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Matched-Conic Approximation to the Two Fixed Force-Center Problem*

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Planar motion of a particle of negligible mass from the neighborhood of a gravitational center (the "earth") of mass $1-\mu$ to the neighborhood of a second center (the "moon") of mass μ is studied by asymptotic methods for the case $\mu \ll 1$. The calculations are carried out for the case of two fixed centers. It is pointed out, however, that the methods used are also applicable to the case of the two centers rotating around their center of mass, that is, to the limiting case of the restricted three-body problem for which the second mass is much smaller than the first. A uniformly valid solution describing the passage from the earth to the moon and the motion in the neighborhood of the moon is obtained. Each part of the motion is in the first approximation a Keplerian conic relative to the earth and moon, respectively. However, these conics cannot be matched directly: In order to determine the second part, as well as the subsequent motion, it is necessary to compute a correction of order μ to the first part. This statement is equally true for the restricted three-body problem.

I. INTRODUCTION

In describing the motion of a particle traveling from the vicinity of a massive celestial body to a much smaller one (say from earth to moon) it has been commonly accepted that a crude first approximation could be had by patching two Keplerian conics appropriate to the motions in the vicinity of the individual bodies.

In this work, this intuitive idea is re-examined with the point of view of deriving a uniformly valid asymptotic approximation for the solution, applying the methods of singular perturbation theory. It is shown that there is no mathematical justification for the direct patching of conics, and that a higher approximation to the motion for the initial leg must be used in order to derive a correct approximate description of the entire motion.

Consider the restricted three-body problem and let the mass μ of one gravitational center, the "moon," be much smaller than that of the other center, the "earth" of mass $(1-\mu)$. Approximations to the motion of a particle moving under the influence of these two centers may be obtained by developing the solutions in powers of μ. The influence of the moon then appears as a correction of order μ to a motion which is Keplerian relative to the earth. This scheme fails, however, when the particle is in a region sufficiently close to the moon for the attraction of the moon to be dominant. As discussed, for example, in Kevorkian (1962), the motion in this region is to first order Keplerian relative to the moon. To study this motion "blown-up" coordinates are used, i.e., distance to the moon as well as time are measured in units which are suitable powers of μ.

For trajectories which leave the earth, approach, and are influenced by, the moon and upon leaving its vicinity again become dominated by the earth's attraction, it will be shown that to first approximation the moon's effect is represented by a discontinuity in the slope of the orbit at the lunar location. The magnitude of this discontinuity depends upon the angular momentum of the hyperbolic orbit associated with the moon; since this quantity depends strongly upon the distance of lunar approach (which is of order μ) it is impossible to isolate the correct return orbit out of the one-parameter family of possibilities unless one carries out the solution for the approach leg to order μ.

Thus the composite solution which approximates the exact complete motion uniformly must include a first correction to the Keplerian orbit for the approach leg.

For the purpose of illustrating the basic ideas discussed above it is sufficient to consider the simpler problem of two *fixed* force centers since the crucial questions of approximation and matching procedures hinge on the nature of the gravitational terms, and do not depend upon the fact that these centers rotate. Actually, the problem of two fixed centers may in principle be solved by quadratures, since two integrals are known. However, instead of starting from the integral representation which formally gives the exact solution, we shall use perturbation methods which proceed from the differential equations and hence apply to the restricted three-body problem for which only one integral is known.

II. BASIC EQUATIONS, INTEGRALS

Only the planar problem is discussed. We consider two gravitational centers in the plane: the "earth" of mass $1-\mu$ and the "moon" of mass μ. A particle, the "spaceship," whose mass is negligible compared to μ (and to $1-\mu$) moves in the plane. Denoting its Cartesian coordinates by x and y and using suitable nondimensional quantities, the equations of motion are

$$\ddot{x}=\frac{\partial}{\partial x}\left(\frac{1-\mu}{r_e}+\frac{\mu}{r_m}\right), \tag{2.1a}$$

$$\ddot{y}=\frac{\partial}{\partial y}\left(\frac{1-\mu}{r_e}+\frac{\mu}{r_m}\right), \tag{2.1b}$$

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where

$$(\xi_e,\eta_e)=\text{position of the earth,} \tag{2.2a}$$

$$(\xi_m,\eta_m)=\text{position of the moon,} \tag{2.2b}$$

$$r_e^2=(x-\xi_e)^2+(y-\eta_e)^2, \tag{2.2c}$$

$$r_m^2=(x-\xi_m)^2+(y-\eta_m)^2. \tag{2.2d}$$

We obtain the equations for the restricted three-body problem by putting {in this case distances are normalized by D, the distance between the earth and the moon; the time is normalized by $[D^3/G(m_e+m_m)]^{\frac{1}{2}}$, where G is the universal gravitational constant and m_m and m_e the dimensional masses of the moon and the earth, respectively; μ is $m_m/(m_m+m_e)$}

$$\xi_e=-\mu\cos(t-\psi), \tag{2.3a}$$

$$\eta_e=-\mu\sin(t-\psi), \tag{2.3b}$$

$$\xi_m=(1-\mu)\cos(t-\psi), \tag{2.3c}$$

$$\eta_m=(1-\mu)\sin(t-\psi), \tag{2.3d}$$

where ψ is a constant defining the position of the earth and moon at $t=0$.

