This volume is published in association with
the Academic Press serial publication
Advances in Space Science
edited by Frederick I. Ordway, III

AN INTRODUCTION TO

# ASTRODYNAMICS

ROBERT M. L. BAKER, Jr. and MAUD W. MAKEMSON
Department of Astronomy, University of California, Los Angeles, California

1960 ACADEMIC PRESS · NEW YORK AND LONDON

# 12 APPLICATION TO INTERPLANETARY ORBITS

In the field of interplanetary orbits, one finds a particularly clear-cut distinction between feasibility and definitive orbit work. The astronomical constants and perturbation techniques discussed in Chapters 5, 8, and 9 are particularly relevant to the analysis of definitive interplanetary orbits and, except for the problem of coordinate transformations, can be applied without further discussion. Two-body relationships, on the other hand, are more varied in nature and owe their usefulness to their ability to yield simple solutions to problems of feasibility in which semiquantitative analysis suffices.* In addition to the definitive and feasibility approaches, the question of interplanetary navigational philosophy is also of paramount importance to the astrodynamicist, and this topic will be dealt with in the last section of this chapter.

## 12.1. Definitive Orbit Studies

### 12.1.1. Transfer Points

Initially, the definitive orbit will be computed (probably with special perturbations) in a geocentric coordinate system. As the vehicle recedes from the Earth, a point will be reached where it becomes computationally efficient to transfer to a heliocentric coordinate system. Finally, at landing or at the terminal maneuver, a planetocentric coordinate system becomes

* Many analysts have, in fact, carried out interplanetary orbit computations by employing such approximations, e.g., Herrick in 1946 and, more recently, Breakwell, *et al.*[159] and Karrenberg and Arthur.[160]

more appropriate. Such transfers should be carried out when, e.g., the vehicle passes from a predominantly geocentric to a predominantly heliocentric "field." The regions in space where one would expect the vehicle to be in a predominantly geocentric, heliocentric, or planetocentric field, can be delineated by calculating the ratio of the perturbative accelerations to the two-body acceleration, i.e., by comparing the geocentric

$$(|\dot{x}_g^{\backprime}| + |\dot{y}_g^{\backprime}| + |\dot{z}_g^{\backprime}|)/(1/r^2_g) \tag{282}$$

with the heliocentric

$$(|\dot{x}_h^{\backprime}| + |\dot{y}_h^{\backprime}| + |\dot{z}_h^{\backprime}|)/(1/r_h^2), \tag{283}$$

where, in this instance, the subscripts $g$ and $h$ refer to perturbations *reckoned in* a geocentric or a heliocentric system respectively, e.g., in a geocentric system $\dot{x}_g^{\backprime}$, $\dot{y}_g^{\backprime}$, and $\dot{z}_g^{\backprime}$ would include solar, lunar, and perhaps, drag perturbations ($r_g$ differs from $r_h$ in that $r_g$ is measured in Earth radii from the center of the Earth, and $r_h$ is measured in astronomical units from the center of the Sun).

At the point where the former ratio (Eq. (282)) exceeds the latter ratio (Eq. (283)) (at about 128 Earth radii), the heliocentric field will predominate. It should be noted that this point is *not* where the force due to the Sun acting on the vehicle exceeds the force due to the Earth acting on the vehicle (at about 40 Earth radii). Adoption of such an erroneous criterion would, e.g., disinherit the Earth of the Moon! A similar analysis holds for heliocentric and planetocentric fields. In this latter case a comparison of

$$(|\dot{x}_h^{\backprime}| + |\dot{y}_h^{\backprime}| + |\dot{z}_h^{\backprime}|)/(1/r_h^2)$$

to

$$(|\dot{x}_p^{\backprime}| + |\dot{y}_p^{\backprime}| + |\dot{z}_p^{\backprime}|)/(1/r_p^2)$$

is carried out (the subscript $p$ refers to the planet). The locus of these points where the solar perturbations dominate forms a roughly spherical surface enclosing each planet that is nearly independent of the specific interplanetary trajectory. The radius of this surface is referred to as the "tidal radius," and any planetary satellite exterior to this surface will eventually stray away from its parent planet and take up its own independent heliocentric orbit. For the purpose of interplanetary orbit computations, however, such a surface is utilized primarily to define the point where coordinate systems should be changed in order to afford the most efficient computation.





### 12.1.2. Perturbative Analyses

Special (or general) perturbation procedures can now be employed in the appropriate coordinate system. In this regard the perturbative forces due to the planets can be reckoned as in Chapter 7 (the coordinates of the planets can be obtained from the references listed in Section 4.6) and a definitive interplanetary orbit determined.

It is, of course, important to change units at the point where the transformation occurs, otherwise a loss in accuracy as well as an increase in formula complexity is incurred. As an example $\dot{s}_\infty$, for a vehicle leaving the Earth on a geocentric orbit, must be changed from Earth circular-satellite speed at one equatorial radius over to Sun circular-satellite speed at 1 a.u. See Section 2.5.

The actual evaluation of Eqs. (282) and (283) and the generation of the $n$-body perturbative forces require tables of the rectangular components $(x, y, z)$ of the following position vectors (see Section 4.6):

$\mathbf{r}_{\oplus\odot}$ = the coordinates of the Sun relative to the Earth in Earth radii and astronomical units,

$\bar{\mathbf{r}}_{\oplus\odot}$ = the coordinates of the Sun relative to the Earth–Moon center of mass in astronomical units,

$\mathbf{r}_{\odot p}$ = the coordinates of the target planet relative to the Sun in astronomical units,

$\mathbf{r}_{\odot\text{♃}}$ = the coordinates of Jupiter relative to the Sun in Earth radii and in astronomical units, and

$\mathbf{r}_{\oplus\text{☾}}$ = the coordinates of the Moon relative to the Earth in Earth radii.

