, the values of the parameters are

$$G = 0.6671 \times 10^{-19}\ \text{km}^3/\text{kg-sec}^2$$

$$A = 0.88746 \times 10^{29}\ \text{kg-km}^2$$

$$B = 0.88764 \times 10^{29}\ \text{kg-km}^2$$

$$C = 0.88801 \times 10^{29}\ \text{kg-km}^2$$

The calling sequences are

(AC) = fractional days past 0^h of epoch

(MQ) = integer days past 0^h January 1, 1950, E.T.

CALL XYZDD (or XYZDD1 for time change check)

PZE 1,,X

PZE ,,X..

X, X + 1, X + 2 contain **R** in the 1950.0 system; X.., X.. + 1, X.. + 2 will contain the perturbing acceleration.

If the entry XYZDD1 is used, the matrix (m_{ij}) is recomputed only after time has changed by $d = 0.01$ day, where d is a program parameter. On the other hand, the entry XYZDD will give recalculation of (m_{ij}) each time. It has been determined by numerical experimentation that $d = 0.01$ day gives the perturbing acceleration to sufficient accuracy to represent faithfully the motion of a

low-altitude satellite in the field of an oblate Moon, as compared with an evaluation of (m_{ij}) at each integration step.

If $R > R_e = 40{,}000$ km, then the contribution from the oblateness is set to zero. R_e is a program parameter.

The subroutine uses six cells of erasable storage starting at COMMON.

BODY, BODY1

The subroutine BODY has been provided to perform the calculation of the n-body perturbation term

$$\mathbf{P} = -\sum_{j=1}^{n} \mu_j \left(\frac{\mathbf{R}_{jp}}{R_{jp}^3} + \frac{\mathbf{R}_j}{R_j^3} \right)$$

where $\mathbf{R}_{jp} = \mathbf{R} - \mathbf{R}_j$.

The subroutine has the execution entry

```
CALL BODY1
```

and the setup entry

```
CALL  BODY
PZE   X,,n
PZE   XN,,KJ
PZE   RJ,,RJP
PZE   X..
```

where the locations X, X + 1, X + 2, contain the vector **R**, the position of the probe with respect to the central body. The maximum number of noncentral bodies is given by n; the gravitational coefficients μ_j for the noncentral bodies are assumed to be stored in the list KJ, ..., KJ + (n − 1) with the convention that a cell containing zero means that the corresponding body is not used in the formation of **P**. The vectors $\mathbf{R}_j$, the positions of the n bodies with respect to the central body, are assumed to be stored in the bank XN, ..., XN + 3(n − 1) + 2 where the ordering is the same as for the μ_j.

The execution entry results in three types of output: $-\mathbf{P}$ is stored in the cells X.., X.. + 1, X.. + 2; the R_j for the effective bodies are stored in locations RJ, ..., RJ + (n − 1), while the R_{jp} for the same bodies are placed in the list RJP, ..., RJP + (n − 1).

The subroutine uses 14 cells of erasable storage starting at COMMON.

6. Variational Equations

VARY, SVARY

The subroutine VARY has been written to provide for the calculation of the derivatives of the first-order variational coefficients, i.e., of the partial derivatives $\partial\mathbf{R}/\partial u_j$ where $\{u_j\} = \{X_0, Y_0, Z_0, \dot{X}_0, \dot{Y}_0, \dot{Z}_0\}$ and all quantities are referred to the mean equator and equinox of 1950.0. The $\partial\ddot{\mathbf{R}}/\partial u_j$ may be expressed in the form $\partial\ddot{\mathbf{R}}/\partial u_j = (A + B)\,\partial\mathbf{R}/\partial u_j$ where the matrix A arises from the central-body term and the n-body perturbation and B approximates the effect of the Earth's oblateness to be used only in the vicinity of the Earth.

The form of A is obtained by differentiating $\ddot{\mathbf{R}}$ with respect to u_j and exchanging the order of differentiation where

$$\ddot{\mathbf{R}} = -\mu\frac{\mathbf{R}}{R^3} - \sum_{k=1}^{n} \mu_k \left\{ \frac{\mathbf{R}_{kp}}{R_{kp}^3} + \frac{\mathbf{R}_k}{R_k^3} \right\}$$

$$\frac{\partial\ddot{\mathbf{R}}}{\partial u_j} = -\sum_{k=0}^{n} \mu_k \left\{ \frac{1}{R_{kp}^3}\frac{\partial\mathbf{R}}{\partial u_j} - \frac{3}{R_{kp}^5}\left(\mathbf{R}_{kp}\cdot\frac{\partial\mathbf{R}}{\partial u_j}\right)\mathbf{R}_{kp} \right\}$$

