a complete and detailed derivation of the formula

(1) $F = \chi_g \, m H \frac{dH}{dx}$ and

its <u>limitations</u>.

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Aug 1959

We begin by Ohms law (and MKS system)

(2) $\vec{j} = \sigma(\vec{E} + \vec{E}')$ where $\vec{j}$ is the true current density, $\vec{E}$ is an irrotational vector field derivable from a scalar potential function (i.e. electric field set up by fixed distribution of charge); $\vec{E}'$ is a field which is not derivable from a scalar potential and which is maintained by some source (e.g., battery) producing a steady ~~the~~ current flow; ~~[illegible]~~. σ is the conductivity of the medium

Hence $\frac{\partial \vec{j}}{\partial \tau} = 0$ for all space and which implies $\frac{\partial \vec{E}'}{\partial \tau} = 0$. ~~[illegible]~~.

Now from (2) $\vec{E}' = \frac{\vec{j}}{\sigma} - \vec{E}$ hence

(3) $\vec{j} \cdot \vec{E}' = \vec{j} \cdot \left(\frac{\vec{j}}{\sigma} - \vec{E}\right) = \frac{\vec{j}^{\,2}}{\sigma} - \vec{j} \cdot \vec{E}$ $\quad (\vec{j}^{\,2} = \vec{j} \cdot \vec{j} = j^2)$

We now integrate (3) over all space

(4) $\iiint_v \vec{j} \cdot \vec{E}' \, dv = \iiint \frac{j^2}{\sigma} dv - \iint \vec{j} \cdot \vec{E} \, dv$

The term on the left hand side of above equation represents the total rate of flow of energy from the source producing $\vec{E}'$. Since this source is the only energy producing source in our system the terms on the right must sum to the total consumption of energy. Now resistance at any point in our media is the reciprocal of it's conductivity at that particular point hence the term $\iiint_v \frac{j^2}{\sigma} dv$ represents the total power dissipated due to resistence of our medium to current flow. The term $\iiint_v \vec{j} \cdot \vec{E} dv$ must then be associated with rate of energy flow used in maintaining the resulting magnetic field (due to the assumption of an electric field $\vec{E}$ existing).

We now make use of Maxwell's field equations:

$$(5) \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$(6) \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$$

For stationary current flow and media, $\frac{\partial \vec{D}}{\partial t} = 0$ hence $\vec{j} = \nabla \times \vec{H}$

Thus $-\iiint_V \vec{J}\cdot\vec{E}\,dv = -\iiint_V \nabla\times\vec{H}\cdot\vec{E}\,dv$ and observing the vector identity

$$\nabla\cdot\vec{E}\times\vec{H} = -\vec{E}\cdot\nabla\times\vec{H} \cancel{+ H\,\nabla\times H} + \vec{H}\cdot\nabla\times\vec{E}$$

we obtain

$$-\iiint_V \vec{J}\cdot\vec{E}\,dv = \iiint_V \nabla\cdot\vec{E}\times\vec{H}\,dv - \iiint_V \vec{H}\cdot\nabla\times\vec{E}\,dv$$

and by (5) we write

$$-\iiint_V \vec{J}\cdot\vec{E}\,dv = \iiint_V \nabla\cdot\vec{E}\times\vec{H}\,dv + \iiint_V \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\,dv$$

By the divergence theorem we obtain

$$\iiint_V \nabla\cdot\vec{E}\times\vec{H}\,dV = \iint_S \vec{E}\times\vec{H}\cdot d\vec{S}$$ where S is the surface bounding V. But since $\vec{E}\times\vec{H}$ falls off at least as $\frac{1}{r^5}$ the term $\iint_S \vec{E}\times\vec{H}\cdot d\vec{S} = 0$

$$\therefore \iiint_V \vec{J}\cdot\vec{E}'\,dv = \iiint_V \frac{J^2}{\sigma}\,dv + \iiint_V \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\,dv$$

Let w denote the work required to maintain our magnetic field. Then

$$\frac{dw}{dt} = \iiint_V \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\,dv$$ During a small time interval

$$\delta w = \iiint_V \vec{H}\cdot\delta\vec{B}\,dv$$ Hence when the field reaches a stationary state the total energy it contains (ie the total amount of

work it took to set it up) is equal to W.

