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NEW METHODS OF CELESTIAL MECHANICS

VOLUME I. PERIODIC SOLUTIONS, THE NON-EXISTENCE OF INTEGRAL INVARIANTS, ASYMPTOTIC SOLUTIONS

By H. Poincaré

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Tome I. Solutions périodiques, Non-existence des intégrals uniformes, Solutions asymptotiques."
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In the second place, we must always have

$$L^2 \geqq G^2 \geqq \Theta^2.$$

Finally, if $G = \pm \Theta$, the old variables and the position of the body m_1 no longer depend on Θ; and if $L = \pm G$, they no longer depend on g. /22

Special Case of the Problem of Three Bodies

9. Let us return to the special case of the Problem of Three Bodies which we considered above.

Two masses, the first equal to $1-\mu$, the second equal to μ, describe two concentric paths about their common center of gravity assumed fixed. The constant distance of these two masses is taken for the unit of length, in such a way that the radii of the two circumferences become respectively μ and $1-\mu$, the mean motion being equal to unity.

Let us now suppose that in the plane of these two circumferences there is a third moving body, infinitely small, and attracted by the first two.

We will take as origin O the common center of the two circumferences, and
21 we will be able to relate the position of the third mass, either to the two fixed rectangular axes Ox_1 and Ox_2, or to the two moving axes $O\xi$ and $O\eta$ defined as in article 2. The mean motion of the first two masses being equal to 1, we can suppose that the angle of $O\xi$ and Ox_1 (i.e., the longitude of the mass μ) is equal to t.

Since the Gaussian constant is assumed equal to 1, the force function reduces to

$$V = \frac{m_1 \mu}{r_1} + \frac{m_1(1-\mu)}{r_2},$$

calling m_1 the infinitely small mass of the third body, r_1 the distance between the two bodies m_1, μ and r_2 the distance from the body m_1 to the body of mass $1-\mu$, such that

$$r_1^2 = \eta^2 + (\xi + \mu - 1)^2 = [x_2 - (1-\mu)\sin t]^2 + [x_1 - (1-\mu)\cos t]^2,$$
$$r_2^2 = \eta^2 + (\xi + \mu)^2 = [x_2 + \mu \sin t]^2 + [x_1 + \mu \cos t]^2.$$

The vis viva equation is then written $\frac{y_1^2}{2m_1} + \frac{y_2^2}{2m_1} - V = \text{const.}$

We agree to call $-m_1R$ the first member of this equation. R will be a function of x_1, x_2, of y_1, y_2 and of t, and the equations of motion will be written /23

$$\frac{dx_1}{dt} = -\frac{d(m_1 R)}{dy_1}, \qquad \frac{dx_2}{dt} = -\frac{d(m_1 R)}{dy_2},$$
$$\frac{dy_1}{dt} = \frac{d(m_1 R)}{dx_1}, \qquad \frac{dy_2}{dt} = \frac{d(m_1 R)}{dx_2}.$$

Let us replace the variables x_1, y_1, x_2, y_2 by their values as functions of the Keplerian variables L, G, l, g, as has been said in the preceding article. R will become a function of L, G, l, g and t, and the equations of motion will be written

$$\frac{dL}{dt} = \frac{dR}{dl}, \quad \frac{dl}{dt} = -\frac{dR}{dL}, \quad \frac{dG}{dt} = \frac{dR}{dg}, \quad \frac{dg}{dt} = -\frac{dR}{dG}.$$

These equations would already be in the canonical form, if R only depended on the four Keplerian variables, but R is also a function of t; it is therefore necessary to transform these equations, so that time does not enter explicitly. To do so, let us see how R depends on t.

It is easily seen that R can be regarded as a function of L, G, l, and g - t. If, in fact, we increase g and t by the same quantity, without touching the other variables, we change neither ξ nor η, r_1, r_2, $y_1^2+y_2^2$, nor consequently R.

This results in

$$\frac{dR}{dt} + \frac{dR}{dg} = 0.$$

If we then set

$$x'_1 = L, \quad x'_2 = G,$$
$$y'_1 = l, \quad y'_2 = g - t,$$
$$F' = R + G,$$

F' will depend only on x'_1, x'_2, y'_1 and y'_2 and the equations of motion, which will be written

$$\frac{dx'_i}{dt} = \frac{dF'}{dy'_i}, \quad \frac{dy'_i}{dt} = -\frac{dF'}{dx'_i}, \tag{1}$$

will be canonical.

It is in this form that we will ordinarily write the equations of this /24
problem.

When mass μ is assumed to be zero, the mass $1-\mu$ becomes equal to 1 and is related to the origin; r_2 reduces to $\sqrt{x_1^2 + x_2^2}$, the force function V reduces to m_1/r_2, and we find

$$R = \frac{1}{2a} = \frac{1}{2L^2} = \frac{1}{2x_1'^2}$$

and

$$F' = \frac{1}{2x_1'^2} + x'_2.$$

When μ is not zero, we see immediately that F' can develop in terms of the increasing powers of μ, which allows us to write

$$F' = F_0 + \mu F_1 + \dots.$$

We see that

$$F_0 = \frac{1}{2x_1'^2} + x_2'$$

is independent of y_1' and of y_2'.

In addition, F_1 will at the same time depend on the four variables; but this function will be periodic with respect to y_1' and y_2', and it will not change when one of these two variables increases by 2π.

We observe finally that if $x_1' = +x_2'$ the eccentricity is zero and the motion is direct, and that F_1 then depends only on x_1', x_2' and $y_1'+y_2'$.

On the contrary, if $x_1' = -x_2'$, the eccentricity is zero, but the motion is retrograde, and F_1 then depends only on x_1', x_2' and $y_1' - y_2'$.

Use of Keplerian Variables

10. Let x_1, x_2, x_3 be the rectangular coordinates of a point; y_1, y_2, y_3 its velocity components; m its mass. Let V_m be the force function, so that the components of the force applied to the point are

$$m\frac{dV}{dx_1}, \quad m\frac{dV}{dx_2}, \quad m\frac{dV}{dx_3}.$$

If we set /25

$$F = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + V,$$

the equations of motion for the point will take the canonical form

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}.$$

In article 8 we defined a certain function

$$S(x_1, x_2, x_3, G, \theta, L).$$

We have seen that if we make the change of variables defined by the equations

$$\frac{dS}{dx_i} = y_i, \qquad \frac{dS}{dG} = g, \qquad \frac{dS}{d\theta} = \theta, \qquad \frac{dS}{dL} = l,$$

the new variables are nothing other than the Keplerian variables we have just defined.

By virtue of the theorem of article 7, the equations will retain the canonical form and will be written

$$\frac{dL}{dt} = -\frac{dF}{dl}, \qquad \frac{dG}{dt} = -\frac{dF}{dg}, \qquad \frac{d\Theta}{dt} = -\frac{dF}{d\theta},$$
$$\frac{dl}{dt} = \frac{dF}{dL}, \qquad \frac{dg}{dt} = \frac{dF}{dG}, \qquad \frac{d\theta}{dt} = \frac{dF}{d\Theta}.$$

In the special case of article 9, it involves 2 degrees of freedom and can be reduced to the second order.

Reduction of the Problem of Three Bodies

15. It is first a question of effectively making this reduction. /38

Let us first envision the case where the three bodies move in the same plane. We have seen that the number of degrees of freedom could then be reduced to 3. Let us attempt to accomplish this reduction effectively.

We have seen that the equations of motion could be written

$$\frac{dL}{dt} = \frac{dF}{\beta\, dl}, \quad \frac{d\Pi}{dt} = \frac{dF}{\beta\, d\varpi}, \quad \frac{dL'}{dt} = \frac{dF}{\beta'\, dl'}, \quad \frac{d\Pi'}{dt} = \frac{dF}{\beta'\, d\varpi'},$$
$$\frac{dl}{dt} = -\frac{dF}{\beta\, dL}, \quad \frac{d\varpi}{dt} = -\frac{dF}{\beta\, d\Pi}, \quad \frac{dl'}{dt} = -\frac{dF}{\beta'\, dL'}, \quad \frac{d\varpi'}{dt} = -\frac{dF}{\beta'\, d\Pi'}.$$

We also have

$$\frac{dF}{d\varpi} + \frac{dF}{d\varpi'} = 0,$$

whence the area integral

$$\beta\Pi + \beta'\Pi' = C,$$

C being a constant.

