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The Application of Lambert's Theorem to the Solution of Interplanetary Transfer Problems

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ABSTRACT

Lambert's equations are stated and discussed for elliptical, hyperbolic, and parabolic trajectories. Numerical results are obtained for Earth-to-Venus and Earth-to-Mars free-flight transfers, with the assumptions of the two-body (Sun-spacecraft) approximation and constant planetary radii. Straightforward methods, to determine which forms of Lambert's equations are valid for a few specific problems, are presented.

I. INTRODUCTION

Suppose a body under the influence of a central gravitational force is observed to travel from a point P_1 on its conic trajectory, to a point P_2 in a time T. The time of flight is related to other variables by Lambert's theorem, which states:

The transfer time of a body moving between two points on a conic trajectory is a function only of the sum of the distances of the two points from the origin of force, the linear distance between the points, and the semimajor axis of the conic.

In mathematical terminology,

$$T = T(r_1 + r_2, c, a) \tag{1}$$

where r_1, r_2, and c are labeled in Fig. 1 and a is the semimajor axis.

In most problems of interplanetary transfer, a space mission is specified, that is, r_1 and r_2 are known. If r_1 and r_2 are known, c can be related to the heliocentric transfer angle θ by the law of cosines:

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta \tag{2}$$

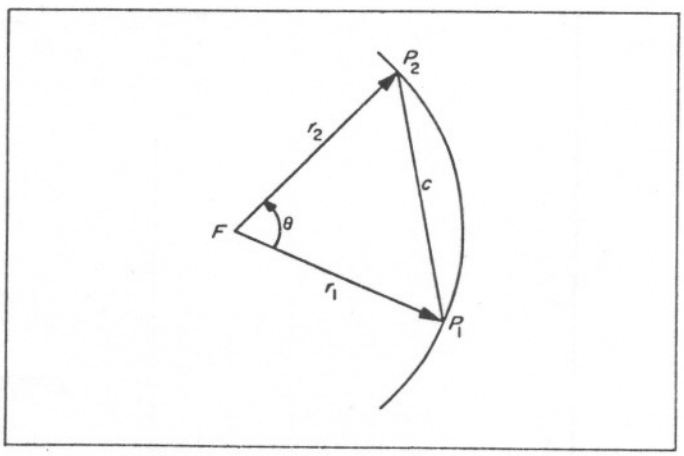

Fig. 1. Geometry for Lambert's theorem

Thus, Lambert's theorem can be stated for a specific mission by

$$T = T(a, \theta) \tag{3}$$

Lambert's theorem can be expressed, of course, as a functional equation, the form of which depends upon the energy per unit mass of the body in motion and certain geometrical conditions. The energy/mass (E) of the body is given by

$$E = \frac{1}{2}V^2 - \frac{\mu}{r} \tag{4}$$

where

r = the distance from the origin of the central force to the body

V = the velocity of the body at r

and

μ = the gravitational constant of the central force

The forms of Lambert's equations are first determined by whether

$$E < 0 \qquad \text{(elliptical conic)}$$

or

$$E > 0 \qquad \text{(hyperbolic conic)}$$

or

$$E = 0 \qquad \text{(parabolic conic)}$$

If the body in question is considered to be a spacecraft launched from the point P_1 in the solar system, the three energy conditions can be expressed alternatively as the launch conditions

$$V_1 < \sqrt{\frac{2\mu}{r_1}}$$

$$V_1 > \sqrt{\frac{2\mu}{r_1}}$$

$$V_1 = \sqrt{\frac{2\mu}{r_1}}$$

where V_1 is the heliocentric velocity at r_1.

II. LAMBERT'S EQUATIONS FOR ELLIPTICAL TRAJECTORIES

For a spacecraft launched from P_1 on an elliptical trajectory to P_2, the launch condition is stated as

$$V_1 < \sqrt{\frac{2\mu}{r_1}}$$

If the magnitudes of V_1 and r_1 are known, the semimajor axis a of the conic can be obtained from

$$E = -\frac{\mu}{2a} \tag{5}$$

The two possible ellipses that exist for given values of r_1, r_2, θ, and a are given in Fig. 2. The proof that no more than two ellipses exist is given in Appendix A. Since clockwise and counterclockwise transfers exist on both ellipses A and B, four different flight times and thus four different forms of Lambert's equations exist for given r_1, r_2, a, and θ, where $0 < \theta < 2\pi$.

The four different forms of Lambert's equations may be established by considering the area enclosed by the

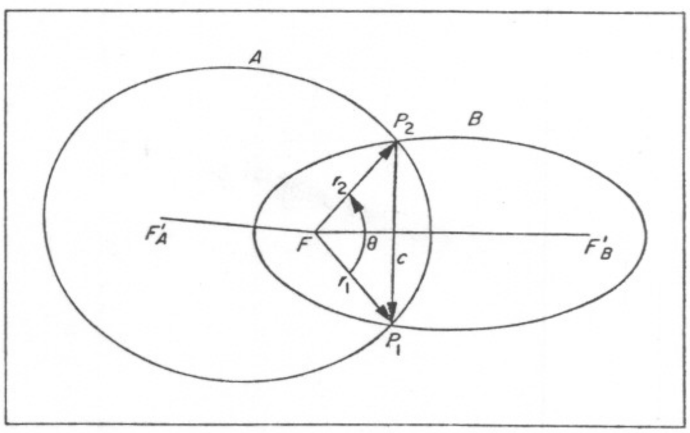

Fig. 2. Geometry of elliptical orbits

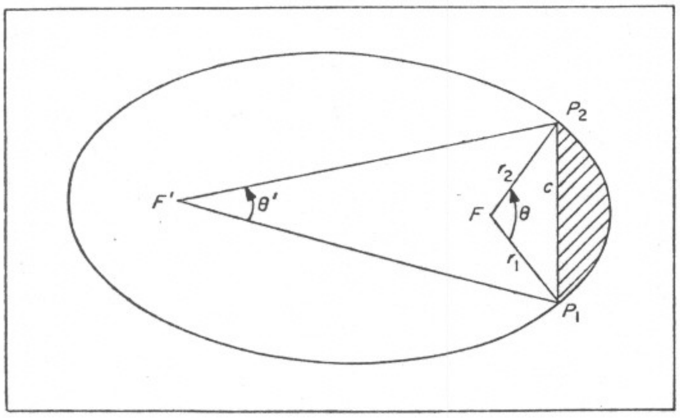

Fig. 3. The chord–flight path area

chord c and the flight path of the moving body. (See Fig. 3.)

1. For the counterclockwise trajectory along A, the chord–flight path area shades neither F, the origin of the central force, nor F'_A, the vacant focus of the ellipse. For this trajectory and all elliptical trajectories where the area shades neither focus, Lambert's equation has the form

$$T(a,\theta) = T = \frac{P}{2\pi}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \tag{6}$$

where P is the period of the ellipse and is given by

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$\cos\alpha = 1 - \frac{s}{a} \qquad (0 \le \alpha \le \pi)$$

and

$$\cos\beta = 1 - \frac{s-c}{a} \qquad (0 \le \beta \le \pi)$$

The semiperimeter of the triangle FP_1P_2 is given by

$$s = \frac{r_1 + r_2 + c}{2}$$

where

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}$$

[For a derivation of Eq. (6), see Ref. 1.]