with $\mu_0 = \mu$ and $\mathbf{R}_{0p} = \mathbf{R}$. Expanding the dot products, the computational form of A results:

$$A_{11} = -\sum_{k=0}^{n} \mu_k \left\{ \frac{1}{R_{kp}^3} - \frac{3X_{kp}^2}{R_{kp}^5} \right\}$$

$$A_{12} = A_{21} = 3\sum_{k=0}^{n} \mu_k \frac{X_{kp}Y_{kp}}{R_{kp}^5}$$

$$A_{13} = A_{31} = 3\sum_{k=0}^{n} \mu_k \frac{X_{kp}Z_{kp}}{R_{kp}^5}$$

$$A_{22} = -\sum_{k=0}^{n} \mu_k \left\{ \frac{1}{R_{kp}^3} - \frac{3Y_{kp}^2}{R_{kp}^5} \right\}$$

$$A_{23} = A_{32} = 3\sum_{k=0}^{n} \mu_k \frac{Y_{kp}Z_{kp}}{R_{kp}^5}$$

$$A_{33} = -\sum_{k=0}^{n} \mu_k \left\{ \frac{1}{R_{kp}^3} - 3\frac{Z_{kp}^2}{R_{kp}^5} \right\}$$

To obtain an approximate expression for the oblateness contribution B, choose the perturbation which retains just the second harmonic term:

$$\left(g_1 \frac{X}{R},\ g_1 \frac{Y}{R},\ g_2 \frac{Z}{R}\right)$$

where

$$g_1 = -\frac{J a_\oplus^2 \mu_\oplus}{R^4}\left(1 - \frac{5Z^2}{R^2}\right)$$

$$g_2 = -\frac{J a_\oplus^2 \mu_\oplus}{R^4}\left(3 - \frac{5Z^2}{R^2}\right)$$

At this point a further approximation is made in that the coordinates are regarded as being expressed in the reference system, the mean equator and equinox of 1950.0.

Forming the partial derivatives

$$\frac{\partial \ddot{X}}{\partial u_j} = g_1 \frac{X}{R}\left(\frac{1}{X}\frac{\partial X}{\partial u_j} - \frac{3}{R^2}\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right) + \frac{\mu_\oplus X}{R^5}\frac{J a_\oplus^2}{R^4}\left\{10Z\frac{\partial Z}{\partial u_j} + 2\left(1 - \frac{10Z^2}{R^2}\right)\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right\}$$

$$\frac{\partial \ddot{Y}}{\partial u_j} = g_1 \frac{Y}{R}\left(\frac{1}{Y}\frac{\partial Y}{\partial u_j} - \frac{3}{R^2}\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right) + \frac{\mu_\oplus Y}{R^5}\frac{J a_\oplus^2}{R^4}\left\{10Z\frac{\partial Z}{\partial u_j} + 2\left(1 - \frac{10Z^2}{R^2}\right)\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right\}$$

$$\frac{\partial \ddot{Z}}{\partial u_j} = g_2 \frac{Z}{R}\left(\frac{1}{Z}\frac{\partial Z}{\partial u_j} - \frac{3}{R^2}\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right) + \frac{\mu_\oplus Z}{R^5}\frac{J a_\oplus^2}{R^4}\left\{10Z\frac{\partial Z}{\partial u_j} + 2\left(3 - \frac{10Z^2}{R^2}\right)\mathbf{R}\cdot\frac{\partial \mathbf{R}}{\partial u_j}\right\}$$

where $\partial \ddot{\mathbf{R}}/\partial u_j$ represents the contribution arising from the oblateness only. The final form of B is obtained by the expansion of the dot products:

$$B_{11} = g_1 \frac{X}{R}\left(\frac{1}{X} - \frac{3X}{R^2}\right) + 2\mu_\oplus \frac{X^2}{R^5}\frac{J a_\oplus^2}{R^4}\left(1 - \frac{10Z^2}{R^2}\right)$$

$$B_{12} = g_1 \frac{X}{R}\left(-\frac{3Y}{R^2}\right) + 2\mu_\oplus \frac{XY}{R^5}\frac{J a_\oplus^2}{R^4}\left(1 - \frac{10Z^2}{R^2}\right)$$