Let us set

$$\beta\Pi = H, \quad \beta'\Pi' = C - H, \quad \varpi - \varpi' = h,$$

whence (if we replace Π and Π' by their values as functions of C and H)

$$\frac{dF}{dH} = \frac{dF}{\beta\, d\Pi} - \frac{dF}{\beta'\, d\Pi'}, \quad \frac{dF}{dh} = \frac{dF}{d\varpi} = -\frac{dF}{d\varpi'}, \tag{1}$$

and the equations of motion will become

$$\frac{d(\beta L)}{dt} = \frac{dF}{dl}, \quad \frac{d(\beta' L')}{dt} = \frac{dF}{dl'}, \quad \frac{dH}{dt} = \frac{dF}{dh},$$
$$\frac{dl}{dt} = -\frac{dF}{d(\beta L)}, \quad \frac{dl'}{dt} = -\frac{dF}{d(\beta' L')}, \quad \frac{dh}{dt} = -\frac{dF}{dH}.$$

There are only 3 degrees of freedom.

16. Let us proceed to the general case where the number of degrees of freedom must be reduced to 4. The equations are then written /39

$$\frac{dL}{dt} = \frac{dF}{\beta\, dl}, \quad \frac{dG}{dt} = \frac{dF}{\beta\, dg}, \quad \frac{d\theta}{dt} = \frac{dF}{\beta\, d\theta},$$
$$\frac{dL'}{dt} = \frac{dF}{\beta'\, dl'}, \quad \frac{dG'}{dt} = \frac{dF}{\beta'\, dg'}, \quad \frac{d\theta'}{dt} = \frac{dF}{\beta'\, d\theta'},$$
$$\frac{dl}{dt} = -\frac{dF}{\beta\, dL}, \quad \frac{dg}{dt} = -\frac{dF}{\beta\, dG}, \quad \frac{d\theta}{dt} = -\frac{dF}{\beta\, d\theta},$$
$$\frac{dl'}{dt} = -\frac{dF}{\beta'\, dL'}, \quad \frac{dg'}{dt} = -\frac{dF}{\beta'\, dG'}, \quad \frac{d\theta'}{dt} = -\frac{dF}{\beta'\, d\theta'}.$$

Moreover, we have the three area integrals which, if we take the plane of the maximum of the areas as first coordinate plane, are written

$$\beta\theta + \beta'\theta' = C, \quad \theta = \theta', \quad \beta^2(G^2 - \theta^2) = \beta'^2(G'^2 - \theta'^2).$$

We then have

$$\frac{dF}{d\theta} + \frac{dF}{d\theta'} = 0,$$

which shows that F depends on θ and θ' only by their difference $\theta - \theta'$; however,
/38 as this difference is zero, by virtue of the area integrals, F can be regarded as no longer depending on either θ or θ'.

We find also

$$\theta = \theta',$$

whence

$$\frac{d\theta}{dt} = \frac{d\theta'}{dt},$$

whence

$$\frac{dF}{\beta\, d\theta} = \frac{dF}{\beta'\, d\theta'}. \qquad (2)$$

Let us now set

$$\left.\begin{array}{c} G = \Gamma, \qquad G' = \Gamma', \\ \text{whence} \\ \beta\theta + \beta'\theta' = C, \qquad \beta^2\Gamma^2 - \beta'^2\Gamma'^2 = C(\beta\theta - \beta'\theta') \end{array}\right\} \qquad (3)$$

and

$$\beta\theta = \frac{C}{2} + \frac{\beta^2\Gamma^2}{2C} - \frac{\beta'^2\Gamma'^2}{2C}, \qquad \beta'\theta' = \frac{C}{2} + \frac{\beta'^2\Gamma'^2}{2C} - \frac{\beta^2\Gamma^2}{2C}, \qquad (4)$$

whence /40

$$\frac{dF}{d\Gamma} = \frac{dF}{dG}\frac{dG}{d\Gamma} + \frac{dF}{d\theta}\frac{d\theta}{d\Gamma} + \frac{dF}{d\theta'}\frac{d\theta'}{d\Gamma}$$

or

$$\frac{dF}{d\Gamma} = \frac{dF}{dG} + \frac{dF}{d\theta}\frac{\beta\Gamma}{C} - \frac{dF}{d\theta'}\frac{\beta^2\Gamma}{\beta' C}$$

or finally, by virtue of equation (2),

$$\frac{dF}{d\Gamma} = \frac{dF}{dG}$$

/39 and similarly

$$\frac{dF}{d\Gamma'} = \frac{dF}{dG'}.$$

The area constant C can be regarded as a given quantity in the problem.

If therefore in F we replace G, G', Θ and Θ' by their values (3) and (4), F depends only on L, L', l, l', g, g', Γ and Γ', and the equations of motion can be written

$$\begin{array}{llll} \dfrac{dL}{dt} = \dfrac{dF}{\beta\, dl}, & \dfrac{d\Gamma}{dt} = \dfrac{dF}{\beta\, dg}, & \dfrac{dL'}{dt} = \dfrac{dF}{\beta'\, dl'}, & \dfrac{d\Gamma'}{dt} = \dfrac{dF}{\beta'\, dg'}, \\ \dfrac{dl}{dt} = -\dfrac{dF}{\beta\, dL}, & \dfrac{dg}{dt} = -\dfrac{dF}{\beta\, d\Gamma}, & \dfrac{dl'}{dt} = -\dfrac{dF}{\beta'\, dL'}, & \dfrac{dg'}{dt} = -\dfrac{dF}{\beta'\, d\Gamma'}, \end{array}$$

and there are now only 4 degrees of freedom.

17. It is important to see what form function F assumes when we adopt the variables of the two preceding articles. Let us first suppose that we take the variables of article 15 and that the three bodies move in the same plane; function F, depending only on the distances of the three bodies, will be developable in terms of the cosine and sine of the multiples of $l - l' + h$; the coefficients of this development will themselves be developable in terms of the increasing powers of

$$e\cos l, \quad e\sin l, \quad e'\cos l', \quad e'\sin l',$$

designating the eccentricities by e and e'; finally, the coefficients of these new developments will themselves be uniform functions of L and L'. /41

For brevity, I will set

$$\beta L = \Lambda, \qquad \beta' L' = \Lambda';$$

we will then have, according to the definition of H,

$$e = \frac{1}{\Lambda}\sqrt{\Lambda^2 - H^2}, \qquad e' = \frac{1}{\Lambda'}\sqrt{\Lambda'^2 - (H - C)^2}.$$

Let us add that F does not change when l, l' and h change sign; consequently, if we develop F in terms of the cosines and sines of the multiples of these three variables, the development can contain only cosines.

We will therefore finally have

$$F = \Sigma A(\Lambda^2 - H^2)^{\frac{p}{2}}[\Lambda'^2 - (H - C)^2]^{\frac{q}{2}}\cos(m_1 l + m_2 l' + m_3 h),$$

p and q are positive integers, m_1, m_2 and m_3 arbitrary integers, A is a coefficient which depends only on Λ and on Λ'. What is more, $|m_3 - m_1|$ is at most equal to p and can differ from it only by an even number; similarly, $|m_3 + m_2|$ is at most equal to q and can differ from it only by an even number.

Such a development is valid when $\Lambda - H$ and $\Lambda' - (C-H)$ are sufficiently small; we see that for

$$\Lambda = H$$

all terms vanish, except those for which $m_3 = m_1$.

Similarly, if we have

$$\Lambda' = C - H,$$

all terms vanish except those for which $m_3 = -m_2$.

If, consequently, we have at the same time,

$$\Lambda = H, \qquad \Lambda' = C - H,$$

all terms will vanish except those for which $m_3 = m_1 = -m_2$, such that F becomes a function of $l - l' + h$.

If, in one of the terms of the development of F, we make /42

$$\Lambda = -H, \quad \Lambda' = H - C,$$

this term will still vanish, unless

$$m_3 = m_1 = -m_2.$$

We might be tempted to conclude that, for

$$\Lambda = -H, \quad \Lambda' = H - C,$$

F is still a function of $l - l' + h$; but this is not so, for the development is valid only for small values of $\Lambda - H$ and $\Lambda' - C + H$. An analogous reasoning to
/41 the one which precedes proves, on the contrary, that for $\Lambda = -H$, $\Lambda' = H - C$, F is a function of $l - l' - h$ and not of $l - l' + h$.

In the case where the value of $\Lambda - H$ is extremely small, it can be advantageous to change the special variables.