2. For the clockwise trajectory along B, the chord–flight path area shades only F, the origin of the central force. For all such elliptical trajectories, Lambert's equation has the form

$$T(a,\theta) = P - \widetilde{T} = \frac{P}{2\pi}\left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right] \tag{7}$$

3. For the counterclockwise trajectory along B, the chord–flight path area shades only F'_B, the vacant focus of the ellipse. For all such trajectories, Lambert's equation has the form

$$T(a,\theta) = \widetilde{T} = P - \frac{P}{2\pi}\left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right] \tag{8}$$

4. For the clockwise trajectory along A, the chord–flight path area shades both foci, F and F'_A. For all such trajectories, Lambert's equation has the form

$$T(a,\theta) = P - T = P - \frac{P}{2\pi}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \tag{9}$$

The question of whether F is shaded or unshaded is equivalent to whether $\theta < \pi$ or $\theta > \pi$. The question of whether F' is shaded or unshaded is equivalent to whether $\theta' < \pi$ or $\theta' > \pi$. (See Fig. 3.) Thus the correct form of Lambert's equation for elliptical trajectories is determined from a knowledge of θ and θ', as follows:

$$T(a,\theta) = T \qquad (\theta < \pi, \theta' < \pi)$$

$$T(a,\theta) = \widetilde{T} \qquad (\theta < \pi, \theta' > \pi)$$

$$T(a,\theta) = P - \widetilde{T} \qquad (\theta > \pi, \theta' < \pi)$$

$$T(a,\theta) = P - T \qquad (\theta > \pi, \theta' > \pi)$$

There are two special cases that should be noted:

1. When the origin of the central force lies on the chord c, $\theta = \pi$; then

$$r_1 + r_2 = c$$

In this case, since

$$s = r_1 + r_2$$

then

$$s - c = 0$$

so

$$\beta = 0$$

Hence

$$T_\pi = (P - \widetilde{T})_\pi \qquad (F' \text{ unshaded}) \qquad (10)$$

and

$$\widetilde{T}_\pi = (P - T)_\pi \qquad (F \text{ shaded}) \qquad (11)$$

and ellipses A and B become identical.

2. When the vacant focus of the transfer ellipse lies on the chord c, $\theta' = \pi$. Since the sum of the distances from a point on an ellipse to the two foci is a constant, equal to $2a$, it may be seen from Fig. 4 that

$$r_1 + r_1' = 2a$$

$$r_2 + r_2' = 2a$$

and

$$r_1' + r_2' = c$$

Therefore,

$$a = a_m = \frac{r_1 + r_2 + c}{4} = \frac{s}{2}$$

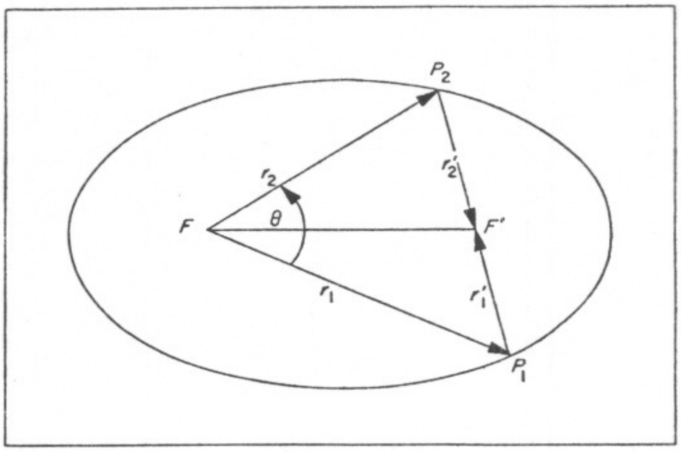

Fig. 4. Geometry of a minimum semimajor axis ellipse

For given values of r_1, r_2, and θ, $(r_1' + r_2')$ has a minimum value when F' lies on c. Hence the minimum value that a can have is given by

$$a_m = \frac{s}{2}$$

If a_m is substituted into the T and $\widetilde{T}$ equations, Eq. (6) and (8), then, for $\theta < \pi$,

$$T(a_m) = T_m = \widetilde{T}_m = \frac{P}{2\pi}\left[\pi - \arccos\left(1 - \frac{s-c}{s/2}\right) - \sqrt{1 - \left(1 - \frac{s-c}{s/2}\right)^2}\right] \qquad (12)$$

and, for $\theta > \pi$,

$$(P - T)_m = (P - \widetilde{T})_m = P - T_m \qquad (13)$$

III. LAMBERT'S EQUATIONS FOR HYPERBOLIC TRAJECTORIES

For a spacecraft launched from P_1 on a hyperbolic trajectory to P_2, the launch condition is stated as

$$V_1 > \sqrt{\frac{2\mu}{r_1}}$$

For given values of V_1 and r_1, the value of a can be determined. Since E is positive, a is negative.

The two possible hyperbolic trajectories that exist for given values of r_1, r_2, θ, and a are given in Fig. 5. The proof that two and only two hyperbolas exist is given in Appendix B.

Two different flight times and thus two different forms of Lambert's equations exist in the hyperbolic case.

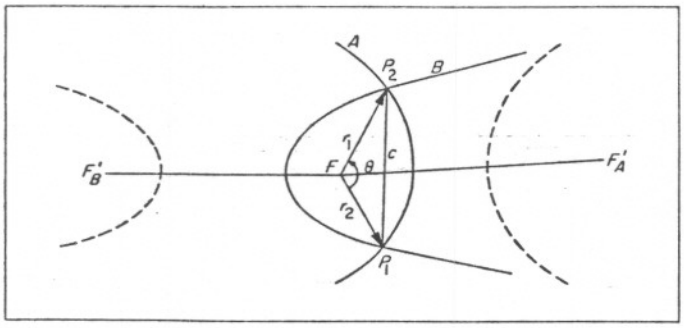

Fig. 5. Geometry of hyperbolic orbits

1. For hyperbola A, the chord–flight path area does not shade F_1, and $\theta < \pi$. Lambert's equation for this type of hyperbola has the form

$$T' = \sqrt{\frac{-a^3}{\mu}}\,[(\sinh\alpha - \alpha) - (\sinh\beta - \beta)] \qquad (\theta < \pi) \tag{14}$$

where

$$\cosh\alpha = 1 - \frac{s}{a}$$

$$\cosh\beta = 1 - \frac{s-c}{a}$$

2. For hyperbola B, the chord–flight path area shades F, and $\theta > \pi$. Lambert's equation for this type of hyperbola is

$$\widetilde{T}' = \sqrt{\frac{-a^3}{\mu}}\,[(\sinh\alpha - \alpha) + (\sinh\beta - \beta)] \qquad (\theta > \pi) \tag{15}$$

In the case where $\theta = \pi$, hyperbolas A and B become identical, so that

$$s = 0 \text{ and } \beta = 0$$

and Lambert's equation reduces to

$$T'_\pi = \widetilde{T}'_\pi = \sqrt{\frac{-a^3}{\mu}}\,(\sinh\alpha - \alpha) \tag{16}$$

IV. LAMBERT'S EQUATIONS FOR PARABOLIC TRAJECTORIES

For a spacecraft launched from P_1 on a parabolic trajectory to P_2, the launch condition is stated as

$$V_1 = \sqrt{\frac{2\mu}{r_1}}$$

Since $E = 0$, a is infinite.

