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Vector Calculus for the Orbit Plane System

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Abstract

The purpose of this note is to demonstrate the usefulness of elementary vector calculus for some topics related to orbital mechanics of an Earth satellite. Calculations for the satellite, approximated by a point mass in the present analysis, are for an active medium altitude (6-hour period) communication satellite. The topics analyzed in this note are the total speed of the satellite along a perturbed elliptic orbit, its angular momentum vector about the geocenter, the products of range and range-rate vectors and a possible application to the orbit determination scheme following the Laplacian method. This note includes perigee and node rates and treats the remaining four orbital elements as invariant. Since the note is subjected to limited space, the symbol list is omitted. Only new shorthand notations introduced are explained and the rest of the symbols can be found in widely accepted books on astrodynamics [1, 2].

Introduction

Several techniques exist for locating the satellite in the geocentric equatorial coordinate system. However, the orbit plane system has proven to be the most effective. The position vector $\mathbf{r}$ of the satellite at point (x, y, z) is given by

$$\mathbf{r} = x_\omega \mathbf{P} + y_\omega \mathbf{Q}. \tag{1}$$

The unit orientation vectors **P** and **Q** entail the Eulerian angles of rotations, Ω, i, and ω. The perigee and node rates are $\dot{\omega}$ and $\dot{\Omega}$, but their specific forms are not given here because of the varieties of $\dot{\omega}$ and $\dot{\Omega}$ that exist in the open literature. The dot above a symbol designates the ordinary derivative with respect to τ, the geocentric characteristic time (one unit of $\tau = 13.447052$ [3] mean solar minutes).

Although analyses published in the past seem to have ignored $\dot{\mathbf{P}}$ and $\dot{\mathbf{Q}}$, the key point of this note is the retention of $\dot{\mathbf{P}}$ and $\dot{\mathbf{Q}}$ for a more exact analysis. As shown later, the introduction of coplanar vectors $\dot{\mathbf{r}}_{PQ}$ and $\ddot{\mathbf{r}}_{PQ}$ will greatly facilitate analytical development. The indicated time derivatives present

$$\begin{pmatrix} \dot{\mathbf{P}} \\ \dot{\mathbf{Q}} \end{pmatrix} = \begin{pmatrix} \dot{\Omega}\mathbf{z}_0 \times & \dot{\omega} \\ -\dot{\omega} & \dot{\Omega}\mathbf{z}_0 \times \end{pmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \tag{2}$$

where $\mathbf{z}_0$ is the unit vector along the Earth's rotational axis. The matrix elements of Eq. (2) are operators for vector products and scalar multipliers. In a separate analysis [4] on wave propagation through a general linear medium, a similar matrix of vectorial operators was found to be useful.

Let $\dot{\mathbf{r}}$ be the total velocity vector and $\ddot{\mathbf{r}}$ be the total acceleration vector of the satellite traveling along the perturbed elliptic orbit. At this point, it should be emphasized that $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ have components perpendicular to the instantaneous orbital plane. In terms of $\dot{\mathbf{r}}_{PQ}$ and $\ddot{\mathbf{r}}_{PQ}$, $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ are given as follows:

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_{PQ} + \dot{\Omega}\mathbf{z}_0 \times \mathbf{r} \tag{3}$$

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{PQ} + 2\dot{\Omega}\mathbf{z}_0 \times \dot{\mathbf{r}}_{PQ} + z\dot{\Omega}^2\mathbf{z}_0 + \ddot{\Omega}\mathbf{z}_0 \times \mathbf{r} \tag{4}$$

$$\dot{\mathbf{r}}_{PQ} = x_{\omega v}\mathbf{P} + y_{\omega v}\mathbf{Q} \tag{5}$$

$$\ddot{\mathbf{r}}_{PQ} = (x_{\omega a} - \dot{\Omega}^2 x_\omega)\mathbf{P} + (y_{\omega a} - \dot{\Omega}^2 y_\omega)\mathbf{Q} \tag{6}$$

where shorthand symbols are

$$\begin{pmatrix} x_{\omega v} \\ y_{\omega v} \end{pmatrix} = \begin{pmatrix} \dfrac{d}{d\tau} & -\dot{\omega} \\ \dot{\omega} & \dfrac{d}{d\tau} \end{pmatrix} \begin{pmatrix} x_\omega \\ y_\omega \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} x_{\omega a} \\ y_{\omega a} \end{pmatrix} = \begin{pmatrix} \dfrac{d}{d\tau} & -\dot{\omega} \\ \dot{\omega} & \dfrac{d}{d\tau} \end{pmatrix} \begin{pmatrix} x_{\omega r} \\ y_{\omega r} \end{pmatrix}. \tag{8}$$

The second subscripts v and a are associated with velocity and acceleration and the crossproducts arise from the coupling and Coriolis terms.