We have identically

$$\Lambda l + Hh = \Lambda(l + h) - h(\Lambda - H);$$

the canonical form, by virtue of article 5, is not therefore altered when we replace the variables

$$\begin{matrix} \Lambda, & \Lambda', & H, \\ l, & l', & h \end{matrix}$$

by the following

$$\begin{matrix} \Lambda, & \Lambda', & \Lambda - H, \\ l + h, & l', & -h. \end{matrix}$$

Let us now set

$$l + h = \lambda^*, \qquad \sqrt{2(\Lambda - H)}\cos h = \xi^*, \qquad -\sqrt{2(\Lambda - H)}\sin h = \eta^*;$$

by virtue of article 6 the canonical form of the equations remains when we take as variables

$$\begin{matrix} \Lambda, & \Lambda', & \xi^*, \\ \lambda^*, & l', & \eta^*. \end{matrix}$$

It is of advantage that function F, which remains periodic in λ^* and l', is developable in terms of the powers of ξ^* and η^* when these two variables are sufficiently small.

18. Let us now take the variables of article 16, i.e., /43

$$\begin{matrix} \beta L = \Lambda, & \beta' L' = \Lambda', & \beta \Gamma = H, & \beta' \Gamma' = H'. \\ l, & l', & g, & g'. \end{matrix}$$

The variables H and H' are manifestly subject to certain inequalities; we have

$$H = \Lambda\sqrt{1 - e^2},$$

CHAPTER 5. NON-EXISTENCE OF UNIFORM INTEGRALS

81. Let us return to our canonical equations /233

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{1}$$
$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots,$$

I first assume that F_0, which does not depend on y_i, depends on n variables x_i and that its Hessian with respect to these n variables is not zero.

I propose to demonstrate that, except in certain exceptional cases which we will study later, equations (1) admit no other analytic and uniform integral than the integral F=const.

This is what I mean to say:

Let Φ be an analytic and uniform function of values s, y, and μ, which must in addition be periodic with respect to y.

I am not required to assume that this function is analytic and uniform for
/232 all values of s, y and μ.

I assume only that this function is analytic and uniform for all real values of y, for sufficiently small values of μ and for the systems of values of x belonging to a certain domain D; the domain D can in addition be arbitrary and as small as desired. Under these conditions, the function Φ can be developed in powers of μ and I may write

$$\Phi = \Phi_0 + \mu\Phi_1 + \mu^2\Phi^2 + \dots,$$

Φ_0, Φ_1, Φ_2, ... being uniform with respect to x and y and periodic with respect to y.

I say that a function Φ in this form cannot be an integral of equations (1). /234

The necessary and sufficient condition for a function Φ to be an integral is written, resuming the notation of article 3,

$$[F, \Phi] = 0,$$

or, replacing F and Φ by their developments,

$$0 = [F_0, \Phi_0] + \mu([F_1, \Phi_0] + [F_0, \Phi_1]) + \mu^2([F_2, \Phi_0] + [F_1, \Phi_1] + [F_0, \Phi_2]) + \dots.$$

We therefore will have separately the following equations, which I will use later

$$[F_0, \Phi_0] = 0 \tag{2}$$

and

$$[F_1, \Phi_0] + [F_0, \Phi_1] = 0. \tag{3}$$

I say that I may always assume that Φ_0 is not a function of F_0.

In fact, let us assume that we have

$$\Phi_0 = \psi(F_0).$$

I say that function ψ generally will be a uniform function, when the variables x remain in the domain D.

We have in fact

$$F_0 = F_0(x_1, x_2, \ldots, x_n).$$

We will be able to solve this equation with respect to x_1 and write

$$x_1 = \theta(F_0, x_2, \ldots, x_n),$$

and θ will be a uniform function unless $\frac{dF_0}{dx_1}$ vanishes within domain D.

Replacing x_1 by its value θ in

$$\Phi_0 \begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix},$$

it follows that /235

$$\Phi_0 \begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix} = \psi \begin{pmatrix} F_0, x_2, \ldots, x_n \\ y_1, y^2, \ldots, y_n \end{pmatrix}.$$

Φ_0 is a uniform function of x and y; if we here replace x_1 by the uniform function θ, we will obtain a uniform function ψ of F_0, of $x_2, \ldots, x_n$ and of y, but by hypothesis this function $\Phi = \psi$ is a uniform function of F_0.

Therefore, $\Phi = \psi$ is a uniform function of F_0.

This result holds provided $\frac{dF_0}{dx_1}$ does not vanish in domain D; this will hold equally well if any one of the derivatives $\frac{dF_0}{dx_1}$ does not vanish in domain D.

This granted, if Φ is a uniform integral, the same will be true for

$$\Phi - \psi(F).$$

$\Phi - \psi(F)$ can be developed in powers of μ and in addition is divisible by μ, since $\Phi_0 - \psi(F_0)$ is zero. Let us therefore set

$$\Phi - \psi(F) = \mu\Phi' :$$

Φ' will be an analytic and uniform integral and it will follow that

$$\Phi' = \Phi'_0 + \mu\Phi'_1 + \mu^2\Phi'_2 + \dots,$$

In general, Φ'_0 will not be a function of F_0; if this were to happen, we would repeat the previous operation.

I say that in repeating this operation, we will in the end arrive at an integral which will not reduce to a function of F_0 for $\mu=0$.

At least this is true whenever Φ is not a function of F, in which case the two integrals F and Φ would not be distinct.

In fact, let J be the Jacobian or functional determinant of Φ and of F with respect to both variables x and y. I may assume that this Jacobian is not identically zero, because if all Jacobians were zero, Φ would be a function of F, which we do not assume.

J will be manifestly developable in powers of μ. In addition, J will vanish with μ, because Φ_0 is a function of F_0. Therefore J will be divisible by a cer- /236
/235 tain power of μ, for example, by μ^p.

Now let J' be the functional determinant or Jacobian of Φ' and of F; we will have

$$J = \mu J',$$

such that J' will only be divisible by μ^{p-1}.

Thus, after p operations at most, we will arrive at a Jacobian which will no longer vanish with μ and which will consequently correspond to an integral which will not reduce to a function of F_0 for $\mu=0$.

Consequently, if there exists an analytic and uniform integral distinct from F, but such that Φ_0 is a function of F_0, we will always be able to find from it another of the same form and which will not reduce to a function of F_0 for $\mu=0$.

We therefore always have the right to assume that Φ_0 is not a function of F_0.

82. I now say that Φ_0 cannot depend on y.

If, in fact, Φ_0 depends on y, it will be a periodic function of these variables, such that we will be able to write

$$\Phi_0 = \Sigma A e^{\sqrt{-1}(m_1 y_1 + m_2 y_2 + \ldots + m_n y_n)} = \Sigma A \zeta,$$

m_i being positive or negative integers, A being functions of x_i and the notation ζ representing for brevity the imaginary exponential which multiplies A.

This granted, we have

$$[F_0, \Phi_0] = \sum \frac{dF_0}{dx_i} \frac{d\Phi_0}{dy_i},$$

since F_0 does not depend on y and values $\frac{dF_0}{dy_i}$ are zero.

On the other hand,

$$\frac{d\Phi_0}{dy_i} = \Sigma \sqrt{-1}\, m_i A \zeta,$$

so that equation (2) may be written

$$\sqrt{-1}\, \Sigma A \left(m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} + \ldots + m_n \frac{dF_0}{dx_n} \right) \zeta = 0,$$

and, as this must be an identity, for all systems of integral values of m_i we will /237
have

$$A \Sigma m_i \frac{dF_0}{dx_i} = 0,$$

such that we must have identically either

$$A = 0, \tag{4}$$

or

$$\Sigma m_i \frac{dF_0}{dx_i} = 0. \tag{5}$$

From identity (5) we may by differentiation deduce

$$\sum_{i=1}^{i=n} m_i \frac{d^2 F_0}{dx_i dx_k} = 0 \qquad (K = 1, 2, \ldots n).$$

Now this can take place in only two manners:

Either

$$m_1 = m_2 = \ldots = m_n = 0,$$

or the Hessian of F_0 is zero.

Now we have assumed at the beginning that the Hessian was not zero.

Therefore A must be identically zero, except for the term where all m_i are zero.

This is the same as saying that Φ_0 reduces to a single term which does not depend on y.