The two parabolic trajectories that exist for given values of r_1, r_2, and θ are given in Fig. 6. Parabolas A and B can be considered the limiting case of ellipses A and B or hyperbolas A and B as $a \to \infty$ or $a \to -\infty$, respectively. Likewise, Lambert's equations for the parabolic case can

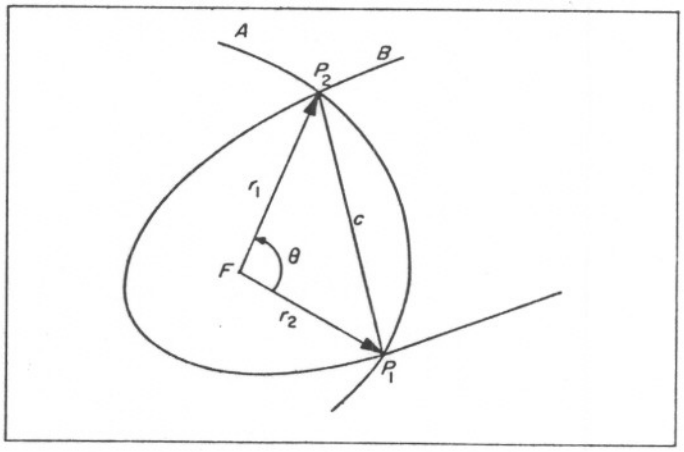

Fig. 6. Geometry of parabolic orbits

be derived, using L'Hopital's rule, from either the elliptical or hyperbolic forms:

for $\theta < \pi$,

$$T_\infty = \lim_{a \to \infty} T = \lim_{a \to -\infty} T' = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} - (s-c)^{3/2}\right] \quad (F \text{ unshaded}) \qquad (17)$$

for $\theta > \pi$,

$$\widetilde{T}_\infty = \lim_{a \to \infty} (P - \widetilde{T}) = \lim_{a \to -\infty} \widetilde{T}'$$

$$= \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} + (s-c)^{3/2}\right] \quad (F \text{ shaded}) \qquad (18)$$

and, for $\theta = \pi$,

$$T_{\infty\pi} = \widetilde{T}_{\infty\pi} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\, s^{3/2} \qquad (19)$$

V. RESULTS

Lambert's equations have been programmed for the IBM 1620 computer at the Jet Propulsion Laboratory for computation of Earth–Venus and Earth–Mars trajectories. The following conditions were assumed:

1. The spacecraft moves from Earth, P_1, to Venus or Mars, P_2, while under the gravitational influence of the Sun only.
2. Planetary orbits are circular.
3. Constants for evaluating Lambert's equations:

   $r_1 = 1$ au

   $r_2 = 0.723332$ au (Venus)

   $= 1.523691$ au (Mars)

   $\mu = 2.959122083 \times 10^{-4} \dfrac{\text{au}^3}{\text{day}^2}$

Figures 7 through 10 are plots of $T(a, \theta)$ vs. a for various values of θ for Earth–Venus and Earth–Mars trajectories. The (a_m, T_m) points of the various curves are connected by a dashed line. Note that

$$T < T_m < \widetilde{T}$$

and

$$(P - \widetilde{T}) < (P - T_m) < (P - T)$$

for any given θ.

Figures 11 and 12 are plots of $T(a, \theta)$ vs. θ for constant a for Earth–Venus flights and Earth–Mars flights, respectively. Figure 13 illustrates the regions of the plots in which the various $T(a, \theta)$ forms are valid. In both plots the horizontal dashed line cuts the curves at values of θ for which the particular a is a_m. The intersection of the two dashed lines is the Hohmann point. The value of a at this point is the a_m for $\theta = \pi$. For $\theta = \pi$, since

$$a_m = \frac{r_1 + r_2 + c}{4}$$

and

$$c = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2}$$

then

$$4a_m = r_1 + r_2 + \sqrt{r_1^2 + r_2^2 + 2r_1 r_2}$$

or

$$a_m\,(\text{Hohmann}) = \frac{r_1 + r_2}{2} \qquad (20)$$

The curve marked $a = \infty$ is the parabolic limit of the elliptical and hyperbolic orbits.

Figure 14 is a view of the $T(a, \theta)$ surface sketched in rectangular Cartesian coordinates (T, a, θ) for elliptical Earth–Venus trajectories.