$$B_{13} = g_1 \frac{X}{R}\left(-\frac{3Z}{R^2}\right) + 2\mu_\oplus \frac{XZ}{R^5}\frac{J a_\oplus^2}{R^4}\left(6 - \frac{10Z^2}{R^2}\right)$$

$$B_{21} = g_1 \frac{Y}{R}\left(-\frac{3X}{R^2}\right) + 2\mu_\oplus \frac{XY}{R^5}\frac{J a_\oplus^2}{R^4}\left(1 - \frac{10Z^2}{R^2}\right)$$

$$B_{22} = g_1 \frac{Y}{R}\left(\frac{1}{Y} - \frac{3Y}{R^2}\right) + 2\mu_\oplus \frac{Y^2}{R^5}\frac{J a_\oplus^2}{R^4}\left(1 - \frac{10Z^2}{R^2}\right)$$

$$B_{23} = g_1 \frac{Y}{R}\left(-\frac{3Z}{R^2}\right) + 2\mu_\oplus \frac{YZ}{R^5}\frac{J a_\oplus^2}{R^4}\left(6 - \frac{10Z^2}{R^2}\right)$$

$$B_{31} = g_2 \frac{Z}{R}\left(-\frac{3X}{R^2}\right) + 2\mu_\oplus \frac{XZ}{R^5}\frac{J a_\oplus^2}{R^4}\left(3 - \frac{10Z^2}{R^2}\right)$$

$$B_{32} = g_2 \frac{Z}{R}\left(-\frac{3Y}{R^2}\right) + 2\mu_\oplus \frac{YZ}{R^5}\frac{J a_\oplus^2}{R^4}\left(3 - \frac{10Z^2}{R^2}\right)$$

$$B_{33} = g_2 \frac{Z}{R}\left(\frac{1}{Z} - \frac{3Z}{R^2}\right) + 2\mu_\oplus \frac{Z^2}{R^5}\frac{J a_\oplus^2}{R^4}\left(8 - \frac{10Z^2}{R^2}\right)$$

The vector $(g_1 X/R,\ g_1 Y/R,\ g_2 Z/R)$ is assumed to be calculated externally while the parts of B which do not contain g_1 or g_2 are replaced by zero whenever $R > 3\,a_\oplus$.

The execution entry VARY is preceded by the setup entry SVARY:

CALL SVARY, A, B, C, D, E, F, G, H, I, J, K

where **R**, the position of the probe with respect to the central body, is contained in the cells A − 3, A − 2, A − 1. The block B − 3n, ..., B − 1 contains the noncentral body position vectors $\mathbf{R}_1, \ldots, \mathbf{R}_n$; R is contained in location C while the block D − n, ..., D − 1 contains the quantities $R_1, \ldots, R_n$. μ is in location E and the cells F − n, ..., F − 1 are occupied by $\mu_1, \ldots, \mu_n$; a zero in one of the latter cells is used as a flag to skip the corresponding body in the calculation of A. The oblateness perturbation is assumed to be stored in the locations G − 3, G − 2, G − 1; an internal test is made to determine whether the Earth is the central body, since B is set to zero whenever the calculation is not centered at the Earth. To determine the maximum number of perturbing bodies, the decrement of the cell H contains n. The oblateness parameters $a_\oplus$ and J occupy the locations I and J respectively.

As output from the execution entry, the matrix $A + B$ is deposited in the storage locations K − 9, ..., K − 1. Execution of the subroutine requires 30 cells of erasable storage starting at COMMON.

7. Numerical Integration

MARK

MARK is the subroutine which obtains the numerical stepwise solution of a set of linear first-order differential equations by employing an Adams-Moulton predictor-corrector of virtually arbitrary order which utilizes backwards differences; a Runge-Kutta scheme is used to form the necessary differences of the derivatives to start the integration for the multistep method. The step size is halved or doubled upon external request by subtabulation of the derivatives in the former case and by elimination of intermediate points in the latter; hence it is not necessary to restart with Runge-Kutta to effect a step-size change. MARK has been designed to carry out the auxiliary functions of obtaining the numerical solution at specified values of the double-precision independent variable; i.e., for desired times, or doing the same job whenever a specified dependent variable attains a null value. To permit the main program to determine the desired times and to define the dependent variables, a list of control words called triggers is appended to the calling sequence; the structure of the triggers is described in the explanation of the calling sequence to follow.

To allow the main program to monitor the numerical solution, EOS, a supervisory routine provided by the main program, is called at the end of each step by MARK. Additionally, MARK must be given access to a subroutine for the evaluation of the derivatives and the calculation of all necessary dependent variables so that isolated zeros may be iterated down upon and captured.