Q. E. D.

Let us now examine equation (3). As F_0 and Φ_0 do not depend on y, this equation can be written

$$-\sum \frac{d\Phi_0}{dx_i}\frac{dF_1}{dy_i} + \sum \frac{dF_0}{dx_i}\frac{d\Phi_1}{dy_i} = 0.$$

On the other hand, F_1 and Φ_1 are periodic with respect to y and consequently can be developed as exponentials of the form

$$e^{\sqrt{-1}(m_1y_1+m_2y_2+\ldots+m_ny_n)},$$

m_i being positive of negative integers.

For brevity I will, as above, designate this exponential by ζ and I will write /238

'237 $$F_1 = \Sigma B\zeta, \qquad \Phi_1 = \Sigma C\zeta,$$

B and the C being coefficients depending only on x.
We will then have

$$\frac{dF_1}{dy_i} = \sqrt{-1}\,\Sigma B\, m_i\zeta, \qquad \frac{d\Phi_1}{dy_i} = \sqrt{-1}\,\Sigma C m_i\zeta,$$

such that equation (3), divided by $\sqrt{-1}$, will be written

$$-\Sigma B\zeta\left(\Sigma_i m_i \frac{d\Phi_0}{dx_i}\right) + \Sigma C\zeta\left(\Sigma m_i \frac{dF_0}{dx_i}\right) = 0.$$

Since this equation is an identity, for all systems of integral values of m_i we must have

$$B\Sigma m_i \frac{d\Phi_0}{dx_i} = C\Sigma m_i \frac{dF_0}{dx_i}. \qquad (6)$$

Relation (6) must hold for all values of x. Let us then give x values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0: \tag{7}$$

the second member of (6) vanishes. Whenever values x satisfy equation (7), we must therefore have either

$$B = 0 \tag{8}$$

or

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0. \tag{9}$$

Function F_1 is one of the given conditions of the problem and the case consequently will be the same for coefficient B. Therefore it is easy to recognize if equality (7) implies equality (8). In general, we will state that this is not true and we must conclude that equality (9) is a necessary consequence of equality (7).

Now let $p_1, p_2, \ldots, p_n$ be a certain number of integers. Let us consider that we give x values such that /239

$$\frac{dF_0}{p_1\, dx_1} = \frac{dF_0}{p_2\, dx_2} = \ldots = \frac{dF_0}{p_n\, dx_n}. \tag{10}$$

We will be able to find an infinity of systems of integers, $m_1, m_2, \ldots, m_n$, such that

$$m_1 p_1 + m_2 p_2 + \ldots + m_n p_n = 0.$$

For each of these systems of integers, we must have

$$\Sigma m_i \frac{dF_0}{dx_i} = 0$$

and, consequently,

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0.$$

Comparison of these two equations shows that we must have

$$\frac{\frac{dF_0}{dx_1}}{\frac{d\Phi_0}{dx_1}} = \frac{\frac{dF_0}{dx_2}}{\frac{d\Phi_0}{dx_2}} = \ldots = \frac{\frac{dF_0}{dx_n}}{\frac{d\Phi_0}{dx_n}},$$

i.e., that the Jacobian of F_0 and of Φ_0 with respect to two arbitrary components of the vector x must be zero.

This must hold true for all values of x which satisfy relations of form (10), i.e., for all values such that $\frac{dF_0}{dx_i}$ be commensurable among themselves. In an arbitrary domain, however small it may be, there is therefore an infinity of systems of values x for which this Jacobian vanishes, and as this Jacobian is a continuous function, it must vanish identically.

To say that all Jacobians of F_0 and of Φ_0 are zero is therefore to say that Φ_0 is a function of F_0. Now this is contrary to the hypothesis which we have assumed at the end of the preceding article.

We must therefore conclude that equations (1) admit no other uniform integral than F=C.

Q. E. D.

Case Where the B Vanish

83. In the preceding demonstration we assumed that coefficients B were not zero. If one or several of these coefficients vanished (and especially if infinitely many of them vanished), we would have to examine this reasoning. /240

239 To make possible the statement of the consequence to which I will be led, I will be forced to introduce a new terminology.

To each system of indices $m_1, m_2, \ldots, m_n$ (where m_i are integers) there corresponds a coefficient B. I will say that this coefficient is secular when x_i take on values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0. \qquad (7)$$

The following will justify this definition.

When, in the calculation of perturbations, we assume that the ratios of the mean motions are commensurable, some of the terms of the perturbing function cease being periodic, and we can then say that they become secular; what happens here is completely analogous.

I will say that two systems of indices $(m_1, m_2, \ldots, m_n)$ and $(m'_1, m'_2, \ldots, m'_n)$ belong to the same class when we have

$$\frac{m_1}{m'_1} = \frac{m_2}{m'_2} = \ldots = \frac{m_n}{m'_n}$$

and that two coefficients B belong to the same class when they correspond to two systems of indices belonging to the same class.

In order to demonstrate the theorem of the preceding article, we have assumed that none of the coefficients B vanishes in becoming secular.

In order for the result to be true, it is sufficient that in each of the classes there be at least one coefficient B which does not vanish in becoming secular.

Let us assume in fact that the coefficient B which corresponds to the system $(m_1, m_2, \ldots, m_n)$ vanishes, but that the coefficient B' which corresponds to to the system $(m_1', m_2', \ldots, m_n)$ does not vanish. /241

If we give x values such that

$$\Sigma m_i \frac{dF_0}{dx_i} = 0,$$

we will have equally

$$\Sigma m_i' \frac{dF_0}{dx_i} = 0,$$

and consequently

$$B \Sigma m_i \frac{d\Phi_0}{dx_i} = 0, \qquad B' \Sigma m_i' \frac{d\Phi_0}{dx_i} = 0.$$

From the first of these equalities we cannot deduce

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0$$

because B is zero; but, as B' is not zero, the second equality gives us

$$\Sigma m_i' \frac{d\Phi_0}{dx_i} = 0$$

and, consequently,

$$\Sigma m_i \frac{d\Phi_0}{dx_i} = 0.$$

The rest of the reasoning is carried out as in the preceding article.

Before continuing, let us first consider the particular case where there are only two degrees of freedom.

There will then be only two indices m_1 and m_2 and one class will be entirely defined by the ratio of these two indices. Let λ be an arbitrary commensurable number; let C be the class of indices where $\frac{m_1}{m_2} = \lambda$. For brevity I will say that this class C belongs to domain D, or is in this domain if we can give to x_i a system of values belonging to this domain, and such that

$$\lambda \frac{dF_0}{dx_1} + \frac{dF_0}{dx_2} = 0.$$

/241 I will say that a class is singular when all coefficients of this class vanish in becoming secular, and that it is ordinary in the opposite case. /242

I say that the theorem will still be true if we assume that, in any domain δ being part of D we can find an infinity of ordinary classes.

Let there be, in fact, an arbitrary system of values of x_1 and x_2, such that we have at this point

$$\lambda \frac{dF_0}{dx_1} + \frac{dF_0}{dx_2} = 0.$$

Let us assume that λ is commensurable and that the class which corresponds to this value of λ is ordinary; the reasoning of the preceding article can then be applied to this system of values and one must conclude that, for these values of x_1 and of x_2, the Jacobian of F_0 and of Φ_0 with respect to x_1 and to x_2 vanishes.

However, by hypothesis there exists, in any domain δ so small that it is part of D, an infinity of such systems of values of x_1 and of x_2. Consequently our Jacobian must vanish at all points of D which shows that Φ_0 is a function of F_0. From this we would conclude, as in the preceding article, that there exists no uniform integral distinct from F.

The case would not be the same if we could find a domain D of which all the classes are singular.

We could then ask if there does not exist an integral which remains uniform not for all values of x, but only for those values which do not leave domain D. We would see, in general, that while this would not be true; it would be sufficient, in order to be certain of it, to consider in the equation

$$[F, \Phi] = 0,$$

not any longer only the term independent of μ, and the terms in μ, but the term in μ^2 and the following terms.

I do not insist, this has no interest, for I do not believe that in any problem of Dynamics occurring naturally it happens that all classes of a domain D are singular without all coefficients B vanishing in becoming secular.

Let us now proceed to the case where there are more than 2 degrees of freedom.

The results will be analogous, although their statement will be more complicated. /243

Let

$$p_1, \ p_2, \ \ldots, \ p_n$$

be n arbitrary integral numbers. Let us consider all systems of indices m_1, m_2, ..., m_n which satisfy the condition

$$m_1 p_1 + m_2 p_2 + \ldots + m_n p_n = 0.$$

I will say that all corresponding coefficients belong to the same family.