This paper is mainly concerned with the problem of two fixed centers, in which case we put

$$\xi_e=\eta_e=0, \tag{2.4a}$$

$$\xi_m=1, \tag{2.4b}$$

$$\eta_m=0. \tag{2.4c}$$

In this case two integrals for Eqs. (2.1) may be derived. First of all, the Hamiltonian is an integral, since it does not involve the time explicitly. This gives the well-known energy integral

$$\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-[(1-\mu)/r]-(\mu/r_m)=h=\text{const.} \tag{2.5}$$

(Since the "earth" is now fixed at the orgin, we write r instead of r_e.)

A second integral is derived in Whittaker (1944) by introducing elliptic-hyperbolic coordinates. Actually, a second integral having a very simple form may be derived directly. It is

$$ll_m+(1-\mu)\frac{x}{r}-\mu\frac{(x-1)}{r_m}=\delta=\text{const,} \tag{2.6}$$

where

$l=x\dot{y}-y\dot{x}=$ angular momentum relative to the earth, (2.7a)

$l_m=(x-1)\dot{y}-y\dot{x}=$ angular momentum relative to the moon. (2.7b)

To prove (2.6) we note that by differentiating (2.7) one finds

$$\text{(a)}\quad \dot{l}=x\ddot{y}-y\ddot{x},\quad \dot{l}_m=\dot{l}-\ddot{y}.$$

Using the equations of motion one obtains

$$\dot{l}=-\mu y/r_m^3,\quad \dot{l}_m=(1-\mu)y/r^3. \tag{2.8}$$

Direct differentiation gives the following identities:

$$\text{(b)}\quad \frac{d}{dt}\left(\frac{x}{r}\right)=\frac{-yl}{r^3},\quad \frac{d}{dt}\left(\frac{x-1}{r_m}\right)=\frac{-yl_m}{r_m^3}.$$

From these formulas it follows easily that the time derivative of the left-hand side of Eq. (2.6) is zero.

Note that since two integrals have been found, Eqs. (2.1) may, in principle, be solved by quadratures (Whittaker 1944). However, in the present paper, perturbation methods are used in order to shed light on their application to the restricted three-body problem where only one integral is known.

Asymptotic expansions are studied for the case of very small μ in which the particle passes within a distance of order μ of the moon. Furthermore, we take the simple case when the particle starts its motion very near the earth. Rather than specify position and velocity at $t=0$, it is more convenient to prescribe the value of the total energy

$$h=\text{prescribed,} \tag{2.9a}$$

and also

$$x=0,\quad y=0,\quad dy/dx=-\mu c\quad \text{at}\quad t=0. \tag{2.9b}$$

Here c is a parameter which determines the initial slope. It is clear that for h sufficiently large and for $\mu=0$ the particle will pass along a straight line from the earth to the position of the moon. Hence, for μ small, it is expected that its minimal distance to the moon will be of order μ. It will always be assumed that h is large enough so that for $\mu=0$ the velocity at the position of the moon is >0.

III. TRANSFORMATION OF EQUATIONS, EXPANSION PROCEDURE

As was pointed out earlier, it is expected that the effect of the earth's and moon's gravitational attractions will individually dominate in neighborhoods centered at their respective origins. An expansion procedure in which x, y, and t are held fixed as $\mu\to 0$ relegates the moon's attraction to a higher-order term as can be seen from Eqs. (2.1). Such a limit process will be called an *outer limit*, and the corresponding expansion, which will be valid when the particle is not near the moon, will be called an *outer expansion*.

Since for a sufficiently small neighborhood of the moon the latter's attraction can become dominant, one must determine the appropriate variables for this case by an analysis of the orders of magnitude of the various terms.

Let the variables

$$x^*=\frac{x-1}{\mu^\alpha},\quad y^*=\frac{y}{\mu^\alpha},\quad t^*=\frac{t-\tau}{\mu^\beta}, \tag{3.1}$$

denote the *inner* or lunar variables, where τ is the time elapsed to reach $x=1$. When Eq. (2.1a) is written in terms of the starred variables one obtains

$$\text{(a)}\quad \frac{d^2x^*}{dt^{*2}}=-\frac{(1-\mu)\mu^{2\beta-\alpha}(1+\mu^\alpha x^*)}{[1+2\mu^\alpha x^*+\mu^{2\alpha}(x^{*2}+y^{*2})]^{\frac{3}{2}}}-\mu^{1-3\alpha+2\beta}\frac{x^*}{[x^{*2}+y^{*2}]^{\frac{3}{2}}}.$$

Thus, in order for the lunar attraction to be of the same order as the acceleration, we must set $3\alpha-2\beta=1$. This is satisfied for an infinity of α and β values. Kevorkian (1962), by a similar analysis for the restricted three-body equations, found that for $\alpha=\frac{1}{3}$, $\beta=0$, Hill's equations were obtained for which the Coriolis terms were also of the same order as the lunar attraction, and that a smaller neighborhood of the moon may be defined by $\alpha=\frac{1}{2}$, $\beta=\frac{1}{4}$, in which case the Coriolis terms were of a higher order.

For the present case, the following argument will be used to define the relation between α and β. We are not interested in orbits which stay permanently in a small neighborhood of the moon but rather in orbits which originate at finite distances from the moon and pass close to the moon with a finite velocity measured in the outer variables. Furthermore, matching of outer and inner solutions requires the velocity in inner coordinates to be of the same order as that in outer coordinates. Thus $dx^*/dt^*=O(dx/dt)=O(1)$. This implies that $\alpha=\beta$, and since also $3\alpha-2\beta=1$, we find $\alpha=1$, $\beta=1$.

