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Tables for Rocket and Comet Orbits

Samuel Herrick

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Contents

Introduction

The anticipated development of rocket navigation has directed serious attention, for the first time, to rectilinear motion in the two-body problem. The accompanying tables of sin E and $1-\cos E$ with argument $E-\sin E$ and the parallel tables in the hyperbolic functions make it possible to determine position and velocity from the time, for rectilinear orbits. This is accomplished, moreover, by direct interpolation, without the aid of series expansions or successive approximations. The inverse process—the determination of the time from position or velocity—is solved by inverse interpolation; the tables were designed primarily for the first of these processes, the more common of the two.

The tables may be used also in connection with nearly rectilinear motion, that is, motion in ellipses and hyperbolas whose eccentricities are near unity. Orbits of these types—cometary orbits—have generally been treated as "nearly parabolic"; that is to say, they have been handled by special tables, based upon series expansions, that correct the position and velocity that would be obtained for a parabola having the same perihelion distance. The "worm's eye view" resemblance of these orbits to the parabola near perihelion is illusory; actually they are more closely and simply related to rectilinear orbits of the same major axis.

Astronomers will be tempted, upon first consideration, to suspect that the tables will not work near perihelion for "nearly parabolic" orbits. Suprisingly enough, they do. The critical test is not the "nearly parabolic" orbit but the rectilinear one; for both kinds of orbits the device that makes the table usable is its subdivision into "ranges", the first of which, designated by "$n \geq 5$", may be used over and over again like a logarithm table or a table of powers (which it is), and yields at least eight significant figures, no matter how closely the moving object may approach to perihelion, perigee, or coincidence. Actually, these tables provide a treatment that is more general than the "nearly parabolic" one; it will handle the parabola, whereas the latter cannot handle the rectilinear orbit. The functions tabulated, moreover, have more general interest and broader application than those of the nearly parabolic tables.

Although the tables are designed for eccentricities near unity, they are usable actually for all values thereof, including zero (cf. section 4). Tables of cos E and sin E with argument E will generally be found preferable, of course, when the eccentricity is small.

So far as I have been able to ascertain, tables such as these, giving $1-\cos E$ and sin E with argument $E-\sin E$, etc. have not been previously considered in connection with rectilinear or "nearly parabolic" orbits. I was pleased to discover, however, in a recent study (April 1949) of some papers of Willard Gibbs, which had been stored for 45 years in the vaults of the Yale University Library and recently unearthed by L. P. Wheeler, that his thoughts had been similarly directed. A single sheet in his "astronomical miscellany" indicates that he contemplated a table of $(E-\sin E)/\sin E$, or of $\log (E-\sin E)$, or of $\log [(E-\sin E)/\sin^3 E]$, with argument $\log \sin E$.

1. Rectilinear Motion; Origin of the Functions

The equations relating time, position, and velocity in the two-body problem reduce to the following when the motion is rectilinear:

Ellipse	*Parabola*	*Hyperbola*	
$M=k(m_1+m_2)^{1/2}a^{-3/2}(t-T)$	$M=k(m_1+m_2)^{1/2}(t-T)$	$M=k(m_1+m_2)^{1/2}(-a)^{-3/2}(t-T)$	
$=E-\sin E$	$=\frac{1}{6}D^3$	$=\sinh F-F$	(1)
$r=a(1-\cos E)$	$r=\frac{1}{2}D^2$	$r=-a(\cosh F-1)$	
$r\dot{r}=a^{1/2}\sin E$	$r\dot{r}=D$	$r\dot{r}=(-a)^{1/2}\sinh F$	

where a is the length of the transverse or major axis ($-a>0$ for the hyperbola); k is the Gaussian gravitational constant, or a similar constant for geocentric orbits; m_1 is the mass of the sun or earth and is usually taken as unity; m_2 is the mass of the comet or rocket, usually zero; t is the time; T is the time of coincidence (=perihelion or perigee passage in nonrectilinear orbits); M is the mean anomaly for the ellipse and is not named for the other two; E is the eccentric anomaly for the ellipse; F is the corresponding angle for the hyperbola; D is a new auxiliary for the parabola; r is the radius vector; $\dot{r}=dr/d[k(m_1+m_2)t]$. For those unfamiliar with the idea of rectilinear ellipses, parabolas, and hyperbolas, the first is a rectilinear orbit of finite length $2a$ for which $\dot{r}=0$ when $r=2a$; the second is infinite with $\dot{r}\rightarrow 0$ as $r\rightarrow\infty$; the third is infinite with $\dot{r}>0$, hypothetically, when $r=\infty$.

From eq (1) it is clear that the tabulation of $1-\cos E$ and $\sin E$ with argument $E-\sin E$ will enable us to obtain r/a and $r\dot{r}/a^{1/2}$ by direct interpolation. The accompanying table is designed to give always about the same number of significant figures; whatever the size or "range" of $E-\sin E$. Similar remarks apply to the tabulation of $\cosh F-1$ and $\sinh F$ with argument $\sinh F-F$.

When E and F are sufficiently small, it is evident from the well-known series for sine, cosine, hyperbolic sine, and hyperbolic cosine that eq (1) reduce to:

Ellipse	*Hyperbola*	
$M=\frac{1}{6}E^3$	$M=\frac{1}{6}F^3$	
$r=\frac{1}{2}aE^2$	$r=\frac{1}{2}(-a)F^2$	(2)
$r\dot{r}=a^{1/2}E$	$r\dot{r}=(-a)^{1/2}F$	

For sufficiently small values of E and F, then, it is evident that our elliptic and hyperbolic tables will be the same, and that they will be giving, in fact, $\frac{1}{2}E^2$ and E with argument $\frac{1}{6}E^3$ (or similarly in F). Since the numerical values will repeat, except for the shifting of the decimal point, moreover, it is necessary to give values for only a limited range and to provide a device for shifting the decimal point, as in any other table of powers.

Before discussing this device, however, we shall note that the parabolic eq (1) may be solved by the same tables as are used for small values of E and F. These tables, that is, may be used to obtain $\frac{1}{2}D^2$ and D if entered with the argument $\frac{1}{6}D^3$. In fact, if we replace $a^{3/2}M$ or $(-a)^{3/2}M$ by $M=k(m_1+m_2)^{1/2}(t-T)$, and $a^{1/2}E$ or $(-a)^{1/2}F$ by D, we find that eq (2) reduce exactly to the parabolic equations of (1). Thus for sufficiently small values of E and F elliptic, parabolic, and hyperbolic rectilinear motion are practically indistinguishable. In computation, however, it is preferable to use the elliptic and hyperbolic equations of (1) or (2), since there is no certainty at the start that E or F is sufficiently small to permit use of the parabolic form.