Let there be q classes defined by the following systems of indices

$$\begin{matrix} m_{1,1}, & m_{2,1}, & \ldots, & m_{n,1} \\ m_{1,2}, & m_{2,2}, & \ldots, & m_{n,2} \\ \ldots, & \ldots, & \ldots, & \ldots \\ m_{1,q}, & m_{2,q}, & \ldots, & m_{n,q}. \end{matrix}$$

If one cannot find q integers

$$a_1, \ a_2, \ \ldots, \ a_q,$$

such that one has

$$\sum_{i=1}^{i=q} a_i m_{k,i} = 0 \qquad (k = 1, 2, \ldots, n),$$

I will say that these 2 classes are independent.

I will say that a family is ordinary, if we can find in it n-1 independent and ordinary classes, and that it is singular in the opposite case. It will be singular of the first order, if we can find in it n-2 independent classes, ordinary and singular classes of the q-th order, if we can find in it n-q -1 independent and ordinary classes and no more.

I will say that a family defined by the integers (p_1, p_2, ..., p_n) belongs to a domain D, if there exist in this domain values of x such that

$$\frac{dF_0}{p_1 dx_1} = \frac{dF_0}{p_2 dx_2} = \ldots = \frac{dF_0}{p_n dx_n}.$$

This granted, I say that if we can find in all domains δ which are part of D an infinity of ordinary families, there can exist no uniform distinct integral of F. /244

The reasoning of the preceding article is, in fact, applicable to all systems of values of x which correspond to an ordinary family.

43 The Jacobians of F_0 and Φ_0, with respect to two arbitraries of variables x, must therefore vanish an infinity of times in all domains δ that are part of D, which can occur only if they are identically zero.

I now say that if we can find in any domain δ which is part of D an infinity of singular classes of the q-th order, the number of distinct uniform integrals which equations (1) can have is at most equal to q+1 (including the integral F).

Let us, in fact, assume that there are q+2 distinct integrals; let

$$F, \quad \Phi^1, \quad \Phi^2, \quad \ldots, \quad \Phi^{q+1}$$

be these integrals and let us assume that for μ=0 they reduce to

$$F_0, \quad \Phi_0^1, \quad \Phi_0^2, \quad \ldots, \quad \Phi_0^{q+1}. \qquad (11)$$

Let there be a system of values of x corresponding to an irregular family of the q-th order. Let us set

$$n - q - 1 = p.$$

There will exist in this family p ordinary classes. Let

$$m_{1,k}, m_{2,k}, \ldots, m_{n,k} \qquad (k = 1, 2, \ldots, p)$$

be the systems of indices corresponding to these classes.

We will have for the values of x under consideration

$$\sum_{i=n}^{i=1} m_{i,k} \frac{dF_0}{dx_i} = \sum_{i=n}^{i=1} m_{i,k} \frac{d\Phi_0^h}{dx_i} = 0$$

$$(k = 1, 2, \ldots p, \; h = 1, 2, \ldots, q+1).$$

We will deduce from this that the Jacobians of the q+2 functions (11) with respect to q+2 arbitrary coordinates of x must vanish for the considered values of x.

And since this must take place an infinity of times in each domain δ, we will /245 conclude from it that these Jacobians vanish identically and consequently that our q+2 integrals cannot be distinct.

44 These considerations present no additional practical interest, and I have presented them here only to be complete and rigorous. We can obviously construct problems artificially where these various circumstances are encountered; but in problems of Dynamics occurring naturally it will always occur either that all classes will be singular, or that they all will be ordinary, with the exception of a finite number of them.

Case Where the Hessian is Zero

84. Let us now proceed to the case where F_0 does not depend on all variables $x_1, x_2, \ldots, x_n$.

I will assume that F_0 depends on x_1 and x_2 only, and that its Hessian with respect to these two variables is not zero.

In order to note well the difference between these two variables x_1 and x_2 and their conjugates y_1 and y_2 on one hand, and the other variables x and y on the other hand, I will agree to designate

$$x_3, \quad x_4, \quad \ldots, \quad x_n,$$
$$y_3, \quad y_4, \quad \ldots, \quad y_n$$

by the notation

$$z_1, \quad z_2, \quad \ldots, \quad z_{n-2},$$
$$u_1, \quad u_2, \quad \ldots, \quad u_{n-2}.$$

We will first observe that the conclusions of article 81 obtain and, if there exists a uniform integral Φ distinct from F, it is always permissible to assume that Φ_0 is not a function of F_0.

This granted, we must first have

$$[F_0, \Phi_0] = \frac{dF_0}{dx_1}\frac{d\Phi_0}{dy_1} + \frac{dF_0}{dx_2}\frac{d\Phi_0}{dy_2} = 0.$$

Let us set

$$\zeta = e^{\sqrt{-1}(m_1 y_1 + m_2 y_2)},$$

we can write /246

$$\Phi_0 = \Sigma A \zeta$$

values A being coefficients depending on x_1, x_2, z and μ. It then follows that

$$\sqrt{-1}\,\Sigma A \zeta \left(m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2}\right) = 0.$$

This relation must be an identity and, on the other hand, the Hessian of F_0 being not zero, we cannot have identically

$$m_1 \frac{dF_0}{dx_1} + m_2 \frac{dF_0}{dx_2} = 0,$$

unless m_1 and m_2 are both zero.

From this we would conclude, as in article 82, that Φ_0 depends neither on y_1 nor on y_2.

If we then write equation (3), we will have

$$-\frac{d\Phi_0}{dx_1}\frac{dF_1}{dy_1}-\frac{d\Phi_0}{dx_2}\frac{dF_1}{dy_2}+\frac{dF_0}{dx_1}\frac{d\Phi_1}{dy_1}$$
$$+\frac{dF_0}{dx_2}\frac{d\Phi_1}{dy_2}+\sum\left(\frac{dF_1}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dF_1}{du_i}\frac{d\Phi_0}{dz_i}\right)=0.$$

Let us also set

$$F_1=\Sigma B\zeta, \qquad \Phi_1=\Sigma C\zeta.$$

When it is necessary to indicate the indices, I will write

$$F_1=\Sigma B_{m_1,m_2}e^{\sqrt{-1}(m_1y_1+m_2y_2)}.$$

It will follow that

$$-\Sigma B\zeta\left(\Sigma m_i\frac{d\Phi_0}{dx_i}\right)+\Sigma C\zeta\left(\Sigma m_i\frac{dF_0}{dx_i}\right)+\Sigma\zeta\sum\left(\frac{dB}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dB}{du_i}\frac{d\Phi_0}{dz_i}\right)=0.$$

This relation must be an identity; we can therefore equate to 0 the coefficient of an arbitrary one of exponentials ξ. We will in addition give x values such that

$$m_1\frac{dF_0}{dx_1}+m_2\frac{dF_0}{dx_2}=0, \tag{12}$$

so as to make the terms which depend on C vanish.

It will follow that /247

$$-B\left(m_1\frac{d\Phi_0}{dx_1}+m_2\frac{d\Phi_0}{dx_2}\right)+\sum\left(\frac{dB}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dB}{du_i}\frac{d\Phi_0}{dz_i}\right)=0. \tag{13}$$

46 We will consider two coefficients $B_{m_1,\, m_2}$, $B_{m_1',\, m_2'}$ as belonging to the same class such that

$$m_1m_2'-m_2m_1'=0,$$

and for brevity I will say that the coefficient $B_{m_1,\, m_2}$ belongs to the class $\frac{m_1}{m_2}$.

It follows from this definition that the coefficient $B_{0,0}$ belongs to all classes at the same time.

According to the preceding, if we give x values which satisfy relation (12), relation (13) must hold for the coefficients of B of class $\frac{m_1}{m_2}$.

Then let p and q be the two first integers among those, such that

$$\frac{m_1}{m_2} = \frac{p}{q}.$$

Let us set

$$\zeta = e^{\sqrt{-1}(py_1+qy_2)}$$

and

$$D_\lambda = B_{\lambda p, \lambda q}\zeta^\lambda, \qquad -\zeta H = p\frac{d\Phi_0}{dx_1} + q\frac{d\Phi_0}{dx_2}.$$

If we give x values such that

$$p\frac{dF_0}{dx_1} + q\frac{dF_0}{dx_2} = 0, \tag{12a}$$

we must have

$$H\frac{dD_\lambda}{d\zeta} + \sum\left(\frac{dD_\lambda}{dz_i}\frac{d\Phi_0}{du_i} - \frac{dD_\lambda}{du_i}\frac{d\Phi_0}{dz_i}\right) = 0, \tag{13a}$$

and this for all integral values of λ, positive, negative or zero.