If m is the highest-order difference retained for the Adams-Moulton method, then for starting purposes the Runge-Kutta portion of MARK must integrate ahead m steps of h, at which time the necessary backwards difference tables for the derivatives will have been completed. Assuming one variable for simplicity, the Runge-Kutta formulas are

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

where the differential equation to be solved has the form $\dot{y} = f(t, y)$. The ordinate y_n is accumulated double precision, while the k_j are evaluated and summed in single precision to be added to y_n in double-precision form. The solution at intermediate times is obtained by altering h to step forward to the desired time and resuming the integration with the old step size h upon return from the trigger execution; while homing in on a zero of a dependent variable, the step size may even become negative. The collecting of the derivatives for the difference tables is accomplished by the setting up of internal time stops at the necessary mesh points; thus the Runge-Kutta section will obtain the solution at the necessary times for the tables while carrying on the ordinary functions of MARK.

The Runge-Kutta formulas give results which agree through fourth order in h with a Taylor series expansion as may be seen from the following development:

$$k_1 = hf$$

$$k_i = h\sum_{j=0}^{3}\frac{\lambda_j^{(i)}}{j!}\left(h\frac{\partial}{\partial t} + k_{i-1}\frac{\partial}{\partial y}\right)^j f + O(h^5), \quad i = 2, 3, 4$$

$$\lambda_j^{(i)} = \begin{cases} 1 & \text{for } i = 4 \\ 2^{-j} & \text{for } i = 2, 3 \end{cases}$$

Expanding the operator and using the notation $\partial^k f / \partial t^j \, \partial y^{k-j} = f_{t^j y^{k-j}}$, there results

$$k_2 = hf + \frac{h^2}{2!}(f_t + ff_y) + \frac{h^3}{3!}\left[\frac{3}{4}(f_{t^2} + 2ff_{ty} + f^2 f_{y^2})\right] + \frac{h^4}{4!}\left[\frac{1}{2}(f_{t^3} + 3ff_{t^2y} + 3f^2 f_{ty^2} + f^3 f_{y^3})\right] + O(h^5)$$

$$k_3 = hf + \frac{h^2}{2!}(f_t + ff_y) + \frac{h^3}{3!}\left[\frac{3}{2}f_y(f_t + ff_y) + \frac{3}{4}(f_{t^2} + 2ff_{ty} + f^2 f_{y^2})\right] + \frac{h^4}{4!}\left[3(f_t + ff_y)(f_{ty} + ff_{y^2}) + \frac{3}{2}f_y(f_{t^2} + 2ff_{ty} + f^2 f_{y^2}) + \frac{1}{2}(f_{t^3} + 3ff_{t^2y} + 3f^2 f_{ty^2} + f^3 f_{y^3})\right] + O(h^5)$$

$$k_4 = hf + \frac{h^2}{2!}[2(f_t + ff_y)] + \frac{h^3}{3!}[3f_y(f_t + ff_y) + 3(f_{t^2} + 2ff_{ty} + f^2 f_{y^2})] + \frac{h^4}{4!}[6f_y^2(f_t + ff_y) + 12(f_t + ff_y)(f_{ty} + ff_{y^2}) + 3f_y(f_{t^2} + 2ff_{ty} + f^2 f_{y^2}) + 4(f_{t^3} + 3ff_{t^2y} + 3f^2 f_{ty^2} + f^3 f_{y^3})] + O(h^5)$$

$$\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = hf + \frac{h^2}{2!}(f_t + ff_y) + \frac{h^3}{3!}[f_y(f_t + ff_y) + (f_{t^2} + 2ff_{ty} + f^2 f_{y^2})] + \frac{h^4}{4!}[f_y^2(f_t + ff_y) + 3(f_t + ff_y)(f_{ty} + ff_{y^2}) + f_y(f_{t^2} + 2ff_{ty} + f^2 f_{y^2}) + (f_{t^3} + 3ff_{t^2y} + 3f^2 f_{ty^2} + f^3 f_{y^3})] + O(h^5)$$

But y_{n+1} is given explicitly by the series

$$y_{n+1} = y_n + \sum_{j=1}^{4} \frac{h^j}{j!} y^{(j)} + O(h^5)$$

where

$$y^{(j)} = \left(\frac{d^j y}{dt^j}\right)_{t_n} = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial y}\right)^{j-1} f$$

In particular,

$$y^{(1)} = f$$
$$y^{(2)} = f_t + ff_y$$
$$y^{(3)} = f_y(f_t + ff_y) + (f_{t^2} + 2ff_{ty} + f^2 f_{y^2})$$
$$y^{(4)} = f_y^2(f_t + ff_y) + 3(f_t + ff_y)(f_{ty} + ff_{y^2}) + f_y(f_{t^2} + 2ff_{ty} + f^2 f_{y^2}) + (f_{t^3} + 3ff_{t^2y} + 3f^2 f_{ty^2} + f^3 f_{y^3})$$

Thus the two series expansions agree through terms of h^4.