The following definitions and general notations have been adopted:

Ellipse	*Parabola*	*Hyperbola*	
$U=E-\sin E$	$U=\frac{1}{6}D^3$	$U=\sinh F-F$	
$C_e(U)=1-\cos E$	$C_p(U)=\frac{1}{2}D^2$	$C_h(U)=\cosh F-1$	(3)
$S_e(U)=\sin E$	$S_p(U)=D$	$S_h(U)=\sinh F$	
$X_e(U)=E$	$X_p(U)=D$	$X_h(U)=F$	

The following quantities are tabulated for the argument $u=10^{3n}U$:

Ellipse	*Parabola*	*Hyperbola*	
$c_e=10^{2n}C_e(U)$	$c_p=10^{2n}C_p(U)$	$c_h=10^{2n}C_h(U)$	
$s_e=10^{n}S_e(U)$	$s_p=10^{n}S_p(U)$	$s_h=10^{n}S_h(U)$	(4)
$x_e=10^{n}X_e(U)$	$x_p=10^{n}X_p(U)$	$x_h=10^{n}X_h(U)$	

Since $U=E-\sin E$ goes through 0 and π with E, its tabulation as far as π makes the table complete for the ellipse; multiples of 2π should be rejected to locate U between $-\pi$ and $+\pi$. Then the X's and S's of eq (3) take the sign of U for the ellipse as well as the parabola and hyperbola; the C's are always positive.

The parabolic portion of the table is also complete, since it is essentially a table of powers, usable for any value with only a shift of the decimal point.

The hyperbolic values are carried as far as $U=\sinh F-F=300$. For values of U beyond 300 Dr. Blanch recommends the following series (the terms shown being sufficient for at least the fifth decimal place):

$$\left.\begin{aligned} C_h &= U-1+\log_e 2U+U^{-1}(\tfrac{1}{2}+\log_e 2U)-\tfrac{1}{2}U^{-2}(\log_e 2U)\ (-1+\log_e 2U)+\ldots \\ S_h &= U+(\log_e 2U)\ [1-U^{-1}-U^{-2}(-1+\tfrac{1}{2}\log_e 2U)+\ldots] \\ X_h &= (\log_e 2U)\ [1+U^{-1}-U^{-2}(-1+\tfrac{1}{2}\log_e 2U)+\ldots] \end{aligned}\right\} \quad (5)$$

The tables are arranged to show the first differences

$$\delta c_{i+\frac{1}{2}}=c_{i+1}-c_i \text{ and } \delta s_{i+\frac{1}{2}}=s_{i+1}-s_i$$

on a level midway between the levels of the arguments and functions for subscripts i and $i+1$. The modified second differences,

$$\delta^{2*}=\delta^2-0.184\delta^4, \quad (6)$$

are located as is usual with this central difference notation. That is,

$$\delta^2 c_i=\delta c_{i+\frac{1}{2}}-\delta c_{i-\frac{1}{2}}, \text{ etc., and } \delta^{2*}c_i \text{ and } \delta^{2*}s_i$$

are on the same line as u_i, c_i, and s_i. The tables are designed for use with Everett coefficients for the second differences, and the use of δ^{2*} in place of δ^2 reduces the error due to neglecting fourth differences to less than one-half unit of the last place, over and above the usual interpolation error. The differences of x are not included, since the nature of the functions makes the modified second differences of x the same as the modified second differences of s.

Linear interpolation will yield four-figure accuracy (relative error $\leq 0.000\,041$) anywhere in the table.

2. Direct Interpolation

Given U, to obtain C, S, X, that is, $C_e(U)$, $S_e(U)$, $X_e(U)$ for ellipse, $C_p(U)$, $S_p(U)$, $X_p(U)$ for parabola, or $C_h(U)$, $S_h(U)$, $X_h(U)$ for hyperbola. Select n from the "range table" (schedule A); then

$$u=10^{3n}\,U. \quad (7)$$

Locate u between two tabular arguments, u_0 and u_1 and take note of the functions and differences arranged as follows:

u_0	c_0	$\delta^{2*}c_0$	s_0	$\delta^{2*}s_0$	x_0	
		$\delta c_{\frac{1}{2}}$		$\delta s_{\frac{1}{2}}$		(8)
u_1	c_1	$\delta^{2*}c_1$	s_1	$\delta^{2*}s_1$	x_1	

Then

$$p=\frac{u-u_0}{u_1-u_0}. \quad (9)$$

Interpolate the Everett coefficients E_0^2 and E_1^2 from (p. 99).

Then

$$c=c_0+p\delta c_{1/2}+E_0^2\delta^{2*}c_0+E_1^2\delta^{2*}c_1 \tag{10}$$

$$s=s_0+p\delta s_{1/2}+E_0^2\delta^{2*}s_0+E_1^2\delta^{2*}s_1 \tag{11}$$

$$x=(1-p)x_0+px_1+E_0^2\delta^{2*}s_0+E_1^2\delta^{2*}s_1 \tag{12}$$

$$C=10^{-2n}c \qquad S=10^{-n}s \qquad X=10^{-n}x \tag{13}$$

In the following examples, eq (7), (9), and (10) are employed, and $k=0.017\ 202\ 098\ 95$, $m_1=1.0$, $m_2=0.0$.

Example 1. (*Rectilinear Ellipse*): Given $t-T=20^d.579\ 397$ and $a=18.018\ 456$, to find r, $r\dot{r}$, and E.

$U=M=0.004\ 628\ 4729$ (Schedule A gives $n=1$.)

$u=4.628\ 4729$ $E_0^2=-0.05821$

$p=0.284\ 729$ $E_1^2=-0.04361$

The following portion of the $n=1$ range of the table is used:

u	c_e	δc	$\delta^{2*}c$	s_e	δs	$\delta^{2*}s$	x_e
4.600	4.545 5290		−4839	2.980 6777		−3173	3.026 6777
		65 3341			20 8428		
4.700	4.610 8631		−4703	3.001 5206		−3061	3.048 5206

$c_e=4.564\ 1802$ $s_e=2.986\ 6441$ $x_e=3.032\ 9288$

$C_e=0.045\ 641\ 802$ $S_e=0.298\ 664\ 41$ $X_e=0.303\ 292\ 88$

$r=aC_e=0.822\ 394\ 80$ $r\dot{r}=a^{1/2}S_e=1.267\ 7752$ $E=X_e$ (in radians)

Example 2. (*Rectilinear Hyperbola*): Given $t-T=115^d.328\ 387$, $a=-87.171\ 633$, to find r, $r\dot{r}$, and F.

$U=M=+0.002\ 437\ 5576$ (Schedule A gives $n=1$.)