(7) $$W = \int_0^{\vec{B}} \delta W = \int_0^{B} \iiint_V \vec{H} \cdot \delta \vec{B} \, dv = \iiint_V \left(\int_0^{\vec{B}} \vec{H} \cdot \delta \vec{B} \right) dv$$

Now $\vec{H}$ is defined as:

(8) $$\vec{H} = \frac{1}{\mu_0} (\vec{B} - \mu_0 \vec{M}) = \frac{\vec{B}}{\mu_0} - \vec{M}$$ where $\vec{M}$ is magnetization of the medium. It depends on the applied field $\vec{H}$ by the equation

(9) $$\vec{M} = \chi_m \vec{H}$$ where χ_m is called the magnetic susceptibility.

We now reach a critical point in our reasoning.

For most media $\vec{M}$ is ~~par~~ $\parallel$ to $\vec{H}$. But for so called ferromagnetic media $\vec{M}$ is not $\parallel$ to $\vec{H}$ and hence χ_m becomes a tensor, not a simple scalar. If $\vec{M}$ is approximately $\parallel$ to $\vec{H}$, then χ_m can be assumed to be a scalar. The magnitude of error now introduced as we proceed depends upon the magnitude of ferromagentism of media under consideration.

We take χ_m to be a scalar.

$$\vec{M} = \chi_m \vec{H}$$

Hence from (8) $\quad \vec{B} = \mu_0 \vec{H} + \vec{M}$

$$= \mu_0 \vec{H} + \chi_m \vec{H}$$
$$= \mu_0 (1 + \chi_m) \vec{H}$$
$$= \mu_0 K_m \vec{H}$$

(9) $\quad \vec{B} = \mu_0 K_m \vec{H} \qquad$ where $K_m = 1 + \chi_m$

and is called relative permeability of the medium. Hence employing this result in (7) we write

$$W = \iiint_V \left(\int_0^{\vec{B}} \vec{H} \cdot \delta \vec{B} \right) dv = \iiint_V \left(\int_0^{\vec{B}} \frac{\vec{B}}{\mu_0 K_m} \cdot \delta \vec{B} \right) dv$$

$$= \iiint_V \int_0^{\vec{B}} \frac{\delta \vec{B}^2}{2\mu_0 K_m} \, dv$$

But $\displaystyle \int_0^{\vec{B}} \frac{\delta \vec{B}^2}{2\mu_0 K_m} = \frac{\vec{B}^2}{2\mu_0 K_m} = \frac{\mu_0^2 K_m^2 \vec{H}^2}{2\mu_0 K_m} = \frac{\mu_0}{2} K_m \vec{H}^2$

$$\therefore \quad W = \iiint_V \tfrac{1}{2} \mu_0 K_m \vec{H}^2 \, dv$$

Let us now introduce a small sample of new media with relative permeability equal to K_m and denoting K_m' to be the relative permeability of original media. We assume that

$$\vec{M} = \chi_m \vec{H} \qquad \chi_m \text{ is a scalar}$$

for this new medium even though our new medium will be ferromagnetic!

Thus the energy stored in the magnetic field will change by an amount

$$\Delta W = \frac{1}{2}\mu_0 \left[\iiint_V K'_m \vec{H}^2 dV - \iiint_{V'_s} K'_m \vec{H}^2 dV - \iiint_{V_s} K_m \vec{H}^2 dV \right]$$

where V of course represents all space, $V'_s = V - V_s$, V_s is region occupied by new medium.

$$\therefore \quad \Delta W = \iiint_{V_s} \frac{\mu_0}{2} (K'_m - K_m) \vec{H}^2 dV$$

If V_s is sufficiently small such that $\vec{H}$ is nearly constant throughout it, we may write

$$\Delta W = \frac{1}{2}\mu_0 (K'_m - K_m) H^2 V_s \quad \text{or}$$

$$\Delta W = \frac{1}{2}\mu_0 (\chi'_m - \chi_m) H^2 V_s$$

where V_s is the volume occupied by the new medium. (χ'_m & χ_m are volume sus.)