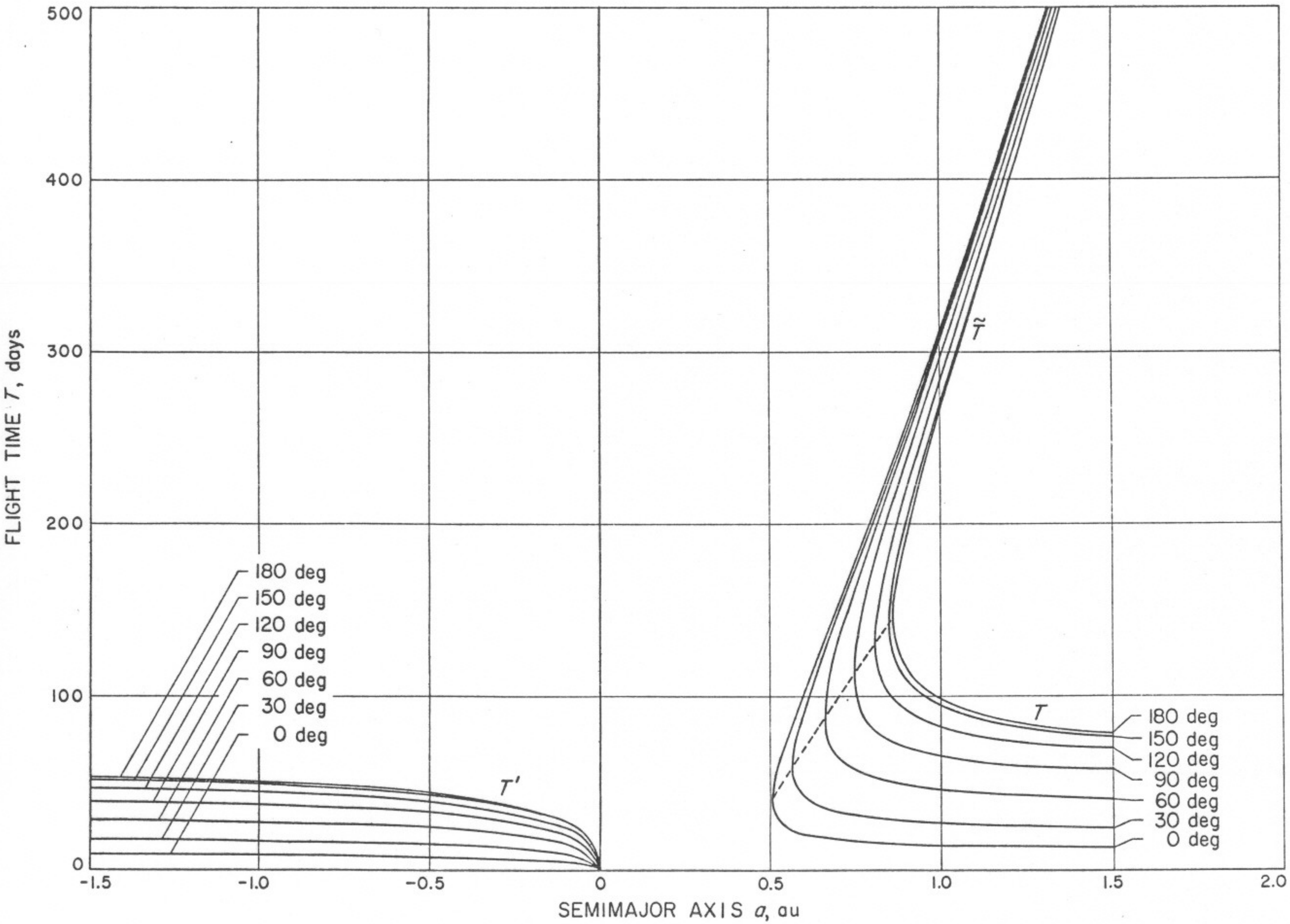

Fig. 7. Venus: flight time vs. semimajor axis for $0 < \theta < 180$ deg

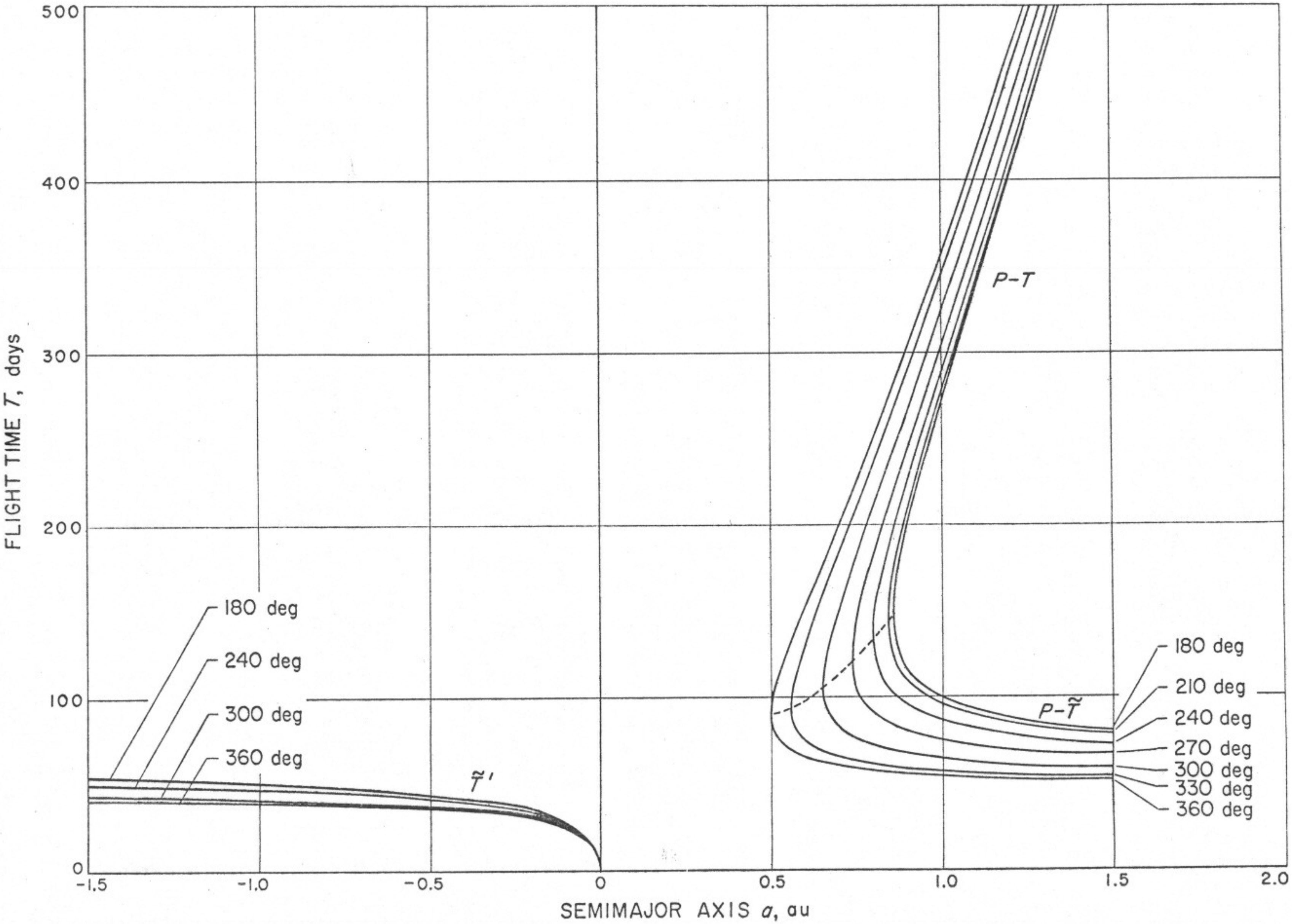

Fig. 8. Venus: flight time vs. semimajor axis for $180 < \theta < 360$ deg

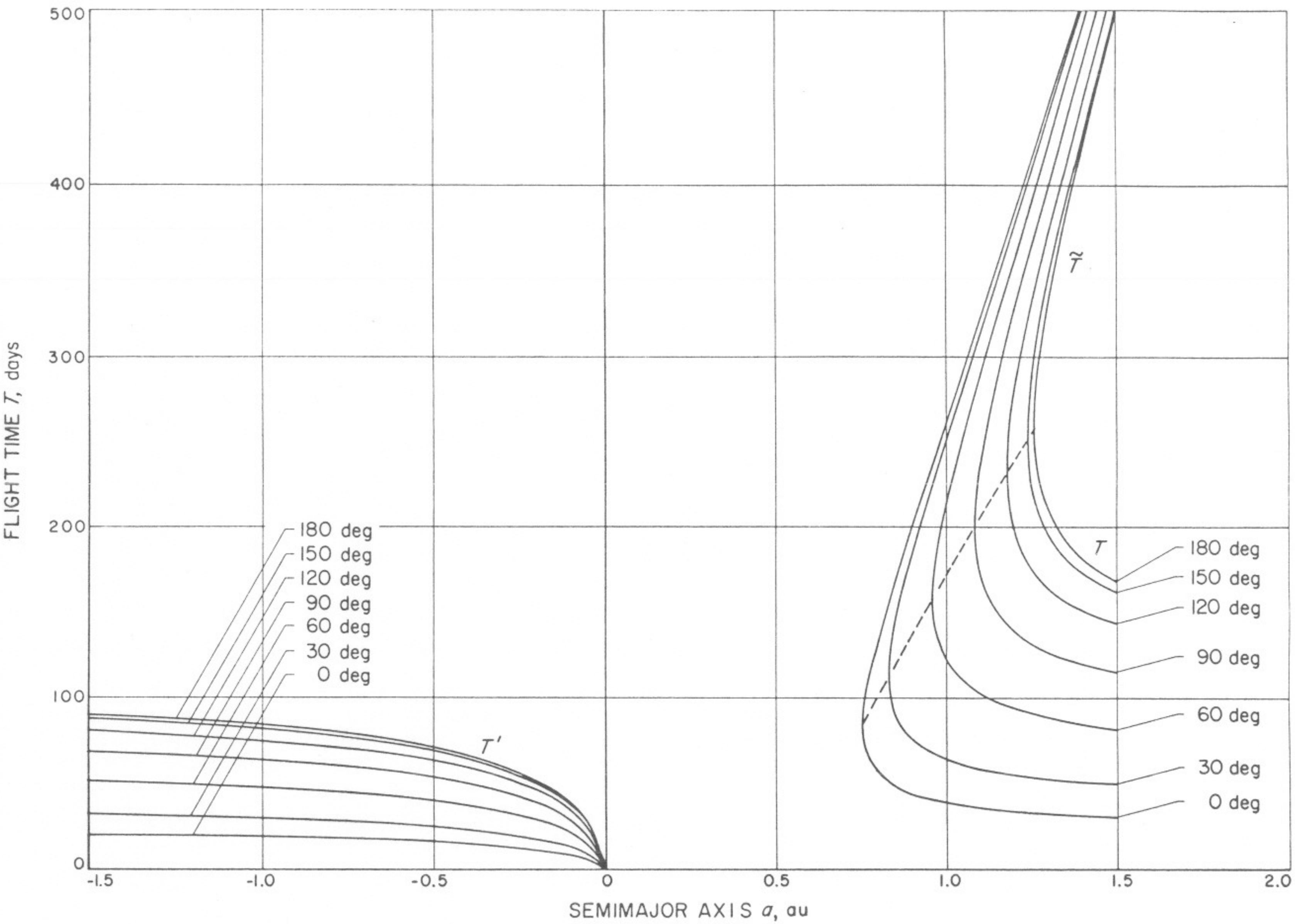

Fig. 9. Mars: flight time vs. semimajor axis for $0 < \theta < 180$ deg

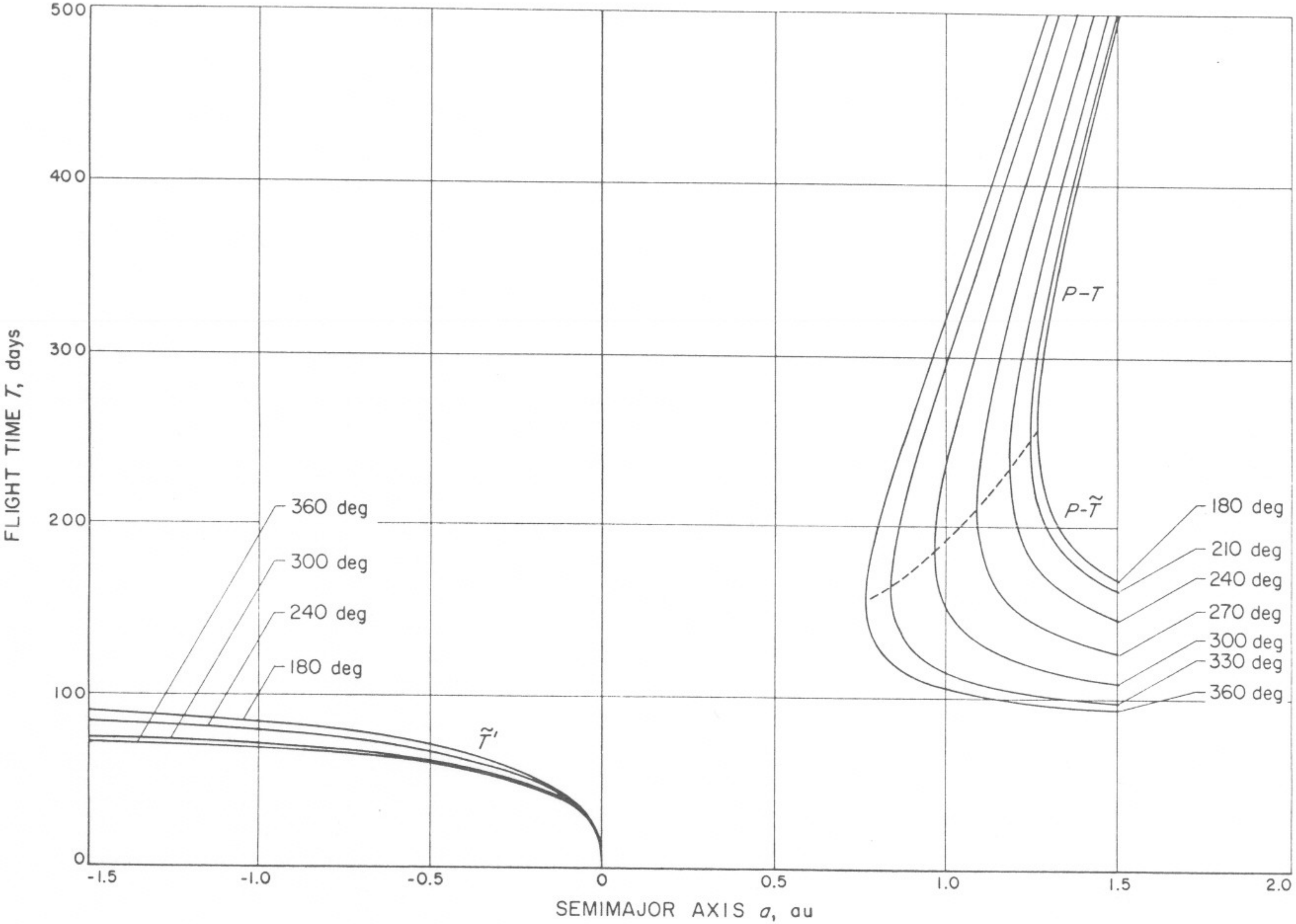

Fig. 10. Mars: flight time vs. semimajor axis for $180 < \theta < 360$ deg

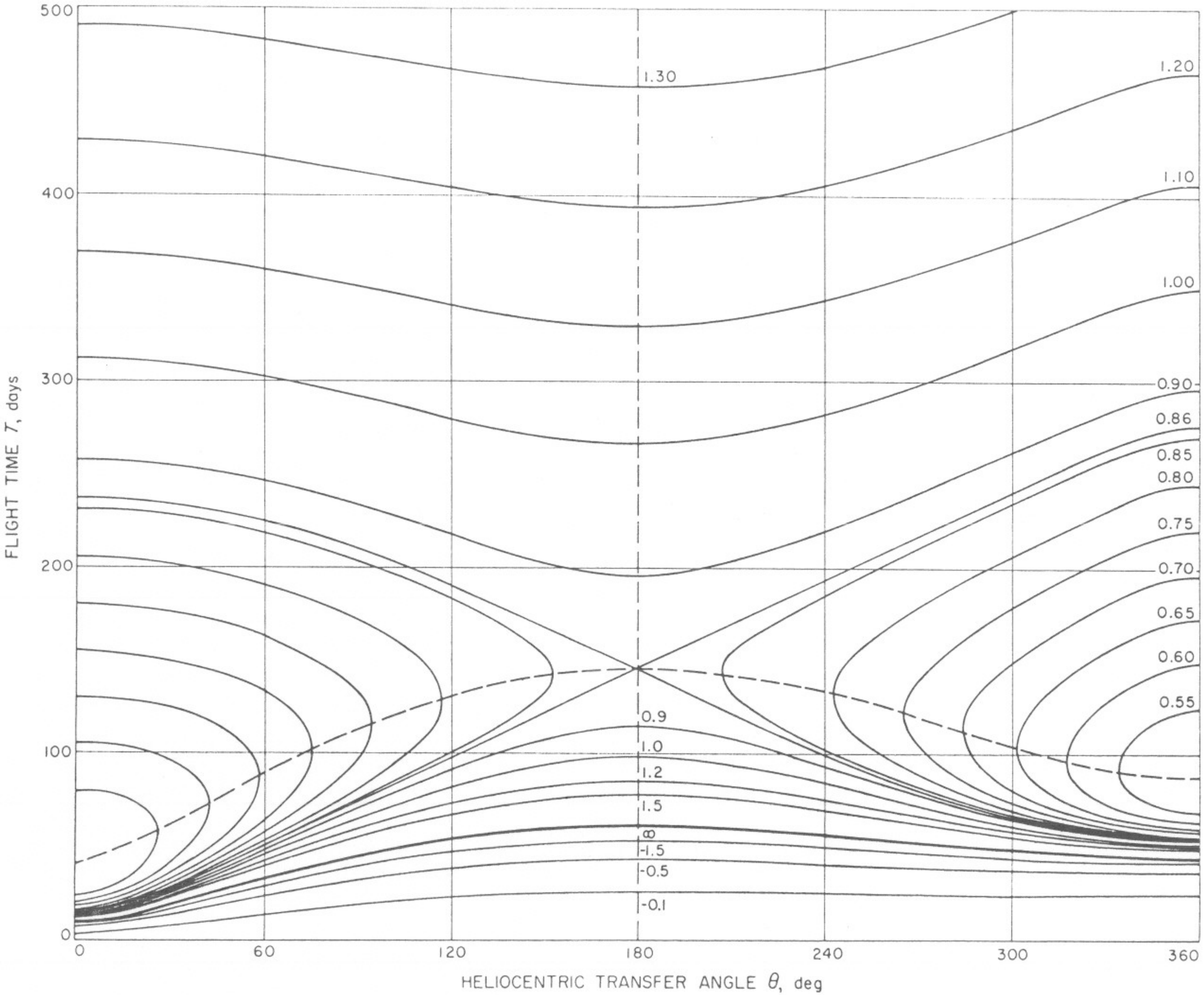

Fig. 11. Venus: flight time vs. heliocentric transfer angle

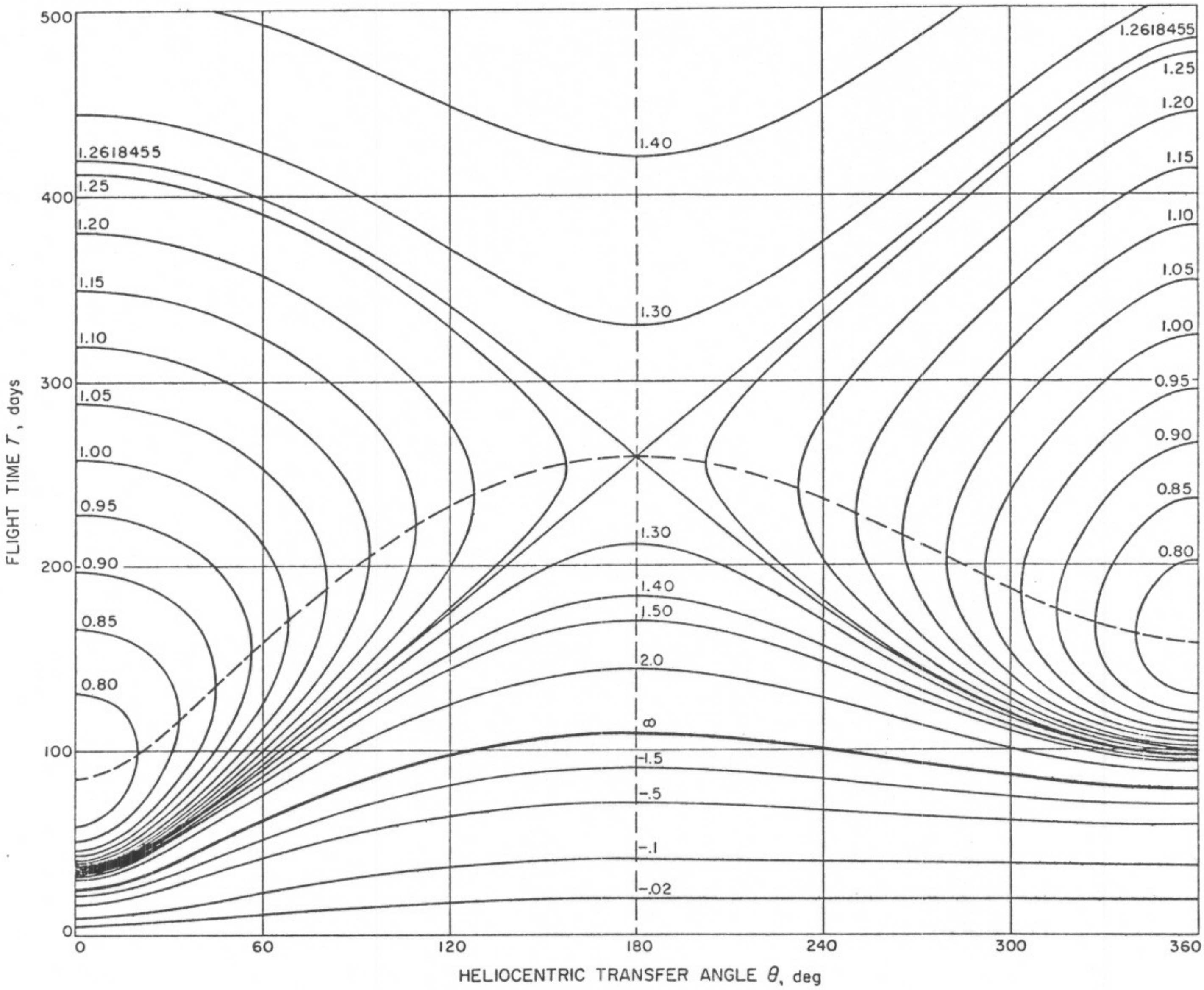

Fig. 12. Mars: flight time vs. heliocentric transfer angle

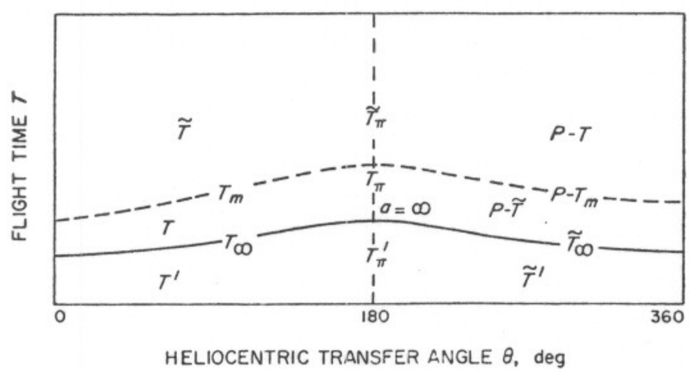

Fig. 13. Flight time vs. heliocentric transfer angle region chart

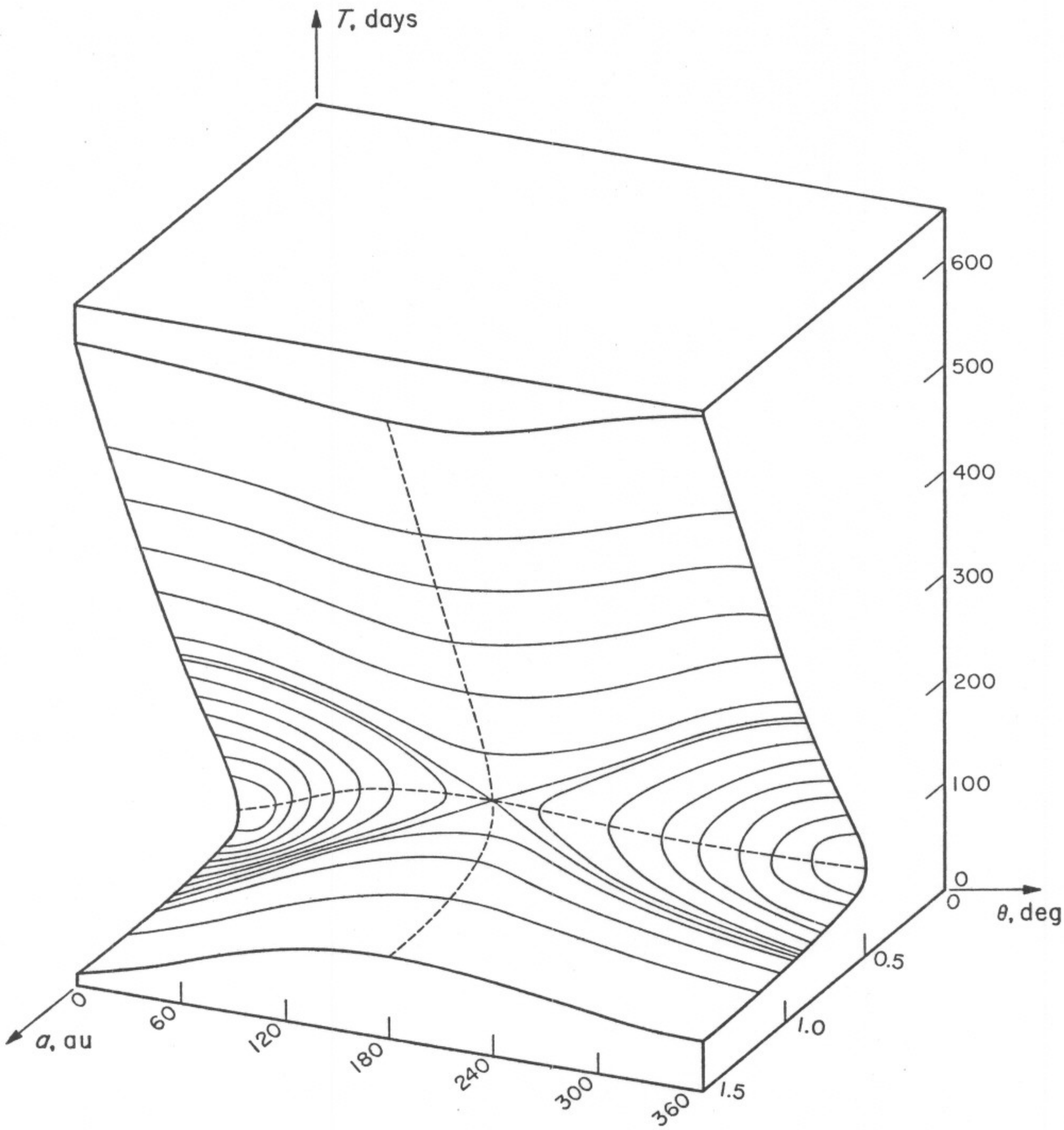

Fig. 14. Earth–Venus elliptical transfers: flight time vs. semimajor axis vs. heliocentric transfer angle

VI. THE APPLICATION OF LAMBERT'S EQUATIONS TO TRANSFER PROBLEMS