$u=+2.437\ 5576$ $E_0^2=-0.03890$

$p=\ 0.751\ 152$ $E_1^2=-0.05455$

The following portion of the $n=1$ range of the table is used:

u	c_h	δc	$\delta^{2*}c$	s_h	δs	$\delta^{2*}s$	x_h
2. 400	2. 968 2018		−2837	2. 454 4869		−2346	2. 430 4869
		41 2058			17 2292		
2. 450	3. 009 4076		−2760	2. 471 7161		−2267	2. 447 2161

$c_h=2.999\ 1797$ $s_h=2.467\ 4501$ $x_h=2.443\ 0746$

$C_h=0.029\ 991\ 797$ $S_h=0.246\ 745\ 01$ $X_h=0.244\ 307\ 46$

$r=-aC_h=2.614\ \ 4339$ $r\dot{r}=(-a)^{1/2}S_h=2.303\ \ 7533$ $F=X_h$ (in radians)

Example 3. (*Rectilinear Parabola*): Given $t-T=5^d.074985$, to find r and $r\dot{r}$.

$$U=M=0.087\ 3004 \qquad \text{(Schedule A gives } n=1.)$$

$$u=87.3004 \qquad E_0^2=-0.05117$$

$$p=0.6502 \qquad E_1^2=-0.06255$$

The following portion of the parabolic (or $n \geq 5$ for ellipses and hyperbolas) range of the table is used:

u	c_p	δc	$\delta^{2*}c$	s_p	δs	$\delta^{2*}s$
86. 000	32. 166 4504		−38653	8. 020 7793		−9637
		496 7914			61 7007	
88. 000	32. 663 2418		−37486	8. 082 4800		−9275

$$c_p=32.489\ 90 \qquad s_p=8.061\ 00$$

$$r=C_p=0.324\ 899 \qquad r\dot{r}=S_p=0.806\ 100 \qquad [D=X_p=S_p]$$

3. Inverse Interpolation

Given C, S, or X, that is, $C_e(U)$, $S_e(U)$, or $X_e(U)$ for ellipse, or $C_h(U)$, $S_h(U)$, or $X_h(U)$ for hyperbola—to obtain U. Select n from the "range table" (schedule A); then, if $n \geq 5$, $U=\frac{1}{6}S^3=\frac{1}{6}X^3=\frac{1}{6}(2C)^{3/2}$ as with the parabola, and it is not necessary to use the tables; if $n<5$,

$$c=10^{2n}C \qquad s=10^nS \qquad x=10^nX \tag{14}$$

Locate c, s, or x between two tabular values, c_0 and c_1, s_0 and s_1, or x_0 and x_1, and take note of the tabular entries arranged as follows:

u_0	c_0		$\delta^{2*}c_0$	s_0		$\delta^{2*}s_0$	x_0
		$\delta c_{1/2}$			$\delta s_{1/2}$		
u_1	c_1		$\delta^{2*}c_1$	s_1		$\delta^{2*}s_1$	x_1

(15)

With an approximation to p,

$$p^*=\frac{c-c_0}{\delta c_{1/2}} \text{ or } p^*=\frac{s-s_0}{\delta s_{1/2}} \text{ or } p^*=\frac{x-x_0}{x_1-x_0} \tag{16}$$

or a modified value based upon these and anticipating the effect of the modified second differences, take E_0^2 and E_1^2, (p. 99). Then

$$p=[c-c_0-E_0^2\delta^{2*}c_0-E_1^2\delta^{2*}c_1]\div\delta c_{1/2} \tag{17}$$

or

$$p=[s-s_0-E_0^2\delta^{2*}s_0-E_1^2\delta^{2*}s_1]\div\delta s_{1/2} \tag{18}$$

or

$$p=[x-x_0-E_0^2\delta^{2*}s_0-E_1^2\delta^{2*}s_1]\div(x_1-x_0). \tag{19}$$

If C, S, and X are all available and are consistent, eq (17), (18), and (19) should check. Then

$$u=u_0+p(u_1-u_0) \qquad U=10^{-3n}u. \tag{20}$$

In the following examples eq (14), (16), and (17) are employed, and $k=0.017\ 202\ 098\ 95$, $m_1=1.0$, $m_2=0.0$.

Example 1. (*Rectilinear Ellipse*): Given $r=0.822\ 394\ 80$ and $a=18.018\ 456$, to find $t-T$.

$$C_e=r/a=0.045\ 641\ 802 \qquad \text{(Schedule A gives } n=1.)$$

$$c_e=4.564\ 1802$$

The following portion of the $n=1$ range of the table is used:

u	c_e	δc	$\delta^{2*}c$
4.600	4.545 5290		−4839
		65 3341	
4.700	4.610 8631		−4703

$$c-c_0=18\ 6512$$

$$p^*=0.285 \qquad \begin{cases} E_0^2=-0.05824 \\ E_1^2=-0.04364 \end{cases}$$

$$p=0.284\ 729$$

$$u=4.628\ 4729$$

$$M=U=0.004\ 628\ 4729$$

$$t-T=20^d.579\ 397$$

Example 2. (*Rectilinear Hyperbola*): Given $r=2.614\ 4339$, and $a=-87.171\ 633$, to find $t-T$.

$$C_h=r/(-a)=0.029\ 991\ 797 \qquad \text{(Schedule A gives } n=1.)$$

$$c_h=2.999\ 1797$$

The following portion of the $n=1$ range of the table is used:

u	c_h	δc	$\delta^{2*}c$
2.400	2.968 2018		−2837
		41 2058	
2.450	3.009 4076		−2760

$$c-c_0=30\ 9779$$

$$p^*=0.751 \qquad \begin{cases} E_0^2=-0.03892 \\ E_1^2=-0.05457 \end{cases}$$

$$p=0.751\ 151$$

$$u=2.437\ 5576$$

$$M=U=0.002\ 437\ 5576$$

$$t-T=115^d.328\ 387$$

4. Nearly Rectilinear or "Nearly Parabolic" Motion

To appreciate the applicability of the tables to "nearly parabolic" orbits, it is useful to compare some of the basic formulas of the two-body problem as they are used for circular orbits (zero eccentricity), elliptic orbits of moderate eccentricity, elliptic orbits with eccentricity near unity ("nearly rectilinear" or "nearly parabolic"), and rectilinear elliptic orbits (unit eccentricity), as follows:

e	0	Moderate	Nearly 1	1	
M	E	$E-e\sin E$	$E-\sin E+(1-e)\sin E$	$E-\sin E$	
r	a	$a(1-e\cos E)$	$a[(1-e)+e(1-\cos E)]$	$a(1-\cos E)$	
x	$a\cos E$	$a(\cos E-e)$	$a[(1-e)-(1-\cos E)]$	$-a(1-\cos E)$	(21)
y	$a\sin E$	$a(1-e^2)^{1/2}\sin E$	$a(1-e^2)^{1/2}\sin E$	0	
$r\dot{r}$	0	$a^{1/2}e\sin E$	$a^{1/2}e\sin E$	$a^{1/2}\sin E$	
$r\dot{x}$	$-a^{1/2}\sin E$	$-a^{1/2}\sin E$	$-a^{1/2}\sin E$	$-a^{1/2}\sin E$	
$r\dot{y}$	$a^{1/2}\cos E$	$a^{1/2}(1-e^2)^{1/2}\cos E$	$a^{1/2}(1-e^2)^{1/2}\cos E$	0	

Definitions:

e=eccentricity
a=semimajor axis or mean distance
M=mean anomaly
E=eccentric anomaly
r=radius vector
$\dot{r}=dr/d[k(m_1+m_2)^{1/2}t]$
x, y=rectangular coordinates referred to the orbit plane, with the x-axis directed to perihelion or perigee or like point. (Note: Transformation formulas for other axes are not pertinent to the discussion.)