The initial geocentric integration of the orbit (see Chapter 8) will yield the components of $\mathbf{r}_{\oplus\Delta}$. From Eq. (164) the geocentric perturbative terms are

$$\dot{x}_g^{\backprime} = m_\odot\left(\frac{x_{\Delta\odot}}{r_{\Delta\odot}{}^3} - \frac{x_{\oplus\odot}}{r_{\oplus\odot}{}^3}\right) + m_{\text{☾}}\left(\frac{x_{\Delta\text{☾}}}{r_{\Delta\text{☾}}{}^3} - \frac{x_{\oplus\text{☾}}}{r_{\oplus\text{☾}}{}^3}\right) + m_{\text{♃}}\left(\frac{x_{\Delta\text{♃}}}{r_{\Delta\text{♃}}{}^3} - \frac{x_{\oplus\text{♃}}}{r_{\oplus\text{♃}}{}^3}\right) \qquad x \rightarrow y, z$$

where

$$x_{\Delta\odot} = x_{\oplus\odot} - x_{\oplus\Delta},$$

$$x_{\Delta\text{☾}} = x_{\oplus\text{☾}} - x_{\oplus\Delta},$$

$$x_{\Delta\text{♃}} = x_{\oplus\text{♃}} + x_{\Delta\oplus} = x_{\odot\text{♃}} + x_{\oplus\odot} - x_{\oplus\Delta},$$

$$x_{\oplus\text{♃}} = x_{\odot\text{♃}} + x_{\oplus\odot};$$

and the masses are in terms of Earth masses, lengths in terms of Earth radii, and time in "$k_e^{-1}$ min." Of course, $k = k_e$ and $\mu = 1$. Similarly, the heliocentric perturbative components are

$$\ddot{x}_h' = (m_\oplus + m_☾)\left(-\frac{x_{\oplus\Delta}}{r_{\oplus\Delta}^3} + \frac{\bar{x}_{\oplus\odot}}{r_{\oplus\odot}^3}\right) + m_p\left(\frac{x_{\Delta p}}{r_{\Delta p}^3} + \frac{x_{\odot p}}{r_{\odot p}^3}\right) + m_{♃}\left(\frac{x_{\Delta ♃}}{r_{\Delta ♃}^3} - \frac{x_{\odot ♃}}{r_{\odot ♃}^3}\right) \qquad x \to y, z$$

where, prior to the transfer to the heliocentric system $x_{\odot\Delta}$ is obtained from

$$x_{\odot\Delta} = x_{\oplus\Delta} - \bar{x}_{\oplus\odot}$$

while

$$x_{\Delta p} = x_{\odot p} + \bar{x}_{\oplus\odot} - x_{\oplus\Delta},$$

$$x_{\Delta ♃} = x_{\odot ♃} - x_{\odot\Delta};$$

and the masses are in terms of solar masses, lengths in terms of astronomical units, and time in $k_s^{-1}$ days. In this case $k = k_s$, but $\mu$ may be taken as equal to $m_\odot + m_♀ + m_☿$. When the transfer to the heliocentric system is accomplished, $\mathbf{r}_{\odot\Delta}$ is obtained directly from the integration.

## 12.2. Two-Body Studies

The following sections are devoted to the application of simple, albeit approximate, two-body relationships to the study of the feasibility of interplanetary missions. The topics included are analyses of launch direction, errors, and sensitivities, and effective gravitational cross sections. Illustrative examples of two-body interplanetary orbits have already been considered in Sections 2.5 and 3.4 and the reader might review these sections prior to his consideration of the following material.

### 12.2.1. Direction of Launch

The question is often raised as to the proper launch direction for an interplanetary orbit. Let us consider the two-body trajectory from Earth to Venus ($\mu$ is assumed equal to unity, i.e., the "restricted" two-body problem). Suppose that a rocket is launched directly towards the Sun. In this case the tangential speed (measured in terms of circular-satellite speed at 1 a.u.), $r\dot{v}$, which in the case of the transfer from an assumed circular





Earth's orbit amounts simply to unity, is not modified at all. Hence if $r\dot{v} = 1$ and $r = 1$ at launch from the Earth (i.e., at 1 a.u.), $r^4\dot{v}^2 = 1 = p = r$. Therefore, the space vehicle, when leaving the Earth, must be at the point where the radius equals the parameter, $p$, of its orbit, i.e., on the $y_\omega$-axis.

From the *vis-viva* integral,

$$\dot{s}^2 = 2/r - 1/a = 2 - 1/a$$

and from Eq. (8)

$$\dot{s}^2 = \dot{r}^2 + r\dot{v}^2 = \dot{r}^2 + 1;$$

hence

$$2 - 1/a = \dot{r}^2 + 1, \qquad \text{or} \qquad 1 = a(1 - \dot{r}^2).$$

Since

$$p = 1 = a(1 - e^2),$$

then

$$e = \dot{r}. \tag{284}$$

Thus, a purely radial launch velocity results directly in a change in eccentricity. The perihelion of the transfer orbit is

$$q = a(1 - e) = a(1 - \dot{r}) = \frac{(1 - \dot{r})}{(1 - \dot{r}^2)} = \frac{1}{1 + \dot{r}}. \tag{285}$$

For the sake of argument, let it be assumed that the thrust directed towards the Sun, $\dot{r}$, is equal to $1/12$ in an astronomical system of units (i.e., in units of circular-orbit speed about the Sun at 1 a.u. = 18.6 mi/sec). From Eq. (285) we can readily find that $q = 12/13 = 0.923$, so that by launching towards the Sun, the vehicle can never get closer to the Sun than 0.923 a.u. at perihelion. Consequently, it could not possibly reach Venus, which describes a nearly circular orbit of radius 0.723 a.u.