Hence we choose as inner variables

$$x^*=\frac{x-1}{\mu},\quad y^*=\frac{y}{\mu},\quad t^*=\frac{t-\tau}{\mu}\tag{3.1a}$$

The basic equations (2.1) expressed in these variables are

$$\frac{d^2x^*}{dt^{*2}}=-\frac{(1-\mu)\mu(1+\mu x^*)}{[1+2\mu x^*+\mu^2(x^{*2}+y^{*2})]^{\frac{3}{2}}}-\frac{x^*}{[x^{*2}+y^{*2}]^{\frac{3}{2}}},\tag{3.2a}$$

$$\frac{d^2y^*}{dt^{*2}}=-\frac{(1-\mu)\mu y^*}{[1+2\mu x^*+\mu^2(x^{*2}+y^{*2})]^{\frac{3}{2}}}-\frac{y^*}{[x^{*2}+y^{*2}]^{\frac{3}{2}}}.\tag{3.2b}$$

We observe that with the starred variables fixed the limit as $\mu\to 0$ for these equations represents motion of the particle under the gravitational influence of *one* center, the moon.

Thus, close to the moon the first approximation to the motion should be a Keplerian orbit relative to the moon.

The nonuniformity in the outer solution is expected to occur in some neighborhood of the moon, and it is hence more convenient to choose a distance coordinate, rather than the time as the independent variable. Although there are more sophisticated ways of choosing the parameters, for the purpose of this example it is sufficient to take x as the independent variable, and treat y and t as functions of x.

It is easy to show that this transformation when applied to Eqs. (2.1) yields

$$-\frac{t''}{t'^3}=-(1-\mu)\frac{x}{r^3}-\mu\frac{(x-1)}{r_m^3},\tag{3.3a}$$

$$\frac{y''}{t'^2}-\frac{t''y'}{t'^3}=-(1-\mu)\frac{y}{r^3}-\mu\frac{y}{r_m^3},\tag{3.3b}$$

where primes denote differentiation with respect to x.

The initial conditions (2.9) are already in the appropriate form if one regards these values to be prescribed at the point $x=0$.

The outer expansion will proceed in the form

$$t(x)=t_0(x)+\mu t_1(x)+O(\mu^2),\tag{3.4a}$$

$$y(x)=\mu y_1(x)+O(\mu^2).\tag{3.4b}$$

Note that in the expansion for y the term of order unity is absent. This is a direct consequence of the special initial value problem chosen with $y(0)=0$ and $y'(0)=O(\mu)$.

The inner expansion is given by

$$t(x^*)-\tau=\mu t_0^*(x^*)+O(\mu^2),\tag{3.5a}$$

$$y(x^*)=\mu y_0^*(x^*)+O(\mu^2).\tag{3.5b}$$

As pointed out earlier, t and y are of order μ near the moon. Hence the leading terms in (3.5a,b) are of order μ.

IV. ONE-DIMENSIONAL PROBLEM

It is instructive to study first the case $c=0$. The motion is then one-dimensional; it is confined to the interval $0\leqslant x\leqslant 1$ and is periodic, as can easily be deduced by studying the integral curves for various values of h in the phase plane of x and $v=dx/dt$.

The energy integral reduces to

$$h=\frac{1}{2}\left(\frac{dx}{dt}\right)^2-\frac{(1-\mu)}{x}+\frac{\mu}{x-1}\quad\text{for}\quad 0\leqslant x\leqslant 1;\tag{4.1a}$$

or if x is regarded as the independent variable, we have

$$h=\frac{1}{2t'^2}-\frac{(1-\mu)}{x}+\frac{\mu}{x-1},\quad 0\leqslant x\leqslant 1.\tag{4.1b}$$

The equilibrium point x_s, which is a saddle point, is located at

$$x_s=\frac{1-\mu-(\mu-\mu^2)^{\frac{1}{2}}}{1-2\mu}=1-\mu^{\frac{1}{2}}+O(\mu),\tag{4.2}$$

and the value of h for the integral curve passing through x_s is

$$h_s = -\frac{(1-2\mu)[\mu^{1/3}+(1-\mu)^{1/3}]}{(1-\mu)^{1/3}-\mu^{1/3}}$$
$$= -[1+2\mu^{1/3}+O(\mu^{2/3})]. \quad (4.3)$$

For values of $h > h_s$, the motion traverses the entire interval, while for $h < h_s$, motion, depending upon the initial position, is confined to one of the two regions bounded by the intersections of the two branches of the integral curve with the x axis (i.e., motion is confined to a neighborhood of the earth or the moon if $h < h_s$).

We shall only consider the special case $h=0$; motion then occurs over the entire interval since $h_s < 0$.

When the expansion (3.4a) is applied to Eq. (3.3a) one obtains

$$\frac{t_0''}{t_0'^3} = \frac{1}{x^2}, \quad (4.4a)$$

$$-\frac{t_1''}{t_0'^3} + \frac{3t_0''t_1'}{t_0'^4} = \frac{1}{x^2} + \frac{1}{(1-x)^2}. \quad (4.4b)$$

The above equations possess the following first integrals:

$$\frac{1}{2t_0'} - \frac{1}{x} = h_0 = \text{const}, \quad (4.5a)$$

$$-\frac{t_1'}{t_0'^3} + \frac{1}{x} - \frac{1}{1-x} = h_1 = \text{const.} \quad (4.5b)$$

It is easy to verify that these are nothing but the first two terms of the expansion of (4.1b).