It should be apparent from eq (21) that the departures of the nearly rectilinear formulas from the rectilinear ones, when e is nearly 1, are of the same nature as the departures of the standard formulas from the circular ones, when e is nearly 0. Accordingly, if tables of $\cos E$ and $\sin E$ with argument E enable us to solve the problem when e is moderate, tables of $1-\cos E$ and $\sin E$ with argument $E-\sin E$ should enable us to do so when e is nearly 1. Techniques for the solution applicable to the one case will generally be applicable to the other.

The technique which we adopt for the solution of the nearly rectilinear problem is modeled upon that developed by L. J. Comrie for the solution of Kepler's equation when the eccentricity is not too near to unity [7].[1] For comparison we briefly sketch this process, slightly modified; it is convenient to divide it into two steps, the "search" and the "interpolation." For the first of these, it is convenient to write Kepler's equation in the form

$$\sin E=(E-M)/e. \quad (22)$$

It should be noted that e is assumed to be exact, so that $1/e$ may be carried to as many significant figures as the calculation requires, even when e is very small. The number $1/e$ is set up on the keyboard of a calculating machine and multiplied by $-M$; $-M/e$ then appears in the product register. The multiplier register is then cleared of $-M$; no further clearances are made until E is located between two adjacent tabular arguments E_0 and E_1.

For simplicity, we shall assume that we are using a table of sines with radian argument and that M and E are, or are treated as being, between 0 and π. From the table select a value of the argument,

[1] Figures in brackets indicate the literature reference on page XXIII.

E_a, slightly greater than M; introduce this as multiplier, so that E_a will appear in the multiplier register and $(E_a-M)/e$ on the product register. If sin E_a is greater than the latter quantity, alter the digits of E_a to those of E_b, a somewhat larger value of the argument, and repeat. If

$$\sin E_i > (E_i - M)/e,$$

increase E_i, but if

$$\sin E_i < (E_i - M)/e,$$

decrease E_i. Eventually, E will thus be located between the two tabular arguments E_0 and E_1.

The interpolation of E between E_0 and E_1 is accomplished by means of

$$E = E_0 + [M - E_0 + e \sin E_0]/(1 - e \cos E_0). \tag{23}$$

As a check we have Kepler's equation in its usual form:

$$M = E - e \sin E. \tag{24}$$

For nearly rectilinear elliptic orbits we may write Kepler's equation

$$M = U + \epsilon S \tag{25}$$

or

$$S = (M - U)/\epsilon, \tag{26}$$

where

$$U = E - \sin E \tag{27}$$

$$\epsilon = 1 - e \tag{28}$$

$$S = S_e(U) = \sin E \tag{29}$$

and, for later use,

$$C = C_e(U) = 1 - \cos E. \tag{30}$$

For simplicity, we shall assume that the accuracy obtainable from the accompanying tables by linear interpolation is sufficient, and that C and S are tabulated directly against U, as in schedule A and the $n=0$ range. The "search" is then as follows:

Set $1/\epsilon$ on the keyboard (carry $1/\epsilon$ to seven or eight significant figures, even when ϵ has fewer); multiply by M and clear the multiplier register; select a tabular $U_a < M$; multiply by $-U_a$ and check against S_a; multiply by larger or smaller U_b by changing the digits of U_a without clearing and check against S_b, etc. If

$$S_i < (M - U_i)/\epsilon$$

increase U_i, but if

$$S_i > (M - U_i)/\epsilon$$

decrease U_i. Eventually, U will be located between the two tabular arguments U_0 and U_1.

The interpolation between U_0 and U_1 is accomplished by means of

$$U = U_0 + [M - U_0 - \epsilon S_0]C_0/(\epsilon + eC_0). \tag{31}$$

For check we have eq (25). This process is somewhat modified in the next two sections to meet the requirements of tabulation by ranges and of nonlinear interpolation.

The experienced computer will note that the foregoing process and the accompanying tables with argument $U = E - \sin E$ eliminate, for all eccentricities, the difficulty encountered in Kepler's equation in its standard form, namely, the lack of sufficient figures in any table of sin E to give accuracy in $M = E - e \sin E$ when E is nearly zero and e nearly unity. A similar cancellation of significant figures never occurs in $M = U + \epsilon S$. He will also observe that modifications are possible in both of the foregoing procedures, and will adopt one to his taste.

The basic hyperbolic formulas that may be handled by the tables are the following:

$$\left.\begin{aligned}
M&=e\sinh F-F=\sinh F-F+(e-1)\sinh F\\
r&=-a(e\cosh F-1)=-a\,[e(\cosh F-1)+(e-1)]\\
x&=a(\cosh F-e)=a\,[(\cosh F-1)-(e-1)]\\
y&=-a(e^2-1)^{1/2}\sinh F\\
r\dot{r}&=(-a)^{1/2}e\sinh F\\
r\dot{x}&=-(-a)^{1/2}\sinh F\\
r\dot{y}&=(-a)^{1/2}(e^2-1)^{1/2}\cosh F,
\end{aligned}\right\}\qquad(32)$$

where M and F are the hyperbolic equivalents of the mean and eccentric anomalies, and the other quantities are defined after eq (21). These equations should be compared with those for the rectilinear hyperbola ($e=1$), eq (1).

The general (nonrectilinear) parabola may also be handled by the tables, when the values encountered lie beyond the ranges or capacities of the tables of Subbotin, Strömgren, and Möller [3a, 4, 5, 6]. For this purpose, we note the following basic parabolic formulas:

$$\left.\begin{aligned}
M&=\tfrac{1}{6}(2q)^{3/2}\left(\tan^3\frac{v}{2}+3\tan\frac{v}{2}\right)&&=\tfrac{1}{6}D^3+qD\\
r&=q\left(1+\tan^2\frac{v}{2}\right)&&=q+\tfrac{1}{2}D^2\\
x&=q\left(1-\tan^2\frac{v}{2}\right)&&=q-\tfrac{1}{2}D^2\\
y&=2q\tan\frac{v}{2}&&=\sqrt{2q}\,D\\
r\dot{r}&=\sqrt{2q}\tan\frac{v}{2}&&=D\\
r\dot{x}&=-\sqrt{2q}\tan\frac{v}{2}&&=-D\\
r\dot{y}&=\sqrt{2q}&&=\sqrt{2q}
\end{aligned}\right\}\qquad(33)$$

where q is the perihelion or perigee distance, and v is the true anomaly. D has been mentioned in connection with the rectilinear parabola ($q=0$); cf. eq (1).