This can only take place in two ways:

(1) Either we have

$$H = 0, \qquad \frac{d\Phi_0}{dz_i} = 0, \qquad \frac{d\Phi_0}{du_i} = 0 \qquad (i = 1, 2, \ldots, n-2),$$

from which /248

$$\frac{dF_0}{dx_1}\frac{d\Phi_0}{dx_2} - \frac{dF_0}{dx_2}\frac{d\Phi_0}{dx_1} = 0.$$

We would deduce from this, by reasoning quite similar to that of article 82, that Φ_0 is a function of F_0, which is contrary to the hypothesis made at the beginning.

(2) Or, on the other hand, if the Jacobian of 2n-3 arbitrary choices of function $D\lambda$ with respect to the 2n-3 variables ξ; z_i and u_i is zero.

From this we would conclude that, if we give x_1 and x_2 constant values satisfying condition (12a), a relation will result between 2n-3 arbitrary choices of functions $D\lambda$, such that all these functions can be expressed by means of 2n-4 from among them.

We can state this result in still another way.

Let us consider the following expressions

$$B^{\lambda'}_{\lambda p, \lambda q} B^{-\lambda}_{\lambda' p, \lambda' q}. \tag{14}$$

If we assume that we give x_1 and x_2 constant values satisfying equation (12a), these expressions (14) depend on 2n-4 variables only, namely z_i and u_i.

If there exists a uniform integral, all these expressions are functions of 2n-5 from among them; or, in other words, we can find a relation among an arbitrary choice 2n-4 from among them.

What is the condition for which there exist three uniform distinct integrals

$$F = \text{const.}, \quad \Phi = \text{const.}, \quad \psi = \text{const.?}$$

Let F_0, Φ_0 and Ψ_0 be the values of these three integrals for μ=0. We could demonstrate, as above, that we can always assume that there is no relation whatever between F_0, Φ_0 and Ψ_0.

We would then find, by setting

$$-H'\zeta = p\frac{d\psi_0}{dx_1} + q\frac{d\psi_0}{dx_2},$$

that we have

$$H'\frac{dD_\lambda}{d\zeta} + \sum\left(\frac{dD_\lambda}{dz_i}\frac{d\psi_0}{du_i} - \frac{dD_\lambda}{du_i}\frac{d\psi_0}{dz_i}\right) = 0. \tag{13b}$$

48 Thus equation (12a) implies as a necessary consequence, not only equation (13a) but equation (13b). By reasoning quite similar to the preceding, we would see that this can occur in only two ways: /249

Either there is a relation between F_0, Φ_0 and Ψ_0, which is contrary to the hypothesis which we have just made;

Or, if the Jacobian of an arbitrary choice 2n-3 of functions $D\lambda$ is zero as well as all its minors of the first order.

From this it would result that, if x_1 and x_2 satisfy condition (12a), there is among an arbitrary choice 2n-3 of $D\lambda$ not one, but two relations.

In other words, expressions (14) can be calculated by means of 2n-3 from among them.

Expressions (14)which depend on the coefficients of the development of the function F_1 are given quantities of the problem, and we will always be able to verify if there are one or two relations among 2n-4 of these expressions.

Generally, we will discover that there is but one, and from this we will conclude that there exists no analytic and uniform integral other than F.

What will happen, however, if this is not so? To be able to state the result in a complete and rigorous manner, I am going to make use of a terminology analogous to that of the preceding article. I will say that a class is ordinary, if there is no relation between 2n-4 of expressions (14) formed with the coefficients of this class, that it is singular of the first order, if there is one, singular of the second order, if there are two, etc. More generally, a class will be singular of order q if there are q relations among an arbitrary choice 2n-3 of quantities $D\lambda$.

Let δ be an arbitrary domain including an infinity of systems of values of x_1, x_2 of z and of u.

If we can find in the domain δ values of x_1 and x_2 satisfying condition (12a), I will say that the class $\frac{p}{q}$ belongs to this domain. I have said of the values of x_1 and of x_2 and not of the values of x_1, x_2, z and of u, because the first member of (12a) depends only on x_1 and x_2.

I will then be able to state the following result:

I will designate by D a domain including an infinity of values of x_1, x_2 of z and u. /250

If in every domain δ that is part of D, we can find an infinity of ordinary classes, we will be certain that there does not exist outside of F any other integral which is analytic and uniform with respect to x, to y, to z and to u, and in addition periodic with respect to y_1 and to y_2 and which remains such for all real values of y_1 and of y_2, for sufficiently small values of μ, and for the values of x_1, x_2 of z and u which belong to the domain D.

If in all domains δ that are part of D, we can find an infinity of singular classes of the q-th order, it will not be possible for them to exist more than q+1 distinct uniform integrals, including F.

Application to the Problem of Three Bodies

85. I shall now concern myself with applying the preceding ideas to the various cases of the Problem of Three Bodies.

Let us begin by the particular case defined in article 9. In this case, we have 2 degrees of freedom only and four variables

$$x_1 = L, \qquad x_2 = G,$$
$$y_1 = l, \qquad y_2 = g - t$$

(cf. article 9); we have, in addition,

$$F_0 = \frac{1}{2x_1^2} + x_2.$$

The Hessian of F_0 is zero, but we can, by the artifice of article 43, reduce the problem to the case where the Hessian is not zero.

If, therefore, a uniform integral were to exist, it would be necessary that in the development of F_1 (which is the perturbing function of the astronomers), in terms of the sines and cosines of the multiples of y_1 and y_2, all coefficients vanish at the moment when they become secular.

Examination of the well-known development of the perturbing function shows that this is not the case.

We must therefore conclude that in the particular case of the Problem of Three Bodies there is no uniform integral distinct from F. /251

In my memoir in Acta mathematica (Vol. XIII), in order to establish the same point I made use of the existence of periodic solutions and of the fact that the characteristic exponents are not zero. The demonstration which I give here differs
250 from that of Acta only in form, but it lends itself better to the generalization which will follow.

Let us now consider a somewhat more general case of the Problem of Three Bodies, that where motion occurs in a plane, and let us assume that we have reduced the number of degrees of freedom to 3, as we have said in article 15.

We then have six conjugate variables, namely

$$\beta L, \quad \beta' L', \quad \beta\Pi = H,$$
$$l, \quad l', \quad h = \varpi - \varpi'.$$

Let us assume that we develop the perturbing function F_1 in the following manner

$$F_1 = \Sigma B_{m_1, m_2} e^{\sqrt{-1}(m_1 l + m_2 l')},$$

the coefficients B_{m_1, m_2} will be functions of βL, $\beta' L'$, H and h.

Let p and q be two arbitrary integers first among them; let us form the expressions

$$B^{\lambda'}_{\lambda p, \lambda q} B^{-\lambda}_{\lambda' p, \lambda' q} \qquad (\lambda, \lambda' = 0, \pm 1, \pm 2, \ldots, \text{ad inf.}). \tag{14}$$

Let us give L and L' values satisfying condition (12a), i.e., such that the relationship of the mean motions is equal to $-\frac{q}{p}$.

In order for the problem to admit a uniform integral other than the vis viva integral, it would be necessary for there to be a relationship between two arbitrarily chosen from among them (n=3, 2n-4=2), i.e., that all these expressions (14)be functions of $B_{0,0}$, i.e., of the secular part of the perturbing function. Now examination of the well-known development of this function shows that this is not the case.

We must conclude that, outside of the vis viva integral, the problem admits no uniform integral of the following form /252

$$\Phi(L, L', H, l, l', h) = \text{const.}$$

periodic in l and l'.

But this is not sufficient for us; we must still demonstrate that the problem admits no integral of the following form

$$\Phi(L, L', \Pi, \Pi', l, l', \varpi, \varpi') = \text{const.},$$

where the function Φ depends in an arbitrary manner on ϖ and on ϖ' instead of depending on the difference $\varpi-\varpi'$.

To do so we must take the problem with 4 degrees of freedom, as we did in article 16.

We will then have eight conjugate variables

$$\begin{matrix} \beta L, & \beta' L', & \beta\Pi, & \beta'\Pi', \\ l, & l', & \varpi, & \varpi'. \end{matrix}$$

The coefficients B_{m_1,m_2} and expressions (14) then depend on L, L', Π, Π', and ϖ'. When we have given L and L' constant values such that the relation of mean motions is equal to $\frac{q}{p}$, expressions (14) will only depend on the four variables Π, Π', ϖ and ϖ'.