When Eqs. (4.5a,b) with $h_0 = h_1 = 0$ are integrated for our initial value problem, the following outer expansion for t is obtained:

$$\sqrt{2}t = \left[\tfrac{2}{3}x^{3/2} + \mu\left(\tfrac{2}{3}x^{3/2} + x^{1/2} - \tfrac{1}{2}\log\frac{1+x^{1/2}}{1-x^{1/2}}\right) + O(\mu^2)\right], \quad (4.6)$$

for motion in the positive x direction.

It is interesting to note that $t_1(x)$, the first correction to the unperturbed motion, possesses a logarithmic singularity at the location of the moon. This is to be expected, since (4.6) cannot be valid when x is close to unity. The role of this singularity is discussed after (4.12).

The inner expansion can be derived from Eq. (3.2a) with $y^* = 0$. Here it is more convenient to let $x^{*\prime} = -x^* = (1-x)/\mu$, which is positive in the interval of interest, be the inner variable.

Since the leading term of the inner expansion for t corresponds to motion in the field of one attractive center, we have the energy integral in the starred variables

$$\tfrac{1}{2}(dx^{*\prime}/dt^*)^2 + 1/x^{*\prime} = h^*. \quad (4.7)$$

The value of h^* is deduced by matching the velocities for the inner and outer limits. General principles for matching of expansions are discussed in Kaplun and Lagerstrom (1957). In the present context matching is most conveniently performed by writing an outer expansion in terms of the inner variable and comparing it to the appropriate order with an inner expansion evaluated for large values of the inner variable.

By comparing Eqs. (4.1b) and (4.7) we see that the velocities match if $h^* = 1$.

Hence in the neighborhood of the moon t is governed by

$$\sqrt{2}\frac{dt}{dx^{*\prime}} = \mp\mu\left(\frac{x^{*\prime}}{1+x^{*\prime}}\right)^{1/2} + O(\mu^2), \quad (4.8)$$

where the negative sign is to be taken when motion is towards the moon (i.e. the negative $x^{*\prime}$ direction).

The integral of (4.8) gives the inner limit for t in the form

$$\sqrt{2}t = (\sqrt{2}\tau_0 + \mu\{\sqrt{2}\tau_1(\mu) - [x^{*\prime}(1+x^{*\prime})]^{1/2}$$
$$+ \sinh^{-1}(x^{*\prime})^{1/2}\}^{1/2} + O(\mu^2)), \quad (4.9)$$

where $\tau_0 + \mu\tau_1$ is the half-period of the motion and should be evaluated by matching (4.9) with (4.6).

Again, if the outer expansion is written correct to order μ in terms of the inner variable, and the inner expansion is evaluated for large $x^{*\prime}$, Eqs. (4.6) and (4.9) give (for the case of motion towards the moon)

Outer expansion near $x=1$:

$$\sqrt{2}t = \tfrac{2}{3} + u[(5/3) - \log 2 + \tfrac{1}{2}\log\mu - x^{*\prime}$$
$$+ \tfrac{1}{2}\log x^{*\prime}] + O(\mu^2); \quad (4.10)$$

Inner expansion for large $x^{*\prime}$:

$$\sqrt{2}t = \sqrt{2}\tau_0 + \mu[\sqrt{2}\tau_1 - x^{*\prime} + \log 2(x^{*\prime})^{1/2}] + O(\mu^2). \quad (4.11)$$

By comparing (4.10) and (4.11) it is seen that these match to order unity if $\tau_0 = \frac{1}{3}\sqrt{2}$ and to order μ if

$$\tau_1 = (1/\sqrt{2})[(5/3) + \tfrac{1}{2}\log\mu - 2\log 2].$$

For this example the composite expansion can be obtained by adding the inner and outer representations of t and subtracting the inner limit of the outer expansion. This gives for motion towards the moon

$$\sqrt{2}t = \tfrac{2}{3}x^{3/2} + \mu\left\{\tfrac{2}{3}x^{3/2} + x^{1/2} - \tfrac{1}{2}\log\frac{1+x^{1/2}}{1-x^{1/2}}\right.$$
$$-\log 2 - [x^{*\prime}(1+x^{*\prime})]^{1/2} + x^{*\prime} - \tfrac{1}{2}\log x^{*\prime}$$
$$\left. + \sinh^{-1}(x^{*\prime})^{1/2}\right\} + O(\mu^2), \quad (4.12)$$

and since the motion is periodic it is only necessary to define t on one leg of the motion.

One can verify that (4.12) is a composite expansion by noting it leads to the correct inner and outer expansion under the appropriate limit processes. Hence, it is uniformly valid on the entire interval. Note that the logarithmic singularity near $x=1$ has been eliminated in (4.12) since the singular contribution of the term

$$\tfrac{1}{2}\log[(1+x^{\frac{1}{2}})/(1-x^{\frac{1}{2}})]$$

is exactly canceled by the term $-\frac{1}{2}\log x^{*\prime}$.