Equations (27), (28), (29), and (30) we shall parallel as follows:

Parabola	*Hyperbola*
$U=\tfrac{1}{6}D^3$	$U=\sinh F-F$
$\epsilon=q$	$\epsilon=e-1$
$S=S_p(U)=D$	$S=S_h(U)=\sinh F$
$C=C_p(U)=\tfrac{1}{2}D^2$	$C=C_h(U)=\cosh F-1$

(34)

with these definitions we shall be able to write identical formulas for the three kinds of orbit in many of the steps of the following sections.

5. Position and Velocity From Time

Given $t-T, e$, and a or $q=a(1-e)$, to find r, x, y, $r\dot{r}$, $r\dot{x}$, $r\dot{y}$ for a parabola or a nearly rectilinear ellipse or hyperbola.

$$\left.\begin{array}{lll}\text{Ellipse:} & M=k(m_1+m_2)^{1/2}a^{-3/2}(t-T), & 1-e=\epsilon \\ \text{Parabola:} & M=k(m_1+m_2)^{1/2}(t-T), & q=\epsilon \\ \text{Hyperbola:} & M=k(m_1+m_2)^{1/2}(-a)^{-3/2}(t-T), & e-1=\epsilon\end{array}\right\} \tag{35}$$

For the determination of n, locate U between two successive values U_0 and U_1 in the appropriate part of schedule A, such that

$$\left.\begin{array}{l}S_0<(M-U_0)/\epsilon \\ S_1>(M-U_1)/\epsilon\end{array}\right\} \tag{36}$$

[Cf. more detailed instructions after eq (30). Restore M/ϵ to the product register, then shift decimal point in this register until it reads $10^{3n}M/\epsilon$. Narrow the search, as above, in the n range of the appropriate part of the main table until u is located between two successive values u_0 and u_1 such that

$$\left.\begin{array}{l}10^{2n}s_0<(10^{3n}M-u_0)/\epsilon \\ 10^{2n}s_1>(10^{3n}M-u_1)/\epsilon\end{array}\right\} \tag{37}$$

Then

$$u=u_0+[10^{3n}M-u_0-10^{2n}\,\epsilon s_0]\div[e+10^{2n}\,\epsilon c_0^{-1}] \tag{38}$$

$$p=\frac{u-u_0}{u_1-u_0}. \tag{39}$$

With this value of u take E_0^2 and E_1^2 from page 99; then interpolate s as in section 2:

$$s=s_0+p\delta s_{1/2}+E_0^2\delta^{2*}s_0+E_1^2\delta^{2*}s_1. \tag{40}$$

Check calculation by

$$10^{3n}M-u-10^{2n}\epsilon s=0. \tag{41}$$

If this check does not indicate that sufficient accuracy has been attained—usually will not the first time except in rough calculations—repeat eq (38), using the approximate u and s in place of u_0 and s_0, in order to obtain an improved value of u. [Note that the first bracket is the left-hand side of eq (41), and that in the second bracket c_0 does not have to be revised.]

Repeat eq (39) with the improved u, but use the original u_0 and u_1.

Repeat eq (40); usually it will not be necessary to revise E_0 and E_1.

Repeat the check, eq (41). This second check should be sufficient if the calculation is free from error; if not, repeat eq (38), etc. Finally,

$$c=c_0+p\delta c_{1/2}+E_0^2\delta^{2*}c_0+E_1^2\delta^{2*}c_1 \tag{42}$$

Then

$$C=10^{-2n}c, \qquad S=10^{-n}s \tag{43}$$

and

$$\left.\begin{array}{l|l|l}\textit{Ellipse} & \textit{Parabola} & \textit{Hyperbola} \\ r=a(\epsilon+eC) & r=q+C & r=-a(\epsilon+eC) \\ x=a(\epsilon-C) & x=q-C & x=-a(\epsilon-C) \\ y=a(1-e^2)^{1/2}S & y=(2q)^{1/2}S & y=-a(e^2-1)^{1/2}S \\ r\dot{r}=a^{1/2}eS & r\dot{r}=S & r\dot{r}=(-a)^{1/2}eS \\ r\dot{x}=-a^{1/2}S & r\dot{x}=-S & r\dot{x}=-(-a)^{1/2}S \\ r\dot{y}=a^{1/2}(1-e^2)^{1/2}(1-C) & r\dot{y}=(2q)^{1/2} & r\dot{y}=(-a)^{1/2}(e^2-1)^{1/2}(1+C)\end{array}\right\} \tag{44}$$

In the following examples $k=0.017\ 202\ 098\ 95$, $m_1=1.0$, $m_2=0.0$.

Example 1. (*Ellipse*): Given $t-T=63^d.54400$, $a=18.018\ 456$, $e=0.967\ 645\ 67$, to find r, x, y, $r\dot{r}$, $r\dot{x}$, $r\dot{y}$. This is a Bauschinger example [3b, 12a] ($q=0.582\ 975\ 07$).

$$\epsilon=\ 0.032\ 354\ 33 \qquad \epsilon^{-1}=\ 30.907\ 764$$

$$M=+0.014\ 291\ 560 \qquad M\epsilon^{-1}=+0.441\ 720\ 16$$

U is now determined to be in the $n=1$ range by means of the following data from the elliptic portion of schedule A:

n	U	S	$(M-U)/\epsilon$
1	0.000 15	0.096 413 944	+0.44
	0.15	0.831 121 61	−4.19

Thus

$$n=1, \qquad 10^{3n}M\ \epsilon^{-1}=+441.720\ 16, \qquad 10^{2n}\epsilon=+3.235\ 433,$$

and since for

$$u_0=4.600 \qquad (10^{3n}M-u_0)/\epsilon=+2.995\times 10^{2n}$$

$$u_1=4.700 \qquad (10^{3n}M-u_1)/\epsilon=+2.965\times 10^{2n},$$

the following portion of the $n=1$ range of the main table is selected:

u	c_e	δc	$\delta^{2*}c$	s_e	δs	$\delta^{2*}s$
4.600	4.545 5290		−4839	2.980 6777		−3173
		65 3341			20 8428	
4.700	4.610 8631		−4703	3.001 5206		−3061

$$e+10^{2n}\epsilon c_0^{-1}=+1.679\ 43$$

$$10^{3n}M-u_0-10^{2n}\epsilon s_0=+0.047\ 777$$

$$u=4.628\ 448$$

$$p=0.284\ 48 \qquad E_0^2=-0.05819$$

$$s=2.986\ 6389 \qquad E_1^2=-0.04357$$

$$10^{3n}M-u-10^{2n}\epsilon s=+0.000\ 042$$

$$u=4.628\ 473$$

$$p=0.284\ 73$$

$$s=2.986\ 6441$$

$$10^{3n}M-u-10^{2n}\epsilon s=+0.000\ 000 \quad \text{[check]}$$

$$c=\ 4.564\ 1802$$

$$C=\ 0.045\ 641\ 802$$

$$S = 0.298\ 664\ 41$$

$$r = 1.378\ 7618 \qquad r\dot{r} = +1.226\ 757\ 21$$

$$x = -0.239\ 419\ 69 \qquad r\dot{x} = -1.267\ 775\ 23$$

$$y = +1.357\ 815\ 28 \qquad r\dot{y} = +1.022\ 138\ 72$$

From these values we may obtain also $v = 100^0.000\ 01$, which is apparently a slightly more accurate result than that of Bauschinger.