Dot and Crossproducts of r, ṙ and r̈

In a Keplerian orbit, the *vis-viva* equation plays an important role. Let $\dot{s}$ be the magnitude of the total velocity vector $\dot{\mathbf{r}}$, then $\dot{s}^2$ is obtained by $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ so that

$$\dot{s} = \dot{r}^2 + r^2(\dot{v} + \dot{\omega}) + 2r^2\dot{\Omega}(\dot{v} + \dot{\omega}) \cos i + \dot{\Omega}^2(x^2 + y^2). \tag{9}$$

The dot product of $\mathbf{r}$ and $\dot{\mathbf{r}}$ provides the scalar product of r and $\dot{r}$. It is observed that the radial component of $\dot{\mathbf{r}}$ is along $\mathbf{r}$ and has a magnitude of $e\sqrt{\mu/p} \sin v$.

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{e\sqrt{\mu p} \sin v}{1 + e \cos v}. \tag{10}$$

The crossproduct of $\mathbf{r}$ and $\dot{\mathbf{r}}$ is the angular momentum vector. Kepler's second law of constant areal velocity is nothing but the conservation of angular momentum. In a perturbed orbit, $\mathbf{r} \times \dot{\mathbf{r}}$ is no longer time invariant as

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$$\mathbf{r} \times \dot{\mathbf{r}} = r^2(\dot{v} + \dot{\omega})\mathbf{W} + r^2\dot{\Omega}\mathbf{z}_0 - z\dot{\Omega}\mathbf{r} \qquad (11)$$

where $\mathbf{W}$ is utilized for $\mathbf{P} \times \mathbf{Q}$. If the orbit is on the equatorial plane, $\dot{\mathbf{r}}$ is on the orbital plane by Eq. (3). The last term of Eq. (11) is zero as z ($\mathbf{z}_0$ – component of $\mathbf{r}$) is zero. Therefore, the angular momentum of a perturbed equatorial orbit is $r^2(\dot{v} + \dot{\omega} + \dot{\Omega})$ directed along $\mathbf{z}_0$.

In general, the physical mechanism of Eq. (11) shows that the normal vector to the orbital plane varies its direction with time as the components of $\mathbf{W}$ have $\sin\Omega$ and $\cos\Omega$. The angular momentum vector $\mathbf{r} \times \dot{\mathbf{r}}$ revolves around this moving $\mathbf{W}$ when the true anomaly v changes from 0 to 2π.

The moment of force acting on the satellite about the geocenter is $m\mathbf{r} \times \ddot{\mathbf{r}}$, where m is the mass of the satellite. In an unperturbed orbit it is zero, but in the present case it is obviously not zero. This, in turn, suggests that there is an acceleration due to the disturbing source (chiefly, the asphericity of the Earth) so that $\ddot{\mathbf{r}}$ should be expressed by

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}(\mathbf{r} + \hat{\mathbf{r}}). \qquad (12)$$

Let the components of $\hat{\mathbf{r}}$ be $\hat{x}$, $\hat{y}$ and $\hat{z}$. The first term of Eq. (12) stands for a pure central force field in a two-body system and $-\frac{\mu}{r^3}\hat{\mathbf{r}}$ is an undetermined acceleration vector due to the asphericity of the Earth. Combining with Eq. (11), $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}} \times \mathbf{r} = -\frac{r^3}{\mu}\frac{d}{d\tau}[r^2(\dot{\omega}\mathbf{W} + \dot{\Omega}\mathbf{z}_0) - z\dot{\Omega}\mathbf{r}]. \qquad (13)$$

Unfortunately, it is impossible to solve Eq. (13) for $\hat{\mathbf{r}}$. However, the above equation relates $\hat{\mathbf{r}}$ to perturbed orbital elements and $\dot{\omega}$ and $\dot{\Omega}$.

Product of Range and Range Rate

Let $\boldsymbol{\varrho}$ and $\dot{\boldsymbol{\varrho}}$ be the range and range-rate vector of the satellite with respect to a particular ground station. Let a vector $\mathbf{R}$ be the station vector which originates at the ground station and terminates at the geocenter. The components of $\mathbf{R}$ are X, Y and Z.

$$\boldsymbol{\varrho} = \mathbf{r} + \mathbf{R} \qquad (14)$$

$$\dot{\boldsymbol{\varrho}} = \dot{\mathbf{r}}_{PQ} + \mathbf{z}_0 \times (\dot{\Omega}\mathbf{r} + \omega_e\mathbf{R}) \qquad (15)$$

where ω_e is the Earth's rotational rate for a known vernal equinox. Just as $\mathbf{r}\cdot\dot{\mathbf{r}}$ is $r\dot{r}$ even for the perturbed elliptic orbit, it is easy to derive the result that $\boldsymbol{\varrho}\cdot\dot{\boldsymbol{\varrho}}$ yields $\rho\dot{\rho}$, which has been extensively used for satellite tracking by the doppler frequency shifts. The product of ρ and $\dot{\boldsymbol{\varrho}}$ furnishes

$$\rho\dot{\rho} = r\dot{r} - (\omega_e - \dot{\Omega})\mathbf{R}\cdot\mathbf{z}_0 \times \mathbf{r} + \mathbf{R}\cdot\dot{\mathbf{r}}_{PQ}. \qquad (16)$$