The exact solution can be derived in a straightforward manner for this simple example. From the energy integral (4.1a) with $h=0$ it follows that

$$\sqrt{2}t=\int_0^x\left[\frac{\xi(1-\xi)}{(1-\mu)-(1-2\mu)\xi}\right]^{\frac{1}{2}}d\xi \quad \text{when} \quad \frac{dx}{dt}>0. \tag{4.13}$$

This can be expressed in terms of elliptic integrals in the form (Byrd and Friedman 1954)

$$\sqrt{2}t=[1/(1-2\mu)^{\frac{1}{2}}]f(x,\gamma), \tag{4.14}$$

where

$$f(x,\gamma)=\frac{2}{3k^4\gamma^{\frac{1}{2}}}\{(2-k^2)E(\sin^{-1}x^{\frac{1}{2}},k) +2(1-k^2)F(\sin^{-1}x^{\frac{1}{2}},k) -k^2[x(1-x)(1-k^2x)]^{\frac{1}{2}}\}, \tag{4.14a}$$

$$k=\gamma^{-\frac{1}{2}}, \tag{4.14b}$$

$$\gamma=1-\mu/(1-2\mu), \tag{4.14c}$$

and where F and E are the elliptic integrals of the first and second kind with amplitude $\sin^{-1}x^{\frac{1}{2}}$ and modulus k.

The approximate formulas obtained by perturbation methods may be checked directly from the exact solution.

V. TWO-DIMENSIONAL PROBLEM

Before carrying out the details of the computations for the special initial conditions (2.9) some intuitive ideas are discussed.

Consider an exact solution of (2.1), depending on the parameter μ, whose trajectory passes within a distance of order μ of the moon. As μ tends to zero in the outer limit, the limiting trajectory will pass through the position of the moon, i.e., $x=1$, $y=0$. The trajectory before and after this point will be called the first and second leg, respectively, of the outer solution to order unity. Each leg will be a Keplerian conic relative to the earth. However, there will be some kind of discontinuity at the position of the moon, similar to a shock wave in fluid flow. The energy relative to the earth will be constant on each leg. Actually, these constants are the same. This follows from the fact that in the exact solution, the total energy, as given by (2.5), is an exact invariant. Hence, in the outer solution the energy reltive to the earth is continuous. This implies that the magnitude of the velocity is continuous. However, we expect the attraction of the moon to alter radically the direction of motion or, equivalently, the angular momentum relative to the earth. To order unity the first leg of the outer solution is a Keplerian orbit defined by the initial conditions to this order: The second leg is then determined to order unity except for its angular momentum. It can be seen that the second integral (2.6) gives no information about the first-order outer solution. The details of the change in velocity direction are given by the leading term of the inner solution, just as the viscous shock-layer solution gives the details of the shock discontinuity occurring across a nonviscous shock wave. Assuming that the first leg of the first-order outer solution is known, a determination of the first-order inner solution is equivalent to a determination of the second leg of the outer solution.

A more detailed consideration shows the following. The first leg of the outer solution gives a certain velocity at the position of the moon. According to general principles of matching this velocity will be the apparent velocity at infinity for the motion around the moon given by the inner solution; an incidental result is that this motion must then be hyperbolic. To determine the inner solution completely we also need the angular momentum relative to the moon. Once this is known the change in velocity direction due to the moon is known and the second leg of the outer solution can be determined. From the first-order outer solution we get the apparent velocity at infinity for the hyperbolic trajectory around the moon. The slope of the first asymptote of this trajectory is then known. In order to determine the angular momentum we also need to know the distance of the moon to this asymptote. However, for the inner solution the length scale is of order μ in terms of the outer variables. Hence in order to define the inner solution we need the distance to order μ from the moon to the trajectory. This requires not only that initial conditions have to be considered to order μ; it is also necessary to compute the first leg of the outer solution correct to order μ, i.e., to compute the first correction to the Keplerian orbit relative to the earth.

To summarize, it is expected that the terms of order unity as well as order μ will be needed in the first leg of the outer solution in order to establish the appropriate matching hyperbolic orbit near the moon and hence the second leg of the outer solution.

When the expansions for y and t given by (3.4) are used in Eqs. (3.3) the following differential equations for t_0, t_1, and y_1 ensue:

$$\frac{t_0''}{t_0'^3}=\frac{1}{x^2}, \tag{5.1a}$$

$$-\frac{t_1''}{t_0'^3}+\frac{3t_0''t_1'}{t_0'^4}=\frac{1}{x^2}+\frac{1}{(1-x)^2}, \tag{5.1b}$$

$$\frac{y_1''}{t_0'^2}-\frac{t_0''y_1'}{t_0'^3}=-\frac{y_1}{x^2}. \tag{5.1c}$$

These define the outer solution to order μ, and if the initial conditions (2.9) are used the first leg of the outer expansion can be computed.

Equations (5.1a,b) are identical to the corresponding one-dimensional equations (4.4a,b) since for this special initial value problem $y_0(x)=0$ and the t equations do not involve y to order μ. For more general initial values this would not be the case and the equations for t_0 and y_0 need to be solved first before one can compute y_1 and t_1.