To show that the truncation error is, in general, at least $O(h^5)$, consider as an example the simple differential equation

$$\dot{y} = y$$

with solution

$$y_{n+1} = y_n\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!}\right] + O(h^6)$$

In comparison, the Runge-Kutta formulas yield

$$k_1 = hy_n$$
$$k_2 = hy_n\left(1 + \frac{h}{2}\right)$$
$$k_3 = hy_n\left(1 + \frac{h}{2} + \frac{h^2}{4}\right)$$
$$k_4 = hy_n\left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right)$$
$$\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = y_n\left[h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!}\right]$$

Thus the two series disagree beginning with terms of h^5.

The formulas used for the Adams-Moulton integration are derived from the expression

$$y_{n-\mu} = y_n + h\frac{(1 - \nabla)^\mu - 1}{-\ln(1 - \nabla)}\dot{y}_n$$

If a series expansion is obtained and differences through order m are retained, then the truncation error is evidently $O(h^{m+2})$ since

$$h\nabla^{m+p}\dot{y}_n = h\{h^{m+p}D^{m+p}\dot{y}_n + O(h^{m+p+1})\}$$

The predictor formula results from $\mu = -1$:

$$y_{n+1} = y_n + h\frac{(1 - \nabla)^{-1} - 1}{-\ln(1 - \nabla)}\dot{y}_n$$

The computational form is obtained by expanding into a series and retaining differences up through mth order.

$$y_{n+1} = y_n + h\left(\sum_{j=0}^{m} a_j \nabla^j\right)\dot{y}_n$$

where the first few coefficients are

$$a_0 = 1$$
$$a_1 = \frac{1}{2}$$
$$a_2 = \frac{5}{12}$$
$$a_3 = \frac{3}{8}$$
$$a_4 = \frac{251}{720}$$
$$a_5 = \frac{95}{288}$$
$$a_6 = \frac{19087}{60480}$$

As the predictor is relatively unstable for $m \geqq 5$, an option has been provided in MARK for the use of a corrector formula which may be obtained by setting $\mu = 1$ in the general expression

$$y_n = y_{n-1} + \frac{h\nabla}{-\ln(1 - \nabla)}\dot{y}_n$$

For purposes of computation this becomes

$$y_n = y_{n-1} + h\left(\sum_{j=0}^{m} b_j \nabla^j\right)\dot{y}_n$$

where the low-order coefficients are

$$b_0 = 1$$

$$b_1 = -\frac{1}{2}$$

$$b_2 = -\frac{1}{12}$$

$$b_3 = -\frac{1}{24}$$

$$b_4 = -\frac{19}{720}$$

$$b_5 = -\frac{3}{160}$$

$$b_6 = -\frac{863}{60480}$$

The predictor and corrector each require separate evaluations of the derivatives; after application of the corrector and the calculation of the derivatives at the new time station t_n, the solution may be obtained at intermediate points t by choice of $\mu = (t_n - t)/h$, where $t_n - h < t < t_n$:

$$y_{n-\mu} = y_n + h\left(\sum_{j=0}^{m} c_j \nabla^j\right)\dot{y}_n$$

where the c_j are obtained by the convolution of the series

$$\frac{(1-\nabla)^{\mu+1}-1}{\nabla} = \sum_{j=0}^{\infty}(-1)^{j+1}\binom{\mu}{j+1}\nabla^j$$

with the series for the corrector

$$\sum_{j=0}^{\infty} b_j \nabla^j$$

The interpolated solution may then be used either for an intermediate time stop or to help find the zero of a dependent variable.

At the return from the execution of a trigger, MARK may be signaled to change step size by powers of 2 over the nominal value; any other type of step-size change must be effected by restarting the numerical solution. Each time a double is called for, MARK sets internal time stops to save the necessary information for doubling during the next m steps as measured from the end of the current step; of course, the necessary past information is regenerated at this time and saved to be adjoined to the future information to form a difference table of derivatives with twice the step size. At the completion of first doubling, further doubles may be executed in sequence as called for by the main program.