Let it next be assumed that we fire along a tangent to the Earth's orbit. In this case, if we move opposite to the direction of the Earth's motion, the heliocentric vehicle orbit will have its aphelion near the Earth at launch and its perihelion near Venus at landing (such an orbit will be tangent both to the Earth's orbit at launch and to the orbit of Venus at landing). If $\Delta\dot{s}$ is defined as the increment of speed relative to the Earth's orbital velocity in heliocentric units, then

$$\dot{s} = 1 - \Delta\dot{s} \quad \text{and} \quad \frac{1}{a} = 2 - (1 - \Delta\dot{s})^2 = 1 + 2\Delta\dot{s} - (\Delta\dot{s})^2,$$

while

$$r\dot{v} = \dot{s} = 1 - \Delta\dot{s}$$

also, because $r = 1$ at injection again (see the bottom of p. 113 with $\mu = 1$)

$$p = r^4\dot{v}^2 = \dot{s}^2 = (1 - \Delta\dot{s})^2$$

and, finally,

$$e^2 = 4[(\Delta\dot{s})^2 - (\Delta\dot{s})^3] + (\Delta\dot{s})^4. \tag{286}$$

For $\Delta\dot{s} = 1/12$ one obtains $q = 0.724$ a.u. Clearly, therefore, it is preferable to fire tangentially to the Earth's orbit rather than normal to it. A more general analysis of this problem can be found in Chapter 8 in Herrick,[1] where it is demonstrated that the least speed and hence energy required for a transfer between two circular orbits is achieved when one employs such doubly-tangent orbits. The first recognition of this principle was due to Hohmann in 1925 and such orbits are, therefore, termed *Hohmann orbits*.[161] Other considerations often preclude the employment of such simple orbits for orbital transfer. The field of orbital transfer is a vast one, however, and the widely published work of D. F. Lawden should be consulted by the interested reader. In particular, if one is willing to initiate a rather large thrust while on the heliocentric transfer orbit, Lawden and others have found by two-body studies that the use of a multiple-orbit transfer is superior to the simple single Hohmann orbital transfer. (See exercise.)

### 12.2.2. Errors and Orbit Sensitivity

In heliocentric ecliptic or equatorial coordinate systems, the distance from the Sun to a space vehicle is defined by $r^2 = x^2 + y^2 + z^2$. Consequently, an error in $r$, $\Delta r$, is a function of the errors in $x$, $y$, $z$, i.e.,

$$\Delta r = (x\Delta x + y\Delta y + z\Delta z)/r. \tag{287}$$

Let us assume for simplified two-body interplanetary orbits that the errors in velocity and position can be considered separately and added linearly when they are combined. Thus, we may assume that the velocity is correct and that an error enters only in the distance. Then from differentiation of the *vis-viva* integral with $\mu$ set equal to unity, we obtain

$$\Delta a = (2a^2/r^2)\Delta r. \tag{288}$$





While

$$\Delta p = 2r(r\dot{v})^2\Delta r + 2r^2(r\dot{v})\Delta(r\dot{v}).$$

If it is assumed that the tangential component of velocity, $r\dot{v}$, is correct, then

$$\Delta p = (2r^3\dot{v}^2)\Delta r.$$

Also

$$\Delta p = (1 - e^2)\Delta a - (2ae)\Delta e,$$

so that the error in eccentricity occasioned by an error in position, $\Delta r$, is simply

$$\Delta e = [(1 - e^2)(a - r)/er^2]\Delta r. \tag{289}$$

Specifically, the error in position as reflected in the perihelion distance becomes

$$\begin{aligned}\Delta q &= (1 - e)\Delta a - a\Delta e\\ &= [2a^2(1 - e)/r^2 - a(1 - e^2)(a - r)/er^2]\Delta r\\ &= \{[2a^2e - a(1 + e)(a - r)](1 - e)/er^2\}\Delta r.\end{aligned}$$

or

$$\Delta q/q = \{[r(1 + e) - q]/er\}\Delta r/r, \tag{290}$$

and in the case of the aphelion distance, $q_2$, the relation is

$$\Delta q_2/q_2 = \{[q_2 - r(1 - e)]/er\}\Delta r/r. \tag{291}$$

In the case of nearly zero eccentricity, the eccentric anomaly must be introduced in order to resolve the indeterminacy of $(a - r)/e$. Thus, the equations for the perihelion and aphelion reduce to

$$\Delta q/q = \{a[2 - (1 + e)\cos E]/r\}\Delta r/r \tag{292}$$

and

$$\Delta q_2/q_2 = \{a[2 + (1 - e)\cos E]/r\}\Delta r/r. \tag{293}$$

Unfortunately, these relations are still indeterminate when $e$ is zero because perihelion is then undefined. Nevertheless, as suggested by Walters, one might assume $E$ to have a value resulting from a random position error and therefore obtain for $e = 0$:

$$\overline{(\Delta q/q)^2} = \overline{(2 - \cos E)^2}\ \overline{(\Delta r/r)^2} = 4.5\ \overline{(\Delta r/r)^2} \qquad q \rightarrow q_2. \tag{294}$$