Example 2. (*Hyperbola*): Given $t-T=216^d.404\ 21$, $a=-87.171\ 633$, $e=1.008\ 658$, to find r, x, y, $r\dot{r}$, $r\dot{x}$, $r\dot{y}$. This is also a Bauschinger example [3c] ($q=0.754\ 732$).

$$\epsilon = +0.008\ 658 \qquad \epsilon^{-1} = 115.500\ 12$$

$$M = +0.004\ 573\ 8759 \qquad M\epsilon^{-1} = +0.528\ 2832$$

U is now determined to be in the $n=1$ range by means of the following data from the hyperbolic portion of schedule A:

n	U	S	$(M-U)/\epsilon$
1	0. 000 15	0. 096 683 944	+0. 51
	0. 15	1. 101 059 31	−16. 79

Thus

$$n=1, \qquad 10^{3n}M\epsilon^{-1} = +528.2832, \qquad 10^{2n}\epsilon = +0.8658,$$

and since for

$$u_0 = 2.400, \qquad (10^{3n}M - u_0)/\epsilon = +2.511 \times 10^{2n},$$

$$u_1 = 2.450 \qquad (10^{3n}M - u_1)/\epsilon = +2.453 \times 10^{2n},$$

the following portion of the $n=1$ range of the main table is selected:

u	c_h	δc	$\delta^{2*}c$	s_h	δs	$\delta^{2*}s$
2. 400	2. 968 2018		−2837	2. 454 4869		−2346
		41 2058			17 2292	
2. 450	3. 009 4076		−2760	2. 471 7161		−2267

$$e + 10^{2n}\epsilon c_0^{-1} = +1.300\ 35$$

$$10^{3n}M - u_0 - 10^{2n}\epsilon s_0 = +0.048\ 781$$

$$u = 2.437\ 514$$

$$p = 0.750\ 28 \qquad E_0^{\prime 2} = -0.03902$$

$$s = 2.467\ 4352 \qquad E_1^{\prime 2} = -0.05466$$

$$10^{3n}M - u - 10^{2n}\epsilon s = +0.000\ 057$$

$$u=2.437\ 557$$

$$p=0.751\ 14$$

$$s=2.467\ 4499$$

$$10^{3n}M-u-10^{2n}\epsilon\, s=+0.000\ 001 \quad \text{[check]}$$

$c=\ 2.999\ 1797$
$C=\ 0.029\ 991\ 797$
$S=\ 0.246\ 745\ 01$
} Result of an additional, hardly necessary, approximation.

$r=\ 3.391\ 8017 \qquad r\dot{r}=\ 2.323\ 6992$

$x=-1.859\ 7019 \qquad r\dot{x}=-2.303\ 7533$

$y=+2.836\ 5167 \qquad r\dot{y}=+1.268\ 1866$

From these values we may obtain also $v=123°.25000$, to check Bauschinger's result.

Example 3. (*Parabola*): Given $t-T=5^d.5436$, $q=0.01$, $e=1.0$, to find r, x, y, $r\dot{r}$, $r\dot{x}$, $r\dot{y}$.

$$\epsilon=q=0.01 \qquad \epsilon^{-1}=100.0$$

$$M=+0.095\ 3616 \qquad M\epsilon^{-1}=+9.536\ 16$$

U is now determined to be in the $n=1$ range by means of the following data from the parabolic portion of schedule A:

n	U	S	$(M-U)/\epsilon$
1	0.000 15	0.096 548 938	+9.521
	0.15	0.965 489 38	−5.463

Thus

$$n=1, \qquad 10^{3n}M\epsilon^{-1}=+9536.16, \qquad 10^{2n}\epsilon=+1.0,$$

and since for

$$u_0=86.000 \qquad (10^{3n}M-u_0)/\epsilon=9.3616\times10^{2n}$$

$$u_1=88.000 \qquad (10^{3n}M-u_1)/\epsilon=7.3616\times10^{2n}$$

the following portion of the parabolic (or $n\geq5$ for ellipses and hyperbolas) range of the main table is selected:

u	c_p	δc	$\delta^{2*}c$	s_p	δs	$\delta^{2*}s$
86. 000	32. 166 4504		−38653	8. 020 7793		−9637
		496 7914			61 7007	
88. 000	32. 663 2418		−37486	8. 082 4800		−9275

$$e+10^{2n}\epsilon c_0^{-1}=+1.0311$$

$$10^{3n}M-u_0-10^{2n}\epsilon s_0=+1.3408$$

$$u=87.3004$$

969895 O—53——2

$$p= 0.6502 \qquad E_0^2=-0.05117$$

$$s= 8.06100 \qquad E_1^2=-0.06255$$

$$10^{3n}M-u-10^{2n}\epsilon s=0.0002 \quad \text{[check]}$$

Note: no further approximation is necessary to the accuracy desired here. Compare examples 1 and 2

$$c= 32.4899$$

$$C= 0.324\,899$$

$$S= 0.806\,100$$

$$r= 0.334\,899 \qquad r\dot{r}= 0.806\,100$$

$$x=- 0.314\,899 \qquad r\dot{x}=-0.806\,100$$

$$y=+ 0.114\,000 \qquad r\dot{y}= 0.141\,421$$

For comparison with the Möller table [3a, 6], we find and check $\tan v/2=5.7000$. This is the last entry in the Möller table, beyond which, or for greater accuracy, the foregoing process is evidently useful.

6. Time from Position or Velocity

Given e, a or $q=a(1-e)$, and some of r, x, y, $r\dot{r}$, $r\dot{x}$, $r\dot{y}$, to find $t-T$ for a parabola or a nearby rectilinear ellipse or hyperbola.