Lambert's equations are very useful in solving transfer problems, especially when the energy of the trajectory is either known or desired, and the eccentricity of the orbit is unknown. In the three cases discussed here, as in most transfer problems, the space mission is specified, that is, r_1 and r_2 are considered known quantities. When $T(a, \theta)$ is known and it is desired to find a or θ, the equations must be solved backwards by an iteration procedure such as Newton's. The techniques of such procedures will not be dealt with here, because of the abundance of such discussions in the numerical analysis literature. In the cases discussed, straightforward methods for determining which forms of Lambert's equations are valid, for a given set of input parameters, are presented.

Case I:

Given: $a, \theta \ (0 \leq \theta \leq 2\pi)$

Find: $T(a, \theta)$

The solutions to Lambert's equations for given a and θ correspond to flight times for a hyperbola, an ellipse, or two ellipses. Under certain conditions, no solution exists.

Test for elliptical or hyperbolic motion:

If: $a < 0$ (hyperbolic conic)

If: $\theta \leq \pi$, $T(a, \theta) = T'$

If: $\theta > \pi$, $T(a, \theta) = \widetilde{T'}$

If: $a > 0$ (elliptical conic)

Compute: $a_m(\theta)$

If: $a < a_m(\theta)$, no solution exists

If: $a \geq a_m(\theta)$