In order for there to be a uniform integral other than that of the vis viva, it is necessary that we have a relation among four arbitrarily chosen (2n-4=4, n=4)of the expressions (14); this is what occurs since all these expressions are only functions of three variables Π, Π' and $\varpi-\varpi'$.

Therefore nothing opposes the fact that there exists an integral other than that of the vis viva, and in fact there does exist one, namely the areal integral.

In order for there to be two integrals, it would be necessary that there be a relationship among an arbitrarily chosen three of these expressions, that is to say, that all these expressions depend only on two of them. This is not the case.

Therefore, outside the vis viva integral and that of the area, the problem admits no other uniform integral.

Let us at last proceed to the most general case of the Problem of Three Bodies, and let us set the problem as in article 11, i.e., with 6 degrees of freedom and with the twelve variables: /253

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$$\begin{matrix} \beta L, & \beta G, & \beta\theta, & \beta' L', & \beta' G', & \beta'\theta'. \\ l, & g, & \theta, & l', & g', & \theta'. \end{matrix}$$

Expressions (14), after we have given L and L' proper constant values chosen as above, still depend on the eight variables G, G', Θ, Θ', g, g', θ, θ'.

In order for these to be q uniform variables distinct from F, it would be necessary for there to be a relationship among 2n-3-2=9-q arbitrarily chosen of expressions (14).

It is easy to verify that these expressions depend only on five variables, namely on

$$G, \; G', \; g, \; g'$$

and on the angle of the planes of the two osculatory orbits.

There is therefore a relationship between an arbitrary 6=9-3 of the expressions (14).

Nothing is therefore in oppostion to the existence of three new integrals and they exist effectively: they are the integrals of area. But there is no relationship between an arbitrary (5=9-4) of expressions (14).

Therefore, the Problem of Three Bodies admits no other uniform integral than those of the vis viva and of area.

I have limited myself, in order not to interrupt the reasoning, to affirming that there exist no relationships among expressions (14); I will later return to this question.

It is known that Bruns has demonstrated (Acta mathematica, Vol. II) that the Problem of Three Bodies admits no new algebraic integral beyond the integrals already known.

The preceding theorem is more general in a sense than that of Bruns, because I demonstrate not only that there exists no algebraic integral, but that there exists not even a transcendental uniform integral, and not only that an integral cannot be uniform for all values of the variables, but that it cannot remain uniform even in a restrained domain defined above.

However, in another sense the theorem of Bruns is more general than mine; /254
I establish only, in effect, that there can exist no algebraic integral for all rather small values of the masses, and Bruns demonstrates that there exist none for any system of values of the masses.

Problems of Dynamics Where There Exists A Uniform Integral

86. There are problems where we know the existence of a uniform integral and where we can propose to verify that the conditions stated in the preceding articles are effectively fulfilled.

Let us take as an example the problem of the motion of a moving point M, attracted by two fixed centers A and B.

I will assume, for simplification, that the motion occurs in a plane; I will assume in addition that the mass A is large, while that of B is equal to a very small quantity μ, in such a manner that one may regard the attraction of B as a perturbing force.

We will then define the location of the point M by the osculating elements of its orbit about A, and we will designate these elements as the letters L, Π, l, ϖ, as in article 10. We will then have

$$F = \frac{1}{2L^2} + \frac{\mu}{MB}, \text{ whence } F_0 = \frac{1}{2L^2}, \quad F_1 = \frac{1}{MB};$$

F_1 can be developed in the following form

$$F_1 = \Sigma B_m e^{\sqrt{-1}ml}.$$

The coefficients B_m then depend on L, Π, and ϖ, and in order for an integral to exist, it is necessary that there be a relationship between three arbitrary quantities of the coefficients of the same class (n-2, 2n-2=2; I say 2n-2 instead of 2n-4 because F_0 depends no longer on two variables x_1 and x_2, as in articles 84 and 85, but on one variable only), when we give L a value satisfying relation (12a).

However, here all coefficients B_m (which have only one index) belong to the same class and one relation (12a) is written simply m $(dF_0/dL)=0$ where $L=\infty$. There could therefore be difficulty only for infinite values of L. If, therefore, we again take up the abbreviated language of the preceding articles, of L, Π, and ω, but such that, for all these systems, the value of L is finite, the class of which all these coefficients B are part will not belong to the domain D; therefore nothing will oppose the existence of an integral which remains uniform in this domain D.

Let us proceed to another problem; that of the motion of a heavy body about a fixed point.

This problem has been integrated in three different particular cases by
/254 Euler, by Lagrange and by Mme. de Kowalevski (cf. Acta mathematica, 12). I believe that Mme. de Kowalevski has discovered other new cases of integrability.

We can therefore ask if, in this problem, the considerations presented in this chapter oppose the existence of a uniform integral other than those of the vis viva and of area.

I will assume that the product of the weight of the body by the distance of the center of gravity to the point of suspension is very small, such that we may write the equations of the problem in the form

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

$$F = F_0 + \mu F_1.$$

Values x_i and y_i form three pairs of conjugate variables; F designates the total energy of the system; F_0 is its semi-vis viva; μ is a very small quantity and μG_1 represents the product of the weight of the body by the distance from the center of gravity to a horizontal plane passing through the point of suspension.

In the case where μ is zero (i.e., where the center of gravity coincides with the point of suspension), the motion of the solid body reduces to a Poinsot
motion. Since we assume μ very small, it is this Poinsot motion which will /256
serve us as first approximation, in the manner of Keplerian motion in the study of the Problem of Three Bodies by successive approximations.

I must, before continuing, define two quantities n and n', which I will call the two mean motions and which will play an important role in what follows. In the Poinsot motion, the ellipsoid of inertia rolls on a fixed plane; let P be the foot of the perpendicular lowered from the point of suspension onto this fixed plane and Q the point of contact. This point of contact belongs to a curve fixed with respect to the ellipsoid and called the polhody. At the end of a certain time T, the same point of the polhody returns to Q' in contact with the fixed plane. Let α be the angle QPQ'. We will set

$$n = \frac{2\pi}{T}, \quad n' = \frac{\alpha}{T}$$

and n and n' will be the two mean motions.

This granted, we will be able to write the equations of Poinsot motion in the following manner:

Let x, y and z be the coordinates of an arbitrary point of the solid body by taking the origin of the coordinates at the point of suspension and the axis of the z vertical.

Let us set

$$l = nt + \varepsilon, \qquad l' = n't + \varepsilon',$$

ϵ and ϵ' being two constants of integration.

Let ξ, η and ψ be three functions of n, n' and l, periodic of period 2π in l (these functions, as is known, depend on the elliptic functions); let θ and φ be two new constants of integration; we will have

$$\begin{aligned} x &= \cos\theta(\xi\cos l' - \eta\sin l') - \sin\theta\cos\varphi(\xi\sin l' + \eta\cos l') + \psi\sin\theta\sin\varphi, \\ y &= \sin\theta(\xi\cos l' - \eta\sin l') + \cos\theta\cos\varphi(\xi\sin l' + \eta\cos l') - \psi\cos\theta\sin\varphi, \\ z &= \sin\varphi(\xi\sin l' + \eta\cos l') + \psi\cdot\cos\varphi. \end{aligned}$$

If we assume that the point (x, y, z) is the center of gravity of the solid body, F_1 reduces within a constant factor to z, so that we will be able to write

$$F_1 = \Sigma B_{m,1} e^{\sqrt{-1}(ml+l')} + \Sigma B_{m,0} e^{\sqrt{-1}(ml)} + \Sigma B_{m,-1} e^{\sqrt{-1}(ml-l')},$$

the coefficients B depending only on n, n' and φ.

When we give n and n' constant values satisfying condition (12a), B will only depend on φ such that there will be a relationship between two arbitrarily chosen from among them.

Values D depend only on φ and ζ in setting, as in the preceding articles,

$$D_\lambda = B_{\lambda p.\lambda q}\zeta^\lambda.$$

There will therefore be a relationship between an arbitrary (2n-3=3) of the $D\lambda$. Every class will therefore be singular of the first order.

Nothing opposes the existence of a uniform integral distinct from that of the vis viva, and we know, in fact, that there exists one, namely that of area.

But the question is to learn whether a third can exist.