Ellipse	*Parabola*	*Hyperbola*
$\epsilon=1-e$	$\epsilon=q$	$\epsilon=e-1$
$C=(ra^{-1}-\epsilon)/e$	$C=r-q$	$C=[r(-a)^{-1}-\epsilon]/e$
$=\epsilon-xa^{-1}$	$=q-x$	$=\epsilon-x(-a)^{-1}$
$=1-r\dot{y}a^{-1/2}(1-e^2)^{-1/2}$		$=r\dot{y}(-a)^{-1/2}(e^2-1)^{-1/2}-1$
$S=r\dot{r}\,a^{-1/2}\,e^{-1}$	$S=r\dot{r}$	$S=r\dot{r}(-a)^{-1/2}e^{-1}$
$=-r\dot{x}\,a^{-1/2}$	$=-r\dot{x}$	$=-r\dot{x}(-a)^{-1/2}$
$=y\,a^{-1}(1-e^2)^{-1/2}$	$=y(2q)^{-1/2}$	$=y(-a)^{-1}(e^2-1)^{-1/2}$
$X=E$	$X=D=S$	$X=F$

(45)

For ellipse and hyperbola refer now to schedule A to find n and the proper range of the main table. For the parabola, and for ellipse and hyperbola if $n\geq 5$, we need not make use of the table, since

$$U=\tfrac{1}{6}S^3=\tfrac{1}{6}X^3=\tfrac{1}{6}(2C)^{3/2} \tag{46}$$

Otherwise,

$$c=10^{2n}C, \qquad s=10^nS, \qquad x=10^nX. \tag{47}$$

By inverse interpolation (cf. section 3) determine u. Then

$$U=10^{-3n}u \tag{48}$$

$$M=U+\epsilon S \tag{49}$$

$$\left.\begin{aligned} &\text{Ellipse:} && t-T=Ma^{3/2}k^{-1}(m_1+m_2)^{-1/2}\\ &\text{Parabola:} && t-T=Mk^{-1}(m_1+m_2)^{-1/2}\\ &\text{Hyperbola:} && t-T=M(-a)^{3/2}k^{-1}(m_1+m_2)^{-1/2}\end{aligned}\right\} \tag{50}$$

In the following examples $k=0.017\ 202\ 098\ 95$, $m_1=1.0$, $m_2=0.0$.

Example 1. (*Ellipse*): Given $e=0.967\ 645\ 67$, $a=18.018\ 456$, $x=-0.239\ 419\ 73$, $y=+1.357\ 815\ 28$, to find $t-T$. This is a Bauschinger example [3*b*, 12*a*] ($q=0.582\ 975\ 07$, $v=100°.0000$).

$\epsilon=0.\ 032\ \ 354\ \ 33$

$C=0.\ 045\ \ 641\ \ 802$ $\qquad S=0.\ 298\ \ 664\ \ 41$ $\qquad$ (Schedule A gives $n=1$.)

$c=4.\ 564\ \ 1802$ $\qquad s=2.\ 986\ \ 6441$

The following portion of the $n=1$ range of the table is used.

u	c_e	δc	$\delta^{2*}c$	s_e	δs	$\delta^{2*}s$
4. 600	4. 545 5290		−4839	2. 980 6777		−3173
		65 3341			20 8428	
4. 700	4. 610 8631		−4703	3. 001 5206		−3061

$c-c_0=18\ \ 6512$ $\qquad s-s_0=5\ \ 9664$

$p^*=0.\ 285\ \ 474$ $\qquad p^*=0.\ 286\ \ 257$

$p^*=0.\ 285$ is adopted because the second difference terms will evidently reduce both of these values.

$E_0^2=-0.\ 058\ \ 24$

$E_1^2=-0.\ 043\ \ 64$

$p=\ 0.\ 284\ \ 729$ $\qquad p=0.\ 284\ \ 730$

$u=\ 4.\ 628\ \ 4729$

$U=\ 0.\ 004\ \ 628\ \ 4729$

$M=\ 0.\ 014\ \ 291\ \ 560$ $\qquad t-T=63^d.\ 54400$

Example 2. (*Hyperbola*): Given $e=1.008\ 658$, $a=-87.171\ 6331$, $x=-1.859\ 7019$, $y=+2.836\ 5167$, to find $t-T$. This is also a Bauschinger example [3c] ($q=0.754\ 732$, $v=123°.25000$).

$\epsilon=0.008\ 658$

$C=0.029\ 991\ 797$ $\qquad S=0.246\ 745\ 01$ $\qquad$ (Schedule A gives $n=1$.)

$c=2.999\ 1797$ $\qquad s=2.467\ 4501$

The following portion of the $n=1$ range of the table is used.

u	c_h	δc	$\delta^{2*}c$	s_h	δs	$\delta^{2*}s$
2.400	2.968 2018		−2837	2.454 4869		−2346
		41 2058			17 2292	
2.450	3.009 4076		−2760	2.471 7161		−2267

$c-c_0=30\ 9779$ $\qquad s-s_0=12\ 9632$

$p^*=\ 0.751\ 785$ $\qquad p^*=0.752\ 397$

$p^*=$ 0.751 is adopted because the second difference terms will evidently reduce both of these values.

$E_0^2=-0.03892$

$E_1^2=\ 0.05457$

$p=\ 0.751\ 151$ $\qquad p=0.751\ 149$

$u=\ 2.437\ 5576$

$U=\ 0.002\ 437\ 5576$

$M=\ 0.004\ 573\ 8759$ $\qquad t-T=216^d.40421$

7. Method of Computation of the Tables [2]

Given

$$U=E-\sin E,\quad C_e(U)=1-\cos E,\quad S_e(U)=\sin E. \tag{51}$$

The power series for U and $C_e(U)$ in terms of E are

$$\left.\begin{aligned} U&=\frac{E^3}{3!}-\frac{E^5}{5!}+\dots \\ C_e(U)&=\frac{E^2}{2!}-\frac{E^4}{4!}+\dots \end{aligned}\right\} \tag{52}$$

Let $\omega=(6U)^{1/3}$. It follows from (51) that E and $S_e(U)$ have MacLaurin expansions in ω, $C_e(U)$ has a MacLaurin expansion in ω^2, and for ω sufficiently small the series converge rapidly. It is enough to obtain either $C=C_e(U)$ or $S=S_e(U)$ in terms of U, and then to use the relation

$$S=\sqrt{1-(1-C)^2}=\sqrt{2C-C^2}.$$

Actually, the power series for $C_e(U)$ was obtained by inverting (52):

$$C_e(U)=\sum_{n=1}^{8} A_n\omega^{2n}+R,$$

where, for $U\leq 1/6$ (i. e., $\omega^2\leq 1$), $|R|<10^{-10}$ We give the values of A_n:

$$A_1=\frac{1}{2}=.5$$

$$A_2=-\frac{1}{2^3\cdot 5}=-.025$$

$$A_3=-\frac{3}{2^5\cdot 5^2\cdot 7}=-.00053\ 57142\ 85714$$

$$A_4=-\frac{23}{2^7\cdot 3^2\cdot 5^3\cdot 7}=-.00002\ 28174\ 60317$$

$$A_5=-\frac{947}{2^8\cdot 3^2\cdot 5^4\cdot 7^2\cdot 11}=-.00000\ 12201\ 09256$$

$$A_6=-\frac{3293}{2^{11}\cdot 5^5\cdot 7^2\cdot 11\cdot 13}=-.00000\ 00734\ 31033$$

$$A_7=-\frac{604523}{2^{11}\cdot 3^4\cdot 5^6\cdot 7^3\cdot 11\cdot 13}=-.00000\ 00047\ 54969$$

$$A_8=-\frac{11192989}{2^{15}\cdot 3^4\cdot 5^6\cdot 7^3\cdot 11\cdot 13\cdot 17}=-.00000\ 00003\ 23677$$

[2] This secton was written by G. Blanch and R. L. Lipkis of the National Bureau of Standards Institute for Numerical Analysis.