If: $\theta \leq \pi$, $T(a, \theta) = T$

$T(a, \theta) = \widetilde{T}$ (two solutions)

If: $\theta > \pi$, $T(a, \theta) = P - \widetilde{T}$

$T(a, \theta) = P - T$ (two solutions)

Case II:

Given: $T, \theta \ (0 \leq \theta \leq 2\pi)$

Find: a

For given values of T and θ, a unique a exists.

If: $\theta \leq \pi$

Consider: $T, \widetilde{T}, T'$, and T_∞ equations

Compute: $a_m(\theta)$

Compute: $T_m(\theta) = T\left[a_m(\theta)\right]$

Compute: $T_\infty(\theta)$

If: $T < T_\infty(\theta)$,	$T(a, \theta) = T'$	(hyperbolic conic)
If: $T = T_\infty(\theta)$,	$T(a, \theta) = T_\infty$	(parabolic conic)
If: $T_\infty < T \leq T_m(\theta)$,	$T(a, \theta) = T$	(elliptical conic)
If: $T > T_m(\theta)$,	$T(a, \theta) = \widetilde{T}$	(elliptical conic)

If: $\theta > \pi$

Consider: $P - T, P - \widetilde{T}, \widetilde{T'}$, and $\widetilde{T}_\infty$ equations

Compute: $a_m(\theta)$

Compute: $P - T\left[a_m(\theta)\right] = P - T_m(\theta)$

Compute: $T_\infty(\theta)$

If: $T < \widetilde{T}_\infty(\theta)$,	$T(a, \theta) = \widetilde{T'}$	(hyperbolic conic)
If: $T = \widetilde{T}_\infty$,	$T(a, \theta) = \widetilde{T}_\infty$	(parabolic conic)