By straightforward integration the solution of Eqs. (5.1a,b) for t_0 and t_1 gives

$$t=t_0+\mu t_1+O(\mu^2), \tag{5.2}$$

$$\sqrt{2}t_0=\frac{1}{\rho^3}\sin^{-1}\rho x^{\frac{1}{2}}-\frac{1}{\rho^2}[x(1-\rho^2x)]^{\frac{1}{2}}, \tag{5.2a}$$

$$\sqrt{2}t_1=-\frac{h_1}{\rho^3}\left\{\frac{[x(1-\rho^2x)]^{\frac{1}{2}}}{\rho}+\frac{1}{\rho}\left(\frac{x}{1-\rho^2x}\right)^{\frac{1}{2}}-\frac{3}{2\rho^2}\sin^{-1}\rho x^{\frac{1}{2}}\right\}$$
$$-\frac{2}{\rho^3}\sin^{-1}\rho x^{\frac{1}{2}}+\frac{2-\rho^2}{\rho^2(1-\rho^2)}\left(\frac{x}{1-\rho^2x}\right)^{\frac{1}{2}}+\frac{1}{2(1-\rho^2)^{\frac{3}{2}}}$$
$$\times\log\frac{1+(1-2\rho^2)x-2[x(1-\rho^2x)(1-\rho^2)]^{\frac{1}{2}}}{1-x}, \tag{5.2b}$$

where

$$h=h_0+\mu h_1+O(\mu^2),$$
$$h_0=-\rho^2>-1.$$

In order to ensure motion over the entire interval $0\leqslant x\leqslant 1$, the condition $h_0>-1$ is necessary (cf. discussion for one-dimensional problem). In addition, we only consider motions which are elliptic to order unity relative to the earth, hence $h_0<0$.

When the value for t_0' is substituted into Eq. (5.1c) one obtains

$$x^2(1-\rho^2x)y_1''-\tfrac{1}{2}xy_1'+\tfrac{1}{2}y_1=0. \tag{5.3}$$

The general solution of (5.3) is easily found:

$$y_1=c_1\rho[x(1-\rho^2x)]^{\frac{1}{2}}+c_2\rho^2x. \tag{5.4}$$

In view of the initial conditions $y_1(0)=0$, $y_1'(0)=-c$, the above reduces to

$$y_1(x)=-cx.$$

Actually, the straight line is a Keplerian trajectory although the motion along this line, i.e., x as a function of t, will differ from Keplerian motion when terms of order μ are considered. The fact that the trajectory is Keplerian to order μ is a coincidence. If initial conditions more general than (2.9b) had been chosen this would no longer be true. Also, as indicated in Sec. VI in the case of the restricted three-body problem, a trajectory which is a straight line to order unity is not a straight line to order μ.

The next task is the computation of the inner solution. The inner equations, obtained by holding the starred variables fixed as $\mu\to 0$ in (3.2), possess the two well-known integrals

$$h^*=\frac{1}{2}\left[\left(\frac{dx^*}{dt^*}\right)^2+\left(\frac{dy^*}{dt^*}\right)^2\right]-\frac{1}{[x^{*2}+y^{*2}]^{\frac{1}{2}}}=\text{const}, \tag{5.5a}$$

$$l^*=x^*\frac{dy^*}{dt^*}-y^*\frac{dx^*}{dt^*}=\text{const}. \tag{5.5b}$$

First consider l^*. This quantity represents the product of the velocity at $-\infty$ with the distance of the asymptote from the origin. From the outer solution evaluated at $x=1$ we obtain the distance $\mu c+O(\mu^2)$ between the asymptote and the moon, while the velocity at $x=1$ is given from the energy integral as $[2(1-\rho^2)]^{\frac{1}{2}}$. Hence from this intuitive argument it is expected that $l^*=c[2(1-\rho^2)]^{\frac{1}{2}}$. This result can also be derived formally by requiring the outer representation for l to match with the inner. This means that when l^* is expressed in terms of the outer variables and evaluated at the point $x=1$ its value must be defined by the results computed from the outer solution.

By definition of the inner variable and l^* we have

$$l^*=\frac{(x-1)}{\mu}\frac{dy}{dt}-\frac{y}{\mu}\frac{dx}{dt}=(x-1)\frac{dy_1}{dt}-y_1\frac{dx}{dt}$$

but

$$\frac{dy_1}{dt}=\frac{1}{t'}\frac{dy_1}{dx}=-\frac{c}{t_0'}[1+O(\mu)]$$

and

$$\frac{dx}{dt}=\frac{1}{t'}=\frac{1}{t_0'}[1+O(\mu)].$$

Hence

$$l^*=c/t_0'+O(\mu),$$

and since $t_0'(1)=[2(1-\rho^2)]^{-\frac{1}{2}}$, the result

$$l^*=c[2(1-\rho^2)]^{\frac{1}{2}}$$

follows.

A similar argument for h^* gives

$$h^*=h_0+1=1-\rho^2>0.$$

Since h^* is positive, the orbit in the vicinity of the moon is hyperbolic, a result which was anticipated by physical arguments.

The orientation of this hyperbolic orbit is easily deduced by noting that the approach velocity at $x^*=-\infty$ along one of the asymptotes is *parallel* to the x^* axis and that the asymptote is at the distance c below the x^* axis (if $c>0$).

Thus, the hyperbolic motion is completely defined and is most conveniently expressed in parametric form (Wintner 1947).