For our present purposes it would have been enough to carry 10 decimals in the value of A_n, since the series was used for $\omega^2<1$. The more accurate values are being exhibited for their possible usefulness in other investigations.

For $U\geq .17$, 10-place values of E were obtained by inverse interpolation in the function $U=E-\sin E$. The sines were obtained from BAAS, Vol. I[14]. Once E was known, $C_e(U)$ and $S_e(U)$ were computed in turn from (51). The value corresponding to $U=.17$ was computed by both methods and results were in agreement to within 2×10^{-10}.

Similarly, let

$$U=\sinh F-F, \qquad C_h(U)=\cosh F-1, \qquad S_h(U)=\sinh F.$$

Then if $\omega=(6U)^{1/3}$,

$$C_h(U)=\sum_{n=1}^{8} B_n\omega^{2n}+R,$$

where $B_{2n-1}=A_{2n-1}$, $B_{2n}=-A_{2n}$, and for $\omega^2<1$, $|R|<10^{-10}$.

Let $U=u\cdot 10^{-3n}$, then $\omega^2=(6u)^{2/3}10^{-2n}=b\cdot 10^{-2n}$.

Hence

$$C_e(U)=10^{-2n}(C_1+C_2), \qquad C_h(U)=10^{-2n}(C_1-C_2), \tag{53}$$

where

$$\left.\begin{aligned} C_1&=A_1b+A_3b^3\cdot 10^{-4n}+A_5b^5\cdot 10^{-8n}+A_7b^7\cdot 10^{-12n}+\ldots \\ C_2&=A_2b^2\cdot 10^{-2n}+A_4b^4\cdot 10^{-6n}+A_6b^6\cdot 10^{-10n}+A_8b^8\cdot 10^{-14n}+\ldots \end{aligned}\right\} \tag{54}$$

Thus the four terms of C_1 and four terms of C_2 are the same for all n, except for the position of the decimal point. Since the argument u is the same for all values of n, except $n=0$, the individual terms of C_1 and C_2 were computed as accurately as they were needed for $n=0$ or 1. Then by merely shifting the decimal point, the corresponding terms were available for the other values of n. It is clear that with increasing n, several terms dropped out. All work involving the series (the greater bulk of the table) was done by means of IBM equipment—the 602 multiplier and 405 tabulator. The values of $(6u)^{2/3}$ were taken from the NBS Table of Fractional Powers [17] (given to 15D). This table has been extended here to give, alongside every $x^{1/3}$ and $x^{2/3}$, the corresponding values of $(10x)^{1/3}$, $(100x)^{1/3}$, $(10x)^{2/3}$, and $(100x)^{2/3}$.

For $U\geq .17$, the function F was obtained by inverse interpolation in $\sinh F-F$; $\sinh F$ was computed with the aid of Van Orstrand's tables [15] of e^x and e^{-x}, and Holtappel's table [16]. This part of the work was done by means of calculating machines. All values were differenced.

Consider the eq (53) and (54) for $n\geq 5$. Inserting the values of A_n, with $U=10^{-3n}u$, we have

$$c_e=10^{2n}C_e(U)=\tfrac{1}{2}(6u)^{2/3}-\frac{10^{-2n}}{40}(6u)^{4/3}+\cdots, \tag{55}$$

$$c_h=10^{2n}C_h(U)=\tfrac{1}{2}(6u)^{2/3}+\frac{10^{-2n}}{40}(6u)^{4/3}+\cdots. \tag{56}$$

It can also be shown that

$$s_e=10^{n}S_e(U)=(6u)^{1/3}-\frac{9}{10}10^{-2n}u+\cdots, \tag{57}$$

$$s_h=10^{n}S_h(U)=(6u)^{1/3}+\frac{9}{10}10^{-2n}u+\cdots, \tag{58}$$

$$x_e=10^{n}X_e(U)=(6u)^{1/3}+\frac{1}{10}10^{-2n}u+\cdots, \tag{59}$$

$$x_h=10^{n}X_h(U)=(6u)^{1/3}-\frac{1}{10}10^{-2n}u+\cdots. \tag{60}$$

As u ranges between 0 and 150, $\frac{1}{40}(6u)^{4/3}$ ranges between 0 and 218; hence from (55) and (56), $c_e=c_h=\frac{1}{2}(6u)^{2/3}$ to within $2\cdot10^{-10}$ for $n\geq6$ and to within $218\cdot10^{-10}$ for $n=5$. It follows that in the region where seven decimals are given in the entries ($u>.34$), only an occasional correction in rounding is required for $n=5$ if we set $c_e=c_h=\frac{1}{2}(6u)^{2/3}$. It is clear from (57) and (58) that similar rounding corrections for s_e and s_h, when these functions are equated to $(6u)^{1/3}$, are even smaller; and from (59) and (60), no correction a t all is required if we set $x_h=x_e=(6u)^{1/3}$, to within 1.5 units in the ninth decimal for $n\geq5$. For $u\leq.34$, where eight decimals are given in the entries, the term $\frac{1}{40}(6u)^{4/3}$ is less than .065; hence no corrections at all are needed in any of the entries. The rounding corrections, where necessary in c_e, c_h, s_e, s_h for $n=5$ (but not for $n>5$), are indicated alongside the entries by the following symbols:

a: reduce c_e only by one unit in last place
b: increase c_h only by one unit in last place
c: reduce s_e only by one unit in last place
d: increase s_h only by one unit in last place.

In the region where six decimals are given in the function $S_h(U)$, the results are good to within a unit in the last place; where seven or eight places are given, the error is believed to be less than 0.6 units of the last place.

The final manuscript, which was reproduced by a photo-offset process, was prepared on an IBM card-controlled typewriter. Adequate testing of the manuscript was therefore an important problem. We describe below the procedure that was followed.

After the tested entries were rounded, they were listed, summed in groups of ten, and their first differences taken during the same run through an IBM 405 tabulator. Successive differences were then taken of this first difference, up to order six of the original entries. Since the unrounded entries were differenced and presumed correct, the differences of the rounded entries should have revealed few, if any, errors. Actually, a few errors were found, due to improper corrections. It is clear that the differences of the rounded entries were guaranteed to be those corresponding to the entries which were summed. After the final manuscript was completed, the entries on the manuscript were summed on a Sundstrand adding machine in the same groups of ten, and the sums so obtained were subtracted from the recorded sums for the rounded punched cards. A zero total was accepted as proof that the entries on the manuscript were the same as those on the tested punched cards. In addition, the pages were inspected for continuity and for missing digits (since a missing zero would not be picked up by the test applied). Several errors were discovered, most of them due to digits which failed to print. Some were operators' errors, due apparently to improper card replacements when there was machine trouble. All regions where changes were made, either in the listing of the preliminary rounded entries or in the final manuscript, were redifferenced on a Sundstrand adding machine.

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SAMUEL HERRICK.

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