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FOUNDATIONS OF DIFFERENTIAL GEOMETRY

By Shoshichi Kobayashi and Katsumi Nomizu

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FOUNDATIONS OF DIFFERENTIAL GEOMETRY

VOLUME I

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PREFACE

Differential geometry has a long history as a field of mathematics and yet its rigorous foundation in the realm of contemporary mathematics is relatively new. We have written this book, the first of the two volumes of the *Foundations of Differential Geometry*, with the intention of providing a systematic introduction to differential geometry which will also serve as a reference book.

Our primary concern was to make it self-contained as much as possible and to give complete proofs of all standard results in the foundation. We hope that this purpose has been achieved with the following arrangements. In Chapter I we have given a brief survey of differentiable manifolds, Lie groups and fibre bundles. The readers who are unfamiliar with them may learn the subjects from the books of Chevalley, Montgomery-Zippin, Pontrjagin, and Steenrod, listed in the *Bibliography*, which are our standard references in Chapter I. We have also included a concise account of tensor algebras and tensor fields, the central theme of which is the notion of derivation of the algebra of tensor fields. In the *Appendices*, we have given some results from topology, Lie group theory and others which we need in the main text. With these preparations, the main text of the book is self-contained.

Chapter II contains the connection theory of Ehresmann and its later development. Results in this chapter are applied to linear and affine connections in Chapter III and to Riemannian connections in Chapter IV. Many basic results on normal coordinates, convex neighborhoods, distance, completeness and holonomy groups are proved here completely, including the de Rham decomposition theorem for Riemannian manifolds.

In Chapter V, we introduce the sectional curvature of a Riemannian manifold and the spaces of constant curvature. A more complete treatment of properties of Riemannian manifolds involving sectional curvature depends on calculus of variations and will be given in Volume II. We discuss flat affine and Riemannian connections in detail.

In Chapter VI, we first discuss transformations and infinitesimal transformations which preserve a given linear connection or a Riemannian metric. We include here various results concerning Ricci tensor, holonomy and infinitesimal isometries. We then treat the extension of local transformations and the so-called equivalence problem for affine and Riemannian connections. The results in this chapter are closely related to differential geometry of homogeneous spaces (in particular, symmetric spaces) which are planned for Volume II.

In all the chapters, we have tried to familiarize the readers with various techniques of computations which are currently in use in differential geometry. These are: (1) classical tensor calculus with indices; (2) exterior differential calculus of E. Cartan; and (3) formalism of covariant differentiation $\nabla_X Y$, which is the newest among the three. We have also illustrated, as we see fit, the methods of using a suitable bundle or working directly in the base space.

The *Notes* include some historical facts and supplementary results pertinent to the main content of the present volume. The *Bibliography* at the end contains only those books and papers which we quote throughout the book.

Theorems, propositions and corollaries are numbered for each section. For example, in each chapter, say, Chapter II, Theorem 3.1 is in Section 3. In the rest of the same chapter, it will be referred to simply as Theorem 3.1. For quotation in subsequent chapters, it is referred to as Theorem 3.1 of Chapter II.

We originally planned to write one volume which would include the content of the present volume as well as the following topics: submanifolds; variations of the length integral; differential geometry of complex and Kähler manifolds; differential geometry of homogeneous spaces; symmetric spaces; characteristic classes. The considerations of time and space have made it desirable to divide the book in two volumes. The topics mentioned above will therefore be included in Volume II.

In concluding the preface, we should like to thank Professor L. Bers, who invited us to undertake this project, and Interscience Publishers, a division of John Wiley and Sons, for their patience and kind cooperation. We are greatly indebted to Dr. A. J. Lohwater, Dr. H. Ozeki, Messrs. A. Howard and E. Ruh for their kind help which resulted in many improvements of both the content and the presentation. We also acknowledge the grants of the National Science Foundation which supported part of the work included in this book.

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CONTENTS

CHAPTER IV

Riemannian Connections

CHAPTER V

Curvature and Space Forms

CHAPTER VI

Transformations

APPENDICES

NOTES