Let x' and y' be Cartesian inner coordinates centered at the focus (moon) of the hyperbola and symmetrically oriented with respect to the asymptotes, i.e.,

$$x' = x^* \cos\theta - y^* \sin\theta, \tag{5.6a}$$

$$y' = x^* \sin\theta + y^* \cos\theta, \tag{5.6b}$$

where θ is the angle between the x^* axis (or the asymptote to the approaching inner trajectory) and the x' axis. From the geometry of the hyperbola θ is given by

$$\theta = \tan^{-1}(\epsilon^2 - 1)^{\frac{1}{2}},$$

where ϵ is the eccentricity.

Then the hyperbolic orbit is defined by

$$x' = a(\epsilon - \cosh u), \tag{5.7a}$$

$$y' = a(\epsilon^2 - 1)^{\frac{1}{2}} \sinh u, \tag{5.7b}$$

$$t' - t_0' = a^{\frac{3}{2}}(\epsilon \sinh u - u), \tag{5.7c}$$

where a is the semimajor axis, t_0' is the time of pericenter passage, and u is the parameter along the hyperbola. ($u \to -\infty$ corresponds to approach to the moon.)

The relations between a, ϵ and the constants h^* and l^* are the well-known results

$$a = \frac{1}{2h^*} = \frac{1}{2(1-\rho^2)}, \tag{5.8a}$$

$$\epsilon = (1 + 2h^* l^{*2})^{\frac{1}{2}} = [1 + 4c^2(1-\rho^2)^2]^{\frac{1}{2}}. \tag{5.8b}$$

The relation between t' and t is simply

$$t' - t_0' = (1/\mu)\{t - [\tau_0 + \mu\tau_1 + O(\mu^2)]\}, \tag{5.9}$$

where $\tau_0 + \mu\tau_1 = \tau$ is the time elapsed to arrive at the moon.

When the parameter μ is eliminated and y^* is expressed as a function of x^* one obtains

$$y^* = \frac{(\epsilon^2-1)^{\frac{1}{2}}}{2-\epsilon^2}\left\{ -x^* + a(\epsilon^2 - 1) \pm x^*\left[1 - \frac{2a}{x^*} + \frac{a^2(\epsilon^2-1)}{x^{*2}}\right]^{\frac{1}{2}}\right\}, \tag{5.10}$$

where the positive sign is to be taken for motion towards the moon and vice versa.

It is easily seen that Eq. (5.10) for y^* does match with the outer expansion to order μ, for if this function is evaluated for $x^* \to -\infty$ it simply reduces to $-c$, which is the value of y_1 at $x = 1$.

The composite expansion for the orbit is then

$$y(x,\mu) = \mu y^*(x^*) + \mu y_1(x) + c\mu + O(\mu^2). \tag{5.11}$$

The reader can easily verify that the above expression gives the inner and outer expansions for the orbit to order μ under the appropriate limit process, and is hence uniformly valid on the unit interval.

The return leg of the outer expansion is not given explicitly in this paper; however, the procedure for calculating it is discussed.

With h^*, l^*, and the orientation of the asymptote corresponding to $u \to \infty$ (i.e., the beginning of the second leg of the outer solution) known, all the initial conditions necessary to define the conic for this second leg are known, and at this point one only needs the first-order results since the trajectory is no longer headed towards a region of nonuniformity. This trajectory will in general be a nondegenerate ellipse relative to the earth since h corresponding to $h^* > 1$ is negative and the angular momentum relative to the earth, l, is no longer of order μ but a first-order quantity. It is easy to see from the geometry that this value for l is $[2(1-\rho^2)]^{\frac{1}{2}} \sin 2\theta$.

To define the time history for the first leg of the trajectory and completely specify this part of the motion, the first leg of the outer expansion for t must be matched with the inner solution.

Following the procedure employed in the one-dimensional problem, the outer expansion is written in terms of the inner variable correct to order μ. At this point to simplify the calculations we shall take $h_1 = 0$, as this quantity is given by the initial conditions and can be chosen arbitrarily. With the above choice Eq. (5.2) yields

$$\sqrt{2}t = \frac{1}{\rho^3}\sin^{-1}\rho - \frac{(1-\rho^2)^{\frac{1}{2}}}{\rho^2} + \mu\left\{\frac{x^*}{(1-\rho^2)^{\frac{1}{2}}} - \frac{2}{\rho^3}\sin^{-1}\rho + \frac{(2-\rho^2)}{\rho^2(1-\rho^2)^{\frac{1}{2}}} + \frac{1}{2(1-\rho^2)^{\frac{3}{2}}}[\log\mu - \log 4(1-\rho^2) + \log(-x^*)]\right\} + O(\mu^2). \tag{5.12}$$

To express the inner expansion for $x^* \to -\infty$ it is sufficient to evaluate the results given in parametric form by Eqs. (5.6) and (5.7) for $u \to -\infty$, and this gives

$$\sqrt{2}t = \sqrt{2}\tau_0 + \mu\left\{\sqrt{2}\tau_1(\mu) + \frac{1}{2(1-\rho^2)^{\frac{3}{2}}}\log(-x^*) + \frac{x^*}{(1-\rho^2)^{\frac{1}{2}}} + \frac{1}{2(1-\rho^2)^{\frac{3}{2}}}\log 4(1-\rho^2)/[1 + 4c^2(1-\rho^2)^2]^{\frac{1}{2}}\right\} + O(\mu^2). \tag{5.13}$$