If the position is assumed to be perfectly known and all the error arises from the velocity, then the equations are precisely those for variation-of-parameters, and relations for $\Delta a$ and $\Delta e$ (i.e., $a^\backslash$ and $e^\backslash$) may be employed. Consequently, (again assuming $\mu = 1$) from Section 8.3.1 we find that

$$\Delta a = (2a^2\dot{s})\Delta\dot{s} = 2a^2(\dot{x}\Delta\dot{x} + \dot{y}\Delta\dot{y} + \dot{z}\Delta\dot{z}) \tag{295}$$

and

$$e\Delta e = (r^2\dot{r}/a)\Delta\dot{r} + ([p/r - r/a]r\dot{s})\Delta\dot{s} \tag{296}$$

(see Chapter 17 in Herrick[1]), where $r$ is the distance from the Sun *at the time when the errors* $\Delta\dot{r}$ *and* $\Delta\dot{s}$ *are introduced* and

$$r\Delta\dot{r} = x\Delta\dot{x} + y\Delta\dot{y} + z\Delta\dot{z}.$$

For the aphelion and perihelion uncertainties it can be shown that

$$\Delta q = ([r^2 - a^2 + 2a^2e - a^2e^2]\dot{s}^2/e)\Delta\dot{s}/\dot{s} - ([r\dot{r}]^2/e)\Delta\dot{r}/\dot{r} \tag{297}$$

and

$$\Delta q_2 = ([a^2 - r^2 + 2a^2e + a^2e^2]\dot{s}^2/e)(\Delta\dot{s}/\dot{s}) + ([r\dot{r}]^2/e)\Delta\dot{r}/r. \tag{298}$$

Again the indeterminacy for $e$ can be resolved by introducing $r\dot{r} = \sqrt{a}$ ($e$ sin $E$) and computing a mean-square error. Consequently for perihelion

$$\overline{\left(\frac{\Delta q}{q}\right)^2} = \frac{1}{2}\overline{\left(\frac{\Delta\dot{r}}{\dot{s}}\right)^2} + 6\overline{\left(\frac{\Delta\dot{s}}{\dot{s}}\right)^2} \qquad q \rightarrow q_2. \tag{299}$$

As has been noted already, if the errors are small, it is possible to combine the position and velocity errors in a linear fashion. The foregoing error equations can, of course, also be applied to geocentric orbits and in particular are quite useful for the preliminary estimation of injection errors for satellite and lunar trajectories.

In the case of interplanetary trajectories it is important to recognize that the error in speed results both from observational uncertainties and from the uncertainties occasioned by lack of exact knowledge of the laboratory unit in terms of the astronomical unit (see Chapter 5).





Let us consider the case of a launch from Earth on a trip to Venus. One of the first questions to be decided is exactly when and where to determine our initial conditions. If we associate $\dot{s}_\infty$ with the speed that our Venusian probe will have at a considerable distance from the Earth relative to the Earth (i.e., its asymptotic or "hyperbolic excess" speed),

$$\left(\dot{s}_\infty^2 = \frac{2}{\infty} - \frac{1}{a} = -\frac{1}{a}\right),$$

and $\dot{s}_p$ the parabolic or escape speed at any given radial distance from the Earth,

$$\left(\dot{s}_p^2 = \frac{2}{r} - \frac{1}{\infty} = \frac{2}{r}\right),$$

then the *vis-viva* equation reduces to the form

$$\dot{s}^2 = \dot{s}_p^2 + \dot{s}_\infty^2. \tag{300}$$

Consequently an error in launch speed, $\dot{s}$, propagates itself into an error in the speed of recession from the Earth at a great distance, $\dot{s}_\infty$, by

$$\Delta\dot{s}_\infty = \frac{\dot{s}}{\dot{s}_\infty}\Delta\dot{s}.$$

Percentage-wise, we have

$$\left(\frac{\Delta\dot{s}_\infty}{\dot{s}_\infty}\right) = \left(\frac{\dot{s}}{\dot{s}_\infty}\right)^2\left(\frac{\Delta\dot{s}}{\dot{s}}\right). \tag{301}$$

For a Hohmann or doubly-tangent Earth–Venus trajectory, this relationship becomes specifically

$$\frac{\Delta\dot{s}_\infty}{\dot{s}_\infty} \cong 16\frac{\Delta\dot{s}}{\dot{s}},$$

so that $\frac{1}{16}$ of 1% error in launch is reflected by a 1% uncertainty in the speed of the vehicle as it recedes to great distances from the Earth. Since the geocentric speed of the vehicle must be added to the heliocentric speed of the Earth (which is over ten times larger), the heliocentric speed of the vehicle is known to perhaps $\frac{1}{10}$ of 1%. It is clear, however, that in order to circumvent the amplification of launch error, it is advisable to measure the interplanetary probe's speed after it has moved ten or so Earth radii from the Earth (cf. Baker[162]), e.g., Doppler radii measurements of interplanetary and lunar vehicles are more effectively made after the vehicle has receded a significant distance from the Earth!