To ~~account~~ account for this energy change is an easy matter. It goes into the energy of position or simply the potential energy U of the new medium.

$$U = \Delta W = \frac{1}{2}\mu_0 (\chi'_m - \chi_m) H^2 V_s$$

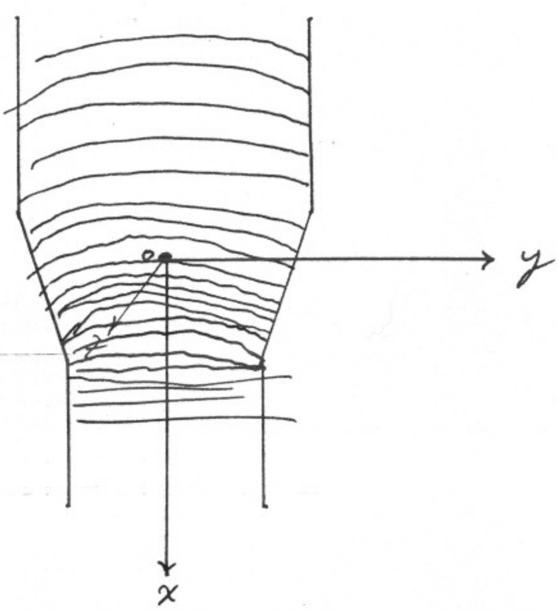
Since this field itself is conservative any force acting on the new medium is equal to the negitive gradient of its potential function U.

$$\therefore \quad \vec{F} = -\nabla U$$

$$= -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}\right)$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in a chosen rectangular coordinate system.

Our field will in general appear as indicated below, hence we take the coordinate system as indicated with the z coordinate comming out of paper.



Hence we have $\frac{\partial H^*}{\partial y} = \frac{\partial H}{\partial z} = 0$ and thus

$$\vec{F} = -\frac{\partial U}{\partial x}\vec{i} = -\frac{\partial}{\partial x}\left[\tfrac{1}{2}\mu_0(\chi'_m - \chi_m)H^2 V_s\right]\vec{i}$$

$$= -\tfrac{1}{2}\mu_0(\chi'_m - \chi_m)V_s \cdot 2H\frac{\partial H}{\partial x}\, i$$

Hence

$$\vec{F} = \mu_0 (\chi_m - \chi'_m) V_s H \frac{\partial H}{\partial x} \, \vec{i}$$

If the original medium ~~[illegible]~~ is air

$\chi'_m \approx 0$ and we write

$$(10) \qquad \vec{F} = \mu_0 \chi_m V_s H \frac{\partial H}{\partial x} \, \vec{i}$$

F is in newtons
μ_0 is $4\pi \times 10^{-7}$ $\frac{\text{henrys}}{\text{m}}$, V_s is in m^3, H is in $\frac{\text{ampere turns}}{\text{m}}$

The dimension of χ_m is then fixed. (mass in kgm)

In cgs (electromagnetic) system μ_0 is unity

F in dynes, H in oersteds, B in gauss, V_s in cm^3

In both systems the volume sus. equals

the gram sus. multiplied by its density.

$$\chi_m = \chi_g d$$

$$\therefore \text{ in cgs} \quad F = \chi_g m H \frac{\partial H}{\partial x}$$

~~In this derivation serious~~

~~limitations exist. and it is impossible for (10) to be true without assuming linearity. The formula cannot be derived without the quad [illegible] tions.~~

~~The other formula~~ Formula (10) ~~[illegible]~~ presupposes linearity between $\vec{M}$ & $\vec{H}$; ~~[illegible]~~ ~~[illegible]~~ and its inability to predict values of susceptibility ~~should~~ of strong ferromagnetic substances should ~~not to~~ be realized.