By comparing (5.12) and (5.13) it is seen that the crucial terms proportional to x^* and $\log(-x^*)$ do match, and that in addition we must set

$$\sqrt{2}\tau_0=\frac{1}{\rho^3}[\sin^{-1}\rho-\rho(1-\rho^2)^{1/2}], \tag{5.14a}$$

$$\sqrt{2}\tau_1=\frac{1}{2(1-\rho^2)^{1/2}}\log\frac{\mu[1+4c^2(1-\rho^2)^2]^{1/2}}{4(1-\rho^2)^2}\cdot$$

$$-\frac{2}{\rho^3}\sin^{-1}\rho+\frac{2-\rho^2}{\rho^2(1-\rho^2)^{1/2}}. \tag{5.14b}$$

The composite expansion can now be obtained according to the general principles already discussed. This is simply the sum of the inner expansion for t defined by (5.9) and the outer expansion defined by (5.2) less the inner limit of the outer expansion given by (5.12). As all the necessary information has been worked out, this somewhat lengthy formula is not exhibited explicitly.

An incidental remark is that for $c=0$, the above results reduce to the one-dimensional solution in the limit $\rho\to 0$.

VI. CONCLUDING REMARKS

The essential purpose of this paper is to point out the following results regarding the structure of an earth–moon trajectory.

The trajectory of a mass particle originating near the earth and passing close to the moon consists to order unity of two separate Kepler orbits with respect to the earth. The determination of the orbit after lunar passage requires the computation of a correction to order μ to the first leg of the trajectory.

Even though the details of lunar passage may be described to lowest order by a Keplerian hyperbola in terms of coordinates and a time scale which are of order μ, this hyperbola *cannot* be defined by only first-order information from the approach orbit. A correction to order μ for the first leg of the earth orbit is needed in order to match it to the hyperbolic moon orbit and hence to determine all the elements of the moon orbit. The determination of the first leg to order μ requires not only use of initial conditions to order μ but also use of equations of motion to order μ. As a result the motion is not Keplerian to this order.

The lowest-order approximation for the second leg of the trajectory relative to the earth is determined if and only if the hyperbolic moon orbit is determined. The limiting value of the velocity along the second asymptote of the moon orbit gives the initial condition at the position of the moon for the second leg of the trajectory relative to the earth.

In the simple example studied here, the first leg of the orbit relative to the earth was a straight line. This happened to coincide with the path traced by a straight-line Kepler orbit even though the time dependence of the coordinates along this path were not Keplerian to order μ. In fact, it was necessary to solve a correction equation to ascertain that the path was a straight line. For different initial conditions for which the first-order orbit would not have been a straight line the correction to order μ would have given a non-Keplerian path.

It is pointed out that the solution for the first correction to the time history along the approach orbit contains a logarithmic singularity at the moon's location. This singularity indicates the fact that an outer solution cannot be valid at a point of non-uniformity, and, in fact, in the uniformly valid composite solution this singularity is eliminated.

The problem of real interest is, of course, the restricted three-body problem. In this case if a straight line is chosen as a first-order orbit, and the initial conditions are adjusted such that the particle would arrive along this straight line to within a distance of order μ to the moon's position, the correction to order μ gives a deviation from the straight line. This is indicated briefly below. A detailed discussion of earth-to-moon trajectories in the restricted three-body problem will be given in Lagerstrom and Kevorkian (1963).

We use an inertial system as described by Eqs. (2.1), (2.2), (2.3), and choose as initial conditions

$$x=0,\quad y=0,\quad dy/dx=0\quad \text{at}\quad t=0 \tag{6.1}$$

for some specified value of energy.

To first order, $y(x)=0$, and $t_0(x)$ is again defined by Eq. (5.2a). To first order the particle will arrive at $x=1$, $y=0$ at the time

$$t=\frac{1}{\sqrt{2}}\left[\frac{1}{\rho^3}\sin^{-1}\rho-\frac{(1-\rho^2)^{1/2}}{\rho^2}\right]. \tag{6.2}$$

We now choose the time parameter ψ in the earth and moon orbits to be the value given by Eq. (6.2) plus a correction term of order μ, say $\mu\psi_1$. This ensures that upon the arrival of the particle at the position $x=1$, $y=0$ according to the first-order solution the moon will be at a distance of order μ from the particle.

The equation for y_1 is then

$$x^2(1-\rho^2x)y_1''-\frac{x}{2}y_1'+\tfrac{1}{2}y_1$$

$$=-\frac{x^3}{2}\sin(t_0-\psi)+\frac{x^3\sin(t_0-\psi)}{2[x^2+1-2x\cos(t_0-\psi)]^{3/2}} \tag{6.3}$$

with the initial conditions $y_1(0)=dy_1(0)/dx=0$. We note first of all that $y_1(x)=0$ is *not* a solution of (6.3). Furthermore since the homogeneous equation corresponding to (6.3) has been solved, and the right-hand side consists of known functions, the general solution can in principle be obtained by quadratures.

The method proposed here will, in general, require numerical integration in order to evaluate the functions occurring in the general solution of (6.3). However, it is believed that a good survey of the various interesting cases may be obtained by relatively simple calculations.

For any specific problem of engineering interest the approximate results will, of course, have to be refined by much more accurate numerical calculations. This task will however be considerably more amenable once an approximate solution, and approximate initial conditions to perform a specific mission have been established by a perturbation theory.

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