Given the hypothetical $\frac{1}{10}$ of 1% error in the initial heliocentric speed of the vehicle and assuming its initial position to be perfectly well-known, we obtain a cumulative error on a Hohmann orbit at Venus amounting to

$$\frac{\Delta q + \Delta q_2}{a} = 4a\dot{s}^2\left(\frac{\Delta \dot{s}}{\dot{s}}\right) = (4)(0.86)(0.916)^2\frac{1}{10} = 0.288\%$$

and for a Hohmann orbit to Mars an error of 0.628% (note that the addition of Eqs. (297) and (298) and division by $a$ will result in the foregoing equation).

### 12.2.3. Effective Gravitational Cross Section

The question of error sensitivity on interplanetary, or for that matter, lunar trajectories cannot be resolved by the computation of the terminal offset distance alone. It is necessary as well to consider the attraction of the planet on the vehicle as it approaches the target. This effect tends to increase the probability of making a landing. As was mentioned in the introduction to this section, it is often useful to divide interplanetary trajectories into three phases and apply two-body considerations to each of these phases individually. In the case of the interplanetary landing phase, for example, it is useful to assign to the planet at landing a size that is greater than its physical size, a size that represents the extent of its predominant gravitational attraction. This concept is employed by atomic and nuclear physicists and is termed the *effective cross section*. Its radius is called the *collision parameter*, $b$, and it is evident that any space vehicle coming towards the planet with an offset distance less than the collision parameter will strike the planet. If we employ a two-body approximation during the landing phase, the radius of this cross section is easily derived from the principle of the conservation of angular momentum and from the *vis-viva* integral (cf. Baker[104]). Equating the angular momentum $b\dot{s}_\infty$ at a point a great distance from the planet to that at a grazing encounter with its surface, we find that

$$b\dot{s}_\infty = q\dot{s}_0, \qquad (302)$$

(see Fig. 45) where $s_\infty$ is now the asymptotic speed of the vehicle as it approaches the planet, $r_0 = q$ is the planet's radius (or the orbital perifocus for grazing encounter with a planetary surface), and $s_0$ is the speed the vehicle would have at a grazing encounter with the surface of the planet.





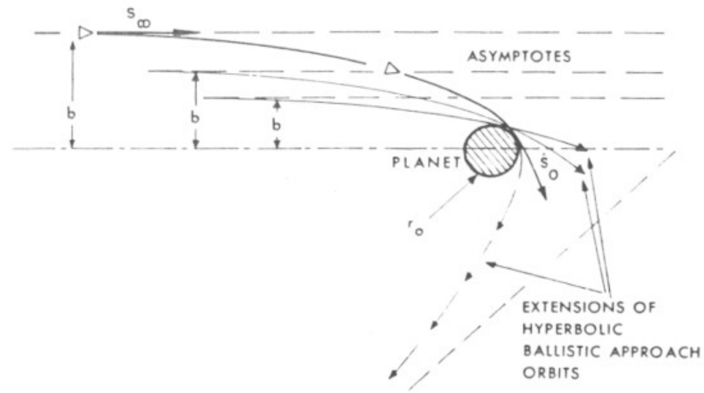


Fig. 45. Collision parameter.

From the *vis-viva* integral (Eq. (13)) applied to the planetocentric orbit

$$\dot{s}_0^2 = \frac{2\mu}{r_0} + \dot{s}_\infty^2 \text{ since } \dot{s}_\infty^2 = -\mu/a \quad (\text{i.e., } \dot{s}_\infty \text{ is the speed as } r \to \infty)$$

so that solving for $b$, we obtain

$$b = r_0\sqrt{\frac{2\mu}{\dot{s}_\infty^2 r_0} + 1}. \qquad (303)$$

With regard to Eq. (303) it should be noted that as $\dot{s}_\infty \to \infty$ (i.e., infinite speed of approach), $b \to r_0$, the physical radius of the planet; and as $\dot{s}_\infty \to 0$ (i.e., no relative velocity at $\infty$) all objects will eventually be attracted to the "lone" planet and $b \to \infty$.

The effective gravitational cross section (sometimes called a "sphere of action" or a "capture arc") represents a rather large target. If one wishes to take advantage of atmospheric retardation, however, a much smaller target must be considered: only a narrow band of the Earth's atmosphere situated at a radial distance $r_0$ from the planet's center (i.e., in this case $r_0$ is not exactly equal to the planet's radius). If the allowable width of this band is $dr_0$, then differentiation of the foregoing equation will yield

$$db = \frac{1}{b}\left[\frac{1}{\dot{s}_\infty^2} + r_0\right] dr_0. \qquad (304)$$

For the trip to Venus $db = 2.5\ dr_0$ so that the effective target will be a thin annulus of radius $b$ and width $db$ as depicted in Fig. 46. Hence, the gravitational field of Venus will not be of much advantage. A more detailed discussion of the braking ellipse maneuver and of other possible alternate lift re-entry orbits will be found in Baker.[104]

In this regard, it is now recognized that, aside from polar orbits, the Van Allen radiation belt hazard will preclude the utilization of the multiple braking ellipse maneuver for the return of manned vehicles. Still, however, the accuracy requirements for either the single-pass ballistic (drag capsule)