Halving is accomplished by the subtabulation of the derivatives according to Newton's formula:

$$\dot{y}_{n-\mu} = (1-\nabla)^{\mu}\dot{y}_n \approx \left(\sum_{j=0}^{m}(-1)^j\binom{\mu}{j}\nabla^j\right)\dot{y}_n$$

for $\mu = 1/2, 1, \ldots, m/2$.

At this point new differences of the derivatives corresponding to half the step size may be generated, and further halving may be accomplished if called for by the main program.

Step-size changes by doubling or halving are executed only at the end of an Adams-Moulton step and after all time stops and dependent variable stops occurring at times inside the current interval have been executed. While integration is being carried out by the Runge-Kutta section, the doubling and halving signals are ignored; both signals given simultaneously will result in internal confusion.

MARK has the calling sequence

```
CALL   MARK
PZE    HBANK,T,EOS
PZE    DER1,,DER2
(ERROR RETURN)
(FIRST TRIGGER)
        .
        .
        .
(LAST TRIGGER)
HTR                 End of calling sequence
```

Most of the information to be shared by the main program and MARK is organized in the following buffer:

HBANK −3	PZE	m	Adams-Moulton order
−2	PZE	NH	Number of initial halves
−1	PZE	ND	Number of initial doubles
+0	DEC	h	Initial Runge-Kutta step
+1	PZE	$N,,n$	Total and effective variables
+2	DEC	T_1	Double precision
+3	DEC	T_2	Time
	BSS	n	y, solution of differential equations
	BSS	$N-n$	Expansion for more equations
	BSS	n	y, derivatives
	BSS	$N-n$	Expansion for more equations
	BSS	$(2m+5)N$	Working area for MARK

T $\neq 0$ flags time as double precision; otherwise, computation would be with single-precision interpretation. At the end of step, MARK calls the routine EOS; return is via TRA 1, 4. For the calculation of the derivatives, MARK calls a routine which may be divided into two parts: DER1 for time-dependent derivatives and DER2 for the other derivatives. If time has just changed, MARK calls the first entry, while the second entry is called if time remains the same as a previous evaluation. The return device is provided by TRA 1, 4.

A generation of a time which is smaller than the current time will cause MARK to give the error return; if the number of active dependent variables exceeds 20, the error return is likewise given. Normally, the main routine controls the integration by means of the subroutine EOS and by the triggers, but MARK does most of the detail work.

While in the Adams-Moulton mode, the main routine must determine how many times the corrector formula is to be applied; the symbolic location NI in MARK must have in its address the desired number of applications of the corrector.

Each trigger has the structure

OP A,,B

PZE C

The trigger is active whenever the sign bit of the first word is plus; otherwise, a minus sign will cause MARK to ignore the trigger. At location A is a subroutine in the main program which MARK calls whenever the condition defined by the trigger has been met. This subroutine returns via TRA 1, 4; if the trigger is not disabled by the subroutine at execution time, then the value of the variable must in general be changed, lest MARK attempt to execute the trigger again upon return of control. The tag of the first word of the trigger is used as a flag by MARK in the case of a dependent variable so that many triggers may be worked on in a single interval.

The variable defined by a trigger may be of two types—independent (time-stop) or dependent. The former case is flagged by B = 0 and C is then the location of the desired double-precision time for execution of the trigger. In the latter case, the dependent variable is defined to be the difference between the contents of B (B $\neq 0$) and the contents of C. In practice, the quantity in C is the desired value of the variable which is computed in the derivative or the end-of-step routine and stored in the location B.

At the end of each step MARK scans the list of triggers and determines the smallest time which will result in a time stop; all the active dependent variable triggers are inspected to determine those variables which have exhibited a sign change over the preceding step. A linear approximation is made to the root of each variable and the variable which apparently has a root at the earliest time in the interval is iterated upon; a new set of linear estimates of the roots for all the pertinent variables is formed at each step of the iterative solution. At convergence, the time stops and the dependent variable stops are executed in proper time sequence.

For purposes of convergence, two times are considered the same whenever agreement is obtained out to approximately the last two bits; the same test is applied to the sequence of times formed by the iteration process in finding the zero of a dependent variable. Each new time generated requires the calculation of the derivatives appropriate to the new time solution. After all the triggers in an interval have been cleaned up, the information at the end of step is restored and the derivatives are recalculated.