Only in the equatorial orbit can $\mathbf{z}_0 \times \mathbf{r}$ be replaced by $\mathbf{r}^*$ (a coplanar vector with $\mathbf{r}$ having a different magnitude as the true anomaly is advanced by 90°). A generalization of the angular momentum vector given by (11) is visualized by $\boldsymbol{\varrho} \times \dot{\boldsymbol{\varrho}}$, though there is little accepted use of $\boldsymbol{\varrho} \times \dot{\boldsymbol{\varrho}}$. If $\mathbf{R}$ goes to zero, $\boldsymbol{\varrho} \times \dot{\boldsymbol{\varrho}}$ becomes equal to $\mathbf{r} \times \dot{\mathbf{r}}$.

$$\begin{aligned}\boldsymbol{\varrho} \times \dot{\boldsymbol{\varrho}} = {} & r^2(\dot{v} + \dot{\omega})\mathbf{W} + \mathbf{R} \times \dot{\mathbf{r}}_{PQ} \\ & + \mathbf{z}_0[r^2\dot{\Omega} + R^2\omega_e + (\omega_e + \dot{\Omega})(\mathbf{r}\cdot\mathbf{R}) \\ & - (z + Z)(\omega_e\mathbf{R} + \dot{\Omega}\mathbf{r})].\end{aligned} \qquad (17)$$

Equation (17) indicates the coupling between the station vector $\mathbf{R}$ and the satellite position vector $\mathbf{r}$, and $\mathbf{R}$ and the velocity vector $\dot{\mathbf{r}}_{PQ}$ on the orbital plane.

Laplacian Method for Orbit Determination

In the Laplacian method, the goal is to determine the range ρ_2 and range-rate $\dot{\rho}_2$ (the subscript 2 for the second date) when there are observed values of the topocentric right ascension and declination for three dates. The assumption used in the Laplacian method is that the time derivatives of the unit range vector $\mathbf{L}$ are numerically obtainable, using the observed data of the three dates. Following the derivation given in many books [5], the Poincaré form vector differential equation is obtained as

$$\ddot{\rho}\mathbf{L} + 2\dot{\rho}\dot{\mathbf{L}} + \rho\left(\frac{\mu\mathbf{L}}{r^3} + \ddot{\mathbf{L}}\right) = \ddot{\mathbf{R}} + \frac{\mu}{r^3}\mathbf{R} - \frac{\mu}{r^3}\hat{\mathbf{r}}. \qquad (18)$$

The geometrical constraint is

$$r = \sqrt{\rho^2 + R^2 - 2\rho(\mathbf{L}\cdot\mathbf{R})} \qquad (19)$$

and the disturbing acceleration equation is

$$\begin{aligned}& -\frac{\mu}{r^3}(\mathbf{r} + \hat{\mathbf{r}}) \\ & = \ddot{\mathbf{r}}_{PQ} + 2\dot{\Omega}\mathbf{z}_0 \times \dot{\mathbf{r}}_{PQ} + z\dot{\Omega}^2\mathbf{z}_0 + \ddot{\Omega}\mathbf{z}_0 \times \mathbf{r}.\end{aligned} \qquad (20)$$

Equations (18) and (20) are dynamical constraints. When vectors are broken down into components, Eqs. (18), (19), and (20) represent the seven equations for seven unknowns $\ddot{\rho}$, $\dot{\rho}$, ρ, r, $\hat{x}$, $\hat{y}$, and $\hat{z}$. Using a high speed digital computer, it appears possible to determine the perturbed equations by an iterative method, provided that $\dot{\omega}$, $\ddot{\omega}$, $\dot{\Omega}$ and $\ddot{\Omega}$ are also available. In the case of preliminary orbit determination, approximate values of the perturbed elements can be used. Because the orbit determination for a perturbed elliptic orbit is subject to a continual orbit improvement process, the expressions given by (18), (19), and (20) may be of some use.

Conclusion

This note has clearly shown how $\dot{\mathbf{P}}$ and $\dot{\mathbf{Q}}$ can be retained with little difficulty by incorporating the two coplanar vectors $\dot{\mathbf{r}}_{PQ}$ and $\ddot{\mathbf{r}}_{PQ}$ with $\mathbf{r}$. Vectorial operations carried out in the preceding examples are actually applicable for some aspects of precision tracking and position fix. Being different from the component oriented celestial mechanics which has strongly been characterized as a "passive" observational science, present day astrodynamics places many man-made devices in outer space. In the "active" astronomy, clearer gross pictures or physical visualizations of complicated dynamical problems are obtained by elementary vectorial calculus.

References

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