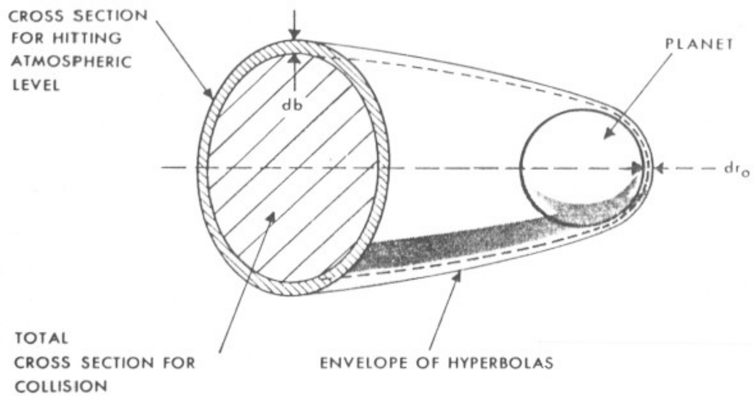


FIG. 46. Effective gravitational cross sections.

or lift (glider) entry orbits are most stringent. Often these single-pass orbits are specified by their *elevation angle*, $\theta$, (an angle between the velocity vector of the vehicle and the local horizontal) reckoned at a particular altitude above the Earth, e.g., 100 miles. (Note that for a nonrotating atmosphere $\theta = 90 - \zeta$, where $\zeta$ is the "entry angle" defined in Section 8.4.) For grazing entry the equation for this elevation angle, from pp. 14, 113, 116

$$\tan\theta = \dot{r}/r_1\dot{v} = er_1 \sin v/p, \tag{305}$$

(see Chapter 4F of Herrick[1]), can be approximated near perigee by

$$\theta \cong er_1 v/p, \tag{306}$$

where $r_1$ is the reference radius from the Earth (e.g., Earth's radius plus 100 miles). Thus, the change in $\theta$ due to a change in $b$ (holding $r_1$ and $\dot{s}_\infty$ constant) is

$$\frac{\partial\theta}{\partial b} = \frac{r_1}{p}\left\{-\frac{ev}{p}\frac{\partial p}{\partial b} + v\frac{\partial e}{\partial b} + e\frac{\partial v}{\partial b}\right\};$$





but, since for a constant $\dot{s}_\infty$, $a$ is a constant also, we find that $\frac{\partial e}{\partial b} = -\frac{\partial p}{\partial b}/2ae$ and for small $v$

$$e\frac{\partial v}{\partial b} = \frac{1}{v}\left(\frac{\partial e}{\partial b} - \frac{1}{r_1}\frac{\partial p}{\partial b}\right) = -\frac{1}{v}\left(\frac{1}{2ae} + \frac{1}{r_1}\right)\frac{\partial p}{\partial b}.$$

Combining these expressions, we find that

$$\frac{\partial\theta}{\partial b} = -\frac{r_1}{p}\left\{\frac{ev}{p} + \frac{v}{2ae} + \frac{1}{v}\left(\frac{1}{2ae} + \frac{1}{r_1}\right)\right\}\frac{\partial p}{\partial b} \cong -\frac{r_1}{pv}\left(\frac{1}{2ae} + \frac{1}{r_1}\right)\frac{\partial p}{\partial b}.$$

Since $p$ = the angular momentum per unit mass squared (see Section 6.1.1), then

$$p = b^2\dot{s}_\infty{}^2 \qquad \text{and} \qquad \frac{\partial p}{\partial b} = 2b\dot{s}_\infty{}^2.$$

Consequently, an accuracy given for elevation angle, $d\theta$, can be translated into a value of $db$, by

$$db = -\{pv/r_1 b\dot{s}_\infty{}^2(1/ae + 2/r_1)\}\,d\theta = -\left\{bv/\left(\frac{r_1}{ae} + 2\right)\right\}d\theta, \tag{307}$$

(the exact relationship is $db = \{-b \sin v \sec^2\theta/[2r_1 e \sin^2 v/p + r_1/ae + 2\cos v]\}\,d\theta$) where $b$ decreases for increasing $\theta$ at the same $\dot{s}_\infty$. Ordinarily pure drag entry vehicles require an accuracy in $d\theta$ of a fraction of a degree, while lift vehicles (with a ratio between lift and drag forces amounting to about two) can have $d\theta$ variations amounting to a few degrees. This advantage of the lifting vehicle (including its maneuverability) must, however, be weighed against an almost two to one mass penalty—a glider being twice as ponderous as a drag capsule for the same payload.

## 12.3. Navigational Philosophy

As was noted in Baker,[163] the navigational problem in space is often quite different from the Earthbound navigational problem. In this concluding section we will address ourselves to the question of a vehicle-born navigational system. Specifically, let us consider the function and utility of an onboard navigational computer. The uses of this onboard navigational computer would be:

(i) To process observational data acquired in flight (it is usually assumed that such data can be more easily, or more accurately obtained from the vehicle than from the ground).

(ii) To serve as a "back-up" for a ground computer (here it is assumed that both data transmitted from the ground and data generated on board the vehicle would be available).

It is clear that if the best data should prove to be obtainable from the ground, then any navigational maneuvers of the vehicle should be executed through a command link between the ground and the vehicle, and the use of an on-board navigational computer would be unwarranted.

It is concluded then that the principal application of on-board navigational computers will be to lunar and interplanetary probes. In particular, in the midcourse guidance of such probes, the employment of a navigational computer may be mandatory. Advantage can be taken of the fact that the midcourse interplanetary and the midcourse lunar orbits are very nearly two-body orbits and, in keeping with the philosophy adopted by Baker,[163] the navigational principle involved would be that of a differential correction to the orbit. The purpose of the on-board navigational computer then would be to process observational data obtained from the vehicle as well as that transmitted from the ground, and to compute the velocity error or corrective velocity that the vehicle must gain in order to accomplish its lunar or interplanetary mission. Before embarking upon this subject of differential correction a word should be included on the non-coplanar nature of the vehicle and the planetary orbit (the vehicle proceeding initially along the ecliptic plane).