If: $\widetilde{T}_\infty < T \leq P - T_m(\theta)$,	$T(a, \theta) = P - \widetilde{T}$	(elliptical conic)
If: $T > P - T_m$,	$T(a, \theta) = P - T$	(elliptical conic)

Case III:

Given: T, a

Find: θ

If: $a < 0$ (hyperbolic conic)

Compute: $T'(a, 0)$

Compute: $T'_\pi(a)$

Compute: $T'(a, 2\pi)$

If: $T < T'(a, 0)$,	no solution exists	
If: $T'(a, 0) \leq T < \widetilde{T'}(a, 2\pi)$,	$T(a, \theta) = T'$	$(\theta < \pi)$
If: $\widetilde{T'}(a, 2\pi) \leq T \leq T'(a, \pi)$,	$T(a, \theta) = T'$	$(\theta < \pi)$
	$T(a, \theta) = \widetilde{T'}$	$(\theta > \pi)$
		(two solutions)
If: $T > T'(a, \pi)$,	no solution exists	

If: $a > 0$ (elliptical conic)

Compute: $a_m(0)$

Compute: $a_m(2\pi)$

Compute: a Hohmann

Test for: $\theta \leq \pi$

If: $a < a_m(0)$, no solution exists

If: $a_m(0) \leq a \leq$ a Hohmann

Compute: $T_m(a), a = a_m$

Compute: $T(a, 0)$

Compute: $\widetilde{T}(a, 0)$

If: $T < T(a, 0)$,	no $\theta < \pi$ exists	
If: $T(a, 0) \leq T \leq T_m(a)$,	$T(a, \theta) = T$	$(\theta < \pi)$
If: $T_m(a) < T \leq \widetilde{T}(a, 0)$,	$T(a, \theta) = \widetilde{T}$	$(\theta < \pi)$
If: $T > \widetilde{T}(a, 0)$,	no $\theta < \pi$ exists	