### 12.3.1. Inclination Change

In the case of the interplanetary trajectory an optimum time to change from the ecliptic plane to an orbit plane that will reach the planet at time of landing (since the planet will not in general be situated at a node of its orbit in the ecliptic plane) is when the true anomaly at the vehicle differs by 90°* from the true anomaly it will have at the time of intercept with the planet. This principle can be gleaned from a consideration of the spherical

* It is emphasized that the position of the node of the planet's orbit plane has *nothing* whatsoever to do with the problem. Only the planet's ecliptic latitude, $\beta_p$, at the time of landing is relevant and the planet's itinerary is irrelevant. The planet could be on any orbit and as long as the angle between the planet's position at the time of landing and the vehicle (at the point of inclination change) $\Delta v$ was 90° (as measured on the heliocentric celestial sphere), the change of inclination, $\Delta i$, would be minimized.[163] The true optimization for minimum fuel expenditure might, however, be at a point slightly different from this due to guidance and the slowly varying speed of the vehicle on its orbit.





geometry involved i.e., $\sin \beta_p = \sin \Delta_i \sin \Delta v$ (see Fig. 32). $\mu \rightarrow \Delta v$, $\beta_p \leq \Delta v \leq \pi/2 - \beta_p$. If, on the other hand, the vehicle cannot maneuver in midcourse, then a single orbit plane passing through the Sun and leading from the Earth at time of take-off to the planet at time of landing must be employed (as shown by Bock and Mundo[135] such a single orbit plane trajectory is not very efficient, e.g., if such a plane is inclined only 10° to the ecliptic a 100% increase in launch weight is required for conventional chemical propellants). In any event, after the last thrust period, whether it be at take-off or in a maneuver to reach the planet's orbit plane, observational data must be taken in order to determine any accumulated error and the orbit corrected.

### 12.3.2. Differential Correction

The differential correction proceeds as follows for both the lunar and interplanetary trajectories: before the flight, precomputations are made of observations *as they would appear from the vehicle* if it traveled along the ideal orbit. As noted by Bock[164] such observations will probably be differential measures of angles between planets and *nearby* stars (long a standard procedure in observational astronomy) and will be supplemented by inertial stabilization of the instruments. Any other data, such as electronic could, however, also be employed, e.g., active Doppler radar, pulsed radar, phase comparison, or even measurement of the Doppler shift of the 21-cm neutral interstellar hydrogen line. Since the vehicle will not in general travel along this ideal orbit because of inaccuracies in initial launch or midcourse maneuvers and errors in the astrodynamical constants, actual observations during the trip will differ from the precomputed observations. From the comparison between the two, residuals are found, and a differential correction is made to obtain an improved orbit. From these "observed minus computed" differences, e.g., $\Delta\rho_i$, (where $\rho_i$ is meant to stand for some typical observational datum) improved elements can be obtained by a least squares solution (see Chapter 6). The question as to which elements to correct must be decided. Either the six components of ideal position and velocity: $x_I$, $y_I$, $z_I$, $\dot{x}_I$, $\dot{y}_I$, and $\dot{z}_I$ or $\mathbf{U}$, $\mathbf{V}$, $a$, and $e_I$, either set of elements being reckoned for the time at which a corrective thrust is to be made (determined before hand), epoch, could be utilized as elements. The following equations (308) based upon the former set of elements must be inverted by least squares (if $n > 6$) to yield improvements, $\Delta\dot{x}_I$, $\Delta\dot{y}_I$, $\Delta\dot{z}_I$, $\Delta x_I$, $\Delta y_I$, and $\Delta z_I$ to the six ideal velocity and position component orbital elements. Specifically velocity and position that the vehicle *will have* at the time of correction (epoch), $\dot{\mathbf{r}}_0$ and $\mathbf{r}_0$, are $\dot{\mathbf{r}}_0 = \dot{\mathbf{r}}_I + \Delta\dot{\mathbf{r}}_I$ and $\mathbf{r}_0 = \mathbf{r}_I + \Delta\mathbf{r}_I$.

$$\Delta\rho_1 = (\partial\rho_1/\partial x_I)_{t_1}\Delta x_I + (\partial\rho_1/\partial y_I)_{t_1}\Delta y_I + (\partial\rho_1/\partial z_I)_{t_1}\Delta z_I$$
$$+ (\partial\rho_1/\partial \dot{x}_I)_{t_1}\Delta \dot{x}_I + (\partial\rho_1/\partial \dot{y}_I)_{t_1}\Delta \dot{y}_I + (\partial\rho_1/\partial \dot{z}_I)_{t_1}\Delta \dot{z}_I$$
$$\Delta\rho_2 = (\partial\rho_2/\partial x_I)_{t_2}\Delta x_I + (\partial\rho_2/\partial y_I)_{t_2}\Delta y_I + \cdots$$
$$\vdots$$
$$\Delta\rho_i = (\partial\rho_i/\partial x_I)_{t_i}\Delta x_I + (\partial\rho_i/\partial y_I)_{t_i}\Delta y_I + \cdots$$
$$\Delta\rho_n = (\partial\rho_n/\partial x_I)_{t_n}\Delta x_I + (\partial\rho_n/\partial y_I)_{t_n}\Delta y_I + \cdots. \qquad (308)$$

where $\rho_i = \rho_i(x_I, y_I, z_I, \dot{x}_I, \dot{y}_I, \dot{z}_I, t_i)$ and the partials are computed at the time of the observation $t_i$ (the sensitivity to clock drift has not been found to be particularly severe). All partials involved in these equations would be evaluated beforehand and stored. In this regard it should be emphasized that the observations, e.g., $\rho_i$, made all along the path at different $t_i$'s, are functions of certain *time constant* orbital parameters and $t_i$—sufficient to say that $a$, $e$, $T$, $i$, $\Omega$, $\omega$; or $\mathbf{a}$, $\mathbf{b}$, $n_0$, $M_0$; or $x$, $y$, $z$, $\dot{x}$, $\dot{y}$, $\dot{z}$, on the ideal orbit reckoned at *any specified time* are all possible alternative sets (that position and velocity are reckoned at some special time makes them nonetheless orbital constants that define the orbit). From these solutions of Eqs. (308) corrections to the elements can be computed; for example, improved values of position and velocity, at a predetermined time for correction (epoch), could be calculated.

Since there exists a unique orbit connecting any two points in space for a given transit time (found by Gauss' method, see Chapter 6), it is possible to compute an orbit that will terminate at the desired target, such as Mars, from every possible space position at a given epoch (the landing time will usually not be the same for all such orbits).

In principle, one could compute a large number of such heliocentric orbits for all the possible path deviations that could be expected in view of the probable uncertainties, and store them on board the vehicle. Subsequently, given the correct position at the moment of correction (epoch), $x_0 = x_I + \Delta x_I$, $y_0 = y_I + \Delta y_I$, $z_0 = z_I + \Delta z_I$, tables could be entered with argument of position to yield the velocity necessary to complete the mission, i.e., with components $\dot{x}_c$, $\dot{y}_c$, $\dot{z}_c$. The improved velocity components of the vehicle could then be subtracted from the velocity components necessary to complete the mission and the required correction to the velocity or *velocity to be gained*, $\dot{\mathbf{r}}_g$, thereby obtained, i.e.,

$$\dot{\mathbf{r}}_g = \dot{\mathbf{r}}_c - \dot{\mathbf{r}}_0 = \dot{\mathbf{r}}_c - (\dot{\mathbf{r}}_I + \Delta\dot{\mathbf{r}}_I).$$





In practice such a tabulation would become an exorbitant burden upon the on-board computer storage. A series expression for the velocity required

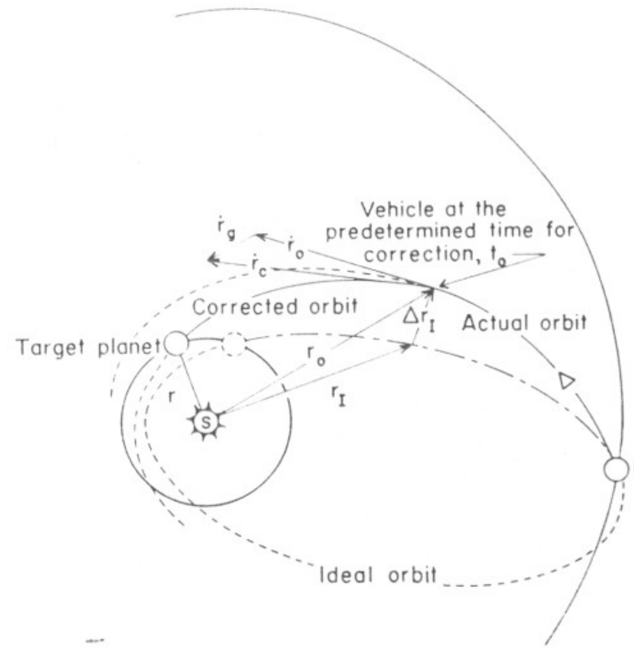


FIG. 47. Interplanetary navigation.

to complete the mission, $\dot{\mathbf{r}}_c$, or more efficiently for a residual velocity $\Delta\dot{\mathbf{r}}_c$ above that of the reference orbit where $\dot{\mathbf{r}}_c = \dot{\mathbf{r}}_I + \Delta\dot{\mathbf{r}}_c$ (i.e., the so-called guidance error coefficients) could be utilized as follows: ($\Delta\dot{\mathbf{r}}_c$ in component form)

$$\Delta\dot{x}_c = \left(\frac{\partial\Delta\dot{x}_c}{\partial x}\right)\Delta x_I + \left(\frac{\partial\Delta\dot{x}_c}{\partial y}\right)\Delta y_I + \left(\frac{\partial\Delta\dot{x}_c}{\partial z}\right)\Delta z_I + \left(\frac{\partial^2\Delta\dot{x}_c}{\partial x\partial y}\right)\Delta x_I\Delta y_I + \cdots$$
$$x \to y,\ z. \qquad (309)$$

If a predetermined change, e.g., in orbital inclination, not just a correction of unpredictable errors is involved, then $\dot{\mathbf{r}}_I$ should be replaced by some precomputed value for the maneuver. Consequently, the velocity to be gained would be simply the difference between this $\Delta\dot{\mathbf{r}}_c$ and the $\Delta\dot{\mathbf{r}}_I$ found for the improved orbit. A more general philosophy that could be applied to low-thrust trajectories is given in Baker.[163] It should be noted that the determination of "lines of position," or position fixes are *not* necessary and serve