If: $a >$ a Hohmann

Compute: $T(a, \pi)$

Compute: $\widetilde{T}(a, \pi)$

If: $T < T(a, 0)$,	no $\theta < \pi$ exists	
If: $T(a, 0) \leq T \leq T(a, \pi)$,	$T(a, \theta) = T$	$(\theta < \pi)$
If: $T(a, \pi) < T < \widetilde{T}(a, \pi)$,	no $\theta < \pi$ exists	
If: $\widetilde{T}(a, \pi) \leq T \leq \widetilde{T}(a, 0)$,	$T(a, \theta) = \widetilde{T}$	$(\theta < \pi)$
If: $T > \widetilde{T}(a, 0)$,	no $\theta < \pi$ exists	

Test for: $\theta > \pi$

If: $a < a_m(2\pi)$, no $\theta < \pi$ exists

If: $a_m(2\pi) \leq a \leq$ a Hohmann

Compute: $P - T_m(a), a = a_m$

Compute: $P - \widetilde{T}(a, 2\pi)$

Compute: $P - T(a, 2\pi)$

If: $T < P - \widetilde{T}(a, 2\pi)$,	no $\theta > \pi$ exists	
If: $P - \widetilde{T}(a, 2\pi) \leq T \leq P - T_m(a)$,	$T(a, \theta) = P - \widetilde{T}$	$(\theta > \pi)$
If: $P - T_m(a) < T \leq P - T(a, 2\pi)$,	$T(a, \theta) = P - T$	$(\theta > \pi)$
If: $T > P - T(a, 2\pi)$,	no $\theta > \pi$ exists	

If: $a >$ a Hohmann

If: $T < P - \widetilde{T}(a, 2\pi)$,	no $\theta > \pi$ exists	
If: $P - \widetilde{T}(a, 2\pi) \leq T \leq T(a, \pi)$,	$T = P - \widetilde{T}$	$(\theta > \pi)$
If: $T(a, \pi) < T < \widetilde{T}(a, \pi)$,	no $\theta > \pi$ exists	
If: $\widetilde{T}(a, \pi) \leq T \leq P - T(a, 2\pi)$,	$T(a, \theta) = P - T$	$(\theta > \pi)$
If: $T > P - T(a, 2\pi)$,	no $\theta > \pi$ exists	

APPENDIX A

Geometrical Properties of Elliptical Trajectories

Take two general points, P_1 and P_2, in the field of a central gravitational force F, and let r_1 and r_2 be the distances from F to P_1 and P_2, respectively. Let θ be the angle subtended by P_1 and P_2. (See Fig. A-1.)

Consider ellipses with semimajor axis a, with one focus at F, which pass through P_1 and P_2. Let r_1' and r_2' be the distances from the vacant focus F' of such an ellipse to P_1 and P_2, respectively. From the definition of an ellipse,

$$r_1 + r_1' = 2a$$

and

$$r_2 + r_2' = 2a$$

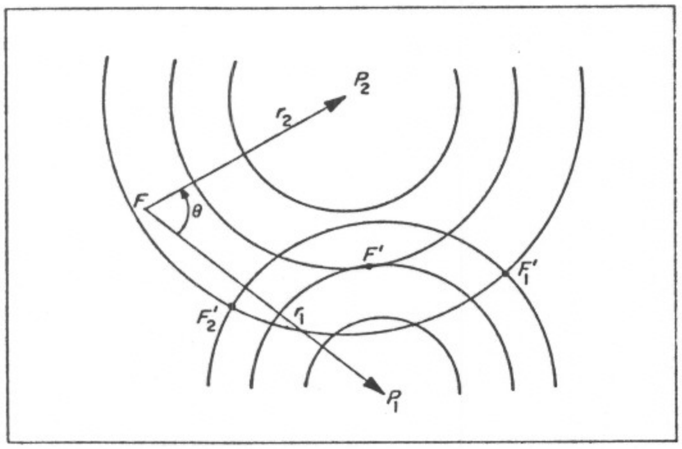

Fig. A-1. Geometrical properties of elliptical trajectories

so that

$$r_1' = 2a - r_1$$

and

$$r_2' = 2a - r_2$$

Construct circles of radii r_1' r_2' (see Fig. A-1) about P_1 and P_2, respectively, for

$$a < \frac{r_1 + r_2 + \sqrt{r_1^2 - 2r_1 r_2 \cos\theta}}{4} = \frac{r_1 + r_2 + c}{4}$$

$$a = \frac{r_1 + r_2 + c}{4}$$

and

$$a > \frac{r_1 + r_2 + c}{4}$$

It may be seen that if

$$a < \frac{r_1 + r_2 + c}{4}$$

no F' exists, and hence no ellipse is defined. If

$$a > \frac{r_1 + r_2 + c}{4}$$

the circles intersect in, at most, two points, F_1' and F_2', and two possible ellipses are defined. In the special case where

$$a = \frac{r_1 + r_2 + c}{4}$$

the circles intersect at one point, F', and one ellipse is defined.

APPENDIX B

Geometrical Properties of Hyperbolic Trajectories

Consider hyperbolas with semimajor axis a, with focus at F, which pass through P_1 and P_2. Let r_1' and r_2' be the distances from the vacant focus F' of such a hyperbola to P_1 and P_2, respectively. (See Fig. B-1.) From the definition of a hyperbola,

$$r_1 - r_1' = 2a$$

and

$$r_2 - r_2' = 2a$$

so that

$$r_1' = r_1 - 2a$$

and

$$r_2' = r_2 - 2a$$

Since $a < 0$,

$$r_1' > r_1$$

and

$$r_2' > r_2$$

Construct circles of radius r_1' and r_2' about P_1 and P_2, respectively. It may be seen that for any a, the circles intersect at two and only two points, F_1' and F_2', so that two hyperbolas are defined.

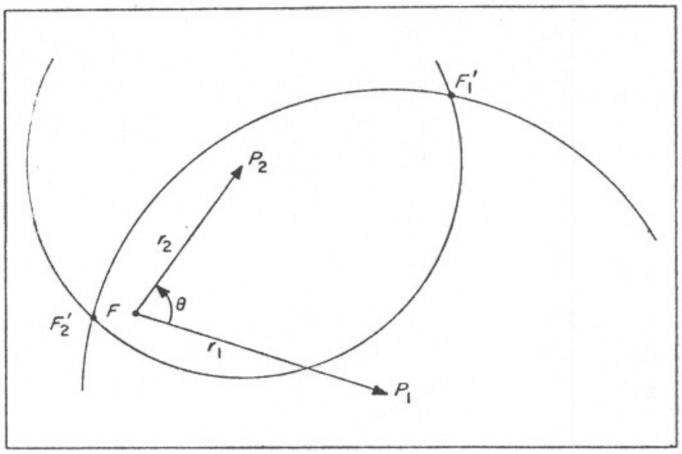

Fig. B-1. Geometrical properties of hyperbolic trajectories

By allowing a to approach $-\infty$, one can see that two and only two parabolas are defined.

REFERENCE

1. Battin, R. H., "The Determination of Round-Trip Planetary Recónnaissance Trajectories," *Journal of the Aero/Space Sciences*, Vol. 26, No. 9, September 1959, pp. 545-567.