For this purpose, let us seek to learn the classes which are singular of the second order. To do this. it is necessary and sufficient that there be among three arbitrarily chosen of D_λ two relationships, and consequently that all D_λ be functions of only one of them. We will thus be led to distinguish several types of classes:

(1) The class $\frac{1}{0}$ which contains all coefficients $B_{m.0}$. This one is singular of the second order. We have in fact,

$$B_{m.0} = C_{m.0}\cos\varphi,$$

$C_{m.0}$ depending only on n and n' and consequently having to be regarded as constant, since we assumed that we gave n and n' constant values. Then we have

$$D_\lambda = C_{\lambda.0} \cos \varphi \zeta^\lambda.$$

In order for $D\lambda$ to be functions of only one of them, all $C_{\lambda.0}$ must vanish with one exception, or the function ψ must reduce to an exponential

$$e^{\sqrt{-1}m.l}.$$

However, in order to satisfy condition (12a), it is necessary to give n the value 0; what is therefore the Poinsot motion for which n=0? A bit of attention shows that it is the one which corresponds to the uniform rotation about one of the axes of inertia. In a similar motion, the function ψ is a constant independent of l. This proves that all $C_{\lambda.0}$ are zero for these particular values of n and of n', /258 with the exception of $C_{0.0}$.

The class is therefore singular of the second order.

(2) The classes of the form $\frac{m}{1}$ which contain only three coefficients

$$B_{m.1}, \quad B_{0.0}, \quad B_{-m.-1}.$$

These classes can be singular of the second order only if

$$B_{m.1} = B_{-m.-1} = 0$$

or, what comes back to the same thing, if in the development of $\xi+i\eta$ and of $\xi-i\eta$ in positive and negative powers of e^{il}, there are no terms in $e^{\pm mil}$ (assuming ξ and η real).

This will not happen, in general, when the ellipsoid of inertia is not one of revolution; but, if this ellipsoid is one of revolution, we will have

$$\xi = A\cos l + B\sin l + C, \qquad \eta = A'\cos l + B'\sin l + C',$$

A, B, C, A', B',C' being constants. The result of this will be that we have

$$B_{m.1} = -B_{-m.-1} = 0,$$

unless m=1, 0 or -1.

All classes $\frac{m}{1}$ will then be singular of the second order, with the exception of classes $\frac{1}{1}$, $\frac{0}{1}$ and $\frac{-1}{1}$.

(3) All other classes reducing to the single coefficient $B_{0.0}$ will be singular of the second order.

In summary, if the ellipsoid is one of revolution, all classes are singular of the second order, with the exception of classes $\frac{1}{1}$, $\frac{0}{1}$ and $\frac{-1}{1}$.

Therefore nothing opposes the existence of a third uniform integral and even that it be algebraic, provided that the Jacobian of the three integrals vanishes when we make $n'=0$ or $n'=\pm n$. (This last condition is not necessary in the case of Lagrange, that is, if the point of suspension is on the axis of revolution, because then ξ and η reduce to constants.)

If, on the contrary, the ellipsoid is not one of revolution, there is an infinity of classes which are not singular of the second order, namely classes $\frac{m}{1}$; but let us consider a domain D containing an infinity of systems of values of n, n', φ and θ and let us assume that for none of these systems is n' a multiple of n; none of the classes $\frac{m}{1}$ will belong to this domain. Therefore still nothing will oppose the existence of a third uniform integral, provided that the Jacobian of the three integrals vanishes when n' is a multiple of n; here the result is that this third integral cannot, in general, be algebraic. /259

The conditions stated in this chapter being necessary, but not sufficient, nothing proves that this third integral exists; it is advisable, before making a statement, to await the complete publication of Mme. de Kowalevski's results.[1]

Nonholomorphic Integrals in μ

87. Until now we have assumed that our uniform integral Φ was developable in integral powers of μ. It is easy to extend the result to the case where we would abandon this hypothesis. Let us assume, for example, that Φ can be developed in integral powers of $\sqrt{\mu}$; we will be able to write

$$\Phi = \Phi' + \sqrt{\mu}\,\Phi'',$$

Φ' and Φ'' being developable in integral powers of μ.

If Φ is an integral, we must have identically

$$[F, \Phi] = [F, \Phi'] + \sqrt{\mu}[F, \Phi''] = 0.$$

Since (F, Φ') and (F, Φ'') can be developed in integral powers of μ, we must have separately

$$[F, \Phi'] = [F, \Phi''] = 0.$$

Therefore Φ' and Φ'' must both be integrals. /26

[1] Since these lines were written, the scientific world has had to mourn the premature death of Mme. de Kowalevski. Those of her notes which were found are unfortunately insufficient to permit reconstructing her demonstrations and calculations.

If, therefore, we have demonstrated that a uniform integral developable in integral powers of μ cannot exist we will have demonstrated that a uniform integral developable in integral powers of $\sqrt{\mu}$ cannot exist either.

More generally, let

$$\theta_1(\mu),\ \theta_2(\mu),\ \ldots,\ \theta_p(\mu) \tag{1}$$

/259 be p arbitrary functions of μ.

Let us assume that Φ is of the form

$$\left.\begin{aligned}\Phi = {} & A_{0,1}\theta_1(\mu) + A_{1,1}\mu\theta_1(\mu) + A_{2,1}\mu^2\theta_1(\mu) + \ldots \\ & + A_{0,2}\theta_2(\mu) + A_{1,2}\mu\theta_2(\mu) + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + A_{0,p}\theta_p(\mu) + A_{1,p}\mu\theta_p(\mu) + \ldots,\end{aligned}\right\} \tag{2}$$

values A being functions of x and y independent of μ.

We can always assume that among the p functions (1) there are no relationships of the form

$$\varphi_1\theta_1 + \varphi_2\theta_2 + \ldots + \varphi_p\theta_p = 0, \tag{3}$$

$\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_p$ being developable in powers of μ. If this were in fact so, one of the functions $\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_p$ will not contain μ as a factor; for, if all these functions contained μ as a factor, the first member of (3) would be divisible by μ and we would make the division.

Let us assume, for example, that φ_1 does not vanish with μ; we will be able to solve equation (3) with respect to θ_1 and we will have

$$\theta_1 = -\frac{\varphi_2}{\varphi_1}\theta_2 - \frac{\varphi_3}{\varphi_1}\theta_3 - \ldots - \frac{\varphi_p}{\varphi_1}\theta_p.$$

$\frac{\varphi_2}{\varphi_1}, \frac{\varphi_3}{\varphi_1}, \ldots$ will be developable in powers of μ, and if we replace θ_1 by this value in expression (2), we will have reduced by unity the number of functions (1).

/260 Let us therefore assume that these functions are not connected by a relation of form (3).

We will be able to write

$$\Phi = \Phi_1\theta_1 + \Phi_2\theta_2 + \ldots + \Phi_p\theta_p,$$

$\Phi_1, \Phi_2, \ldots, \Phi_p$ being developable in powers of μ. If Φ is an integral, we will /261
have

$$[F, \Phi] = \theta_1[F, \Phi_1] + \theta_2[F, \Phi_2] + \ldots + \theta_p[F, \Phi_p] = 0. \quad (4)$$

I say that we will have separately

$$[F, \Phi_1] = [F, \Phi_2] = \ldots = [F, \Phi_p] = 0. \quad (5)$$

For, if this were not so, as quantities (F, Φ_i) $(i=1,2, \ldots, p)$ are developable in powers of μ, relation (4) would be of form (3), which is contrary to the hypothesis we have just made.

Therefore relations (5) hold true.

Therefore $\Phi_1, \Phi_2, \ldots, \Phi_p$ are integrals.

If, therefore, we have demonstrated that there cannot be a uniform integral developable in powers of μ, we will have demonstrated that neither is there a uniform integral of form (2).

I will add that this reasoning applies when functions (1) are finite in number.

Discussion of Expressions (14)

88. I return to the subject which I had reserved above, namely the demonstration of the fact that there exists no relationship between an arbitrary 2n-4 expressions of (14) in the case of the Problem of Three Bodies.

In order to define expressions (14), we have assumed that the perturbative function F_1 had been developed in the following form

$$F_1 = \Sigma B_{m_1 m_2} e^{\sqrt{-1}(m_1 l + m_2 l')}, \quad (1)$$

the coefficients $B_{m_1 m_2}$ being functions of the other variables

$$L, \quad L', \quad \Pi, \quad \Pi', \quad \varpi, \quad \varpi'$$

or

$$L, \quad L', \quad G, \quad G', \quad g, \quad g', \quad \Theta, \quad \Theta', \quad \theta, \quad \theta'.$$

It is not in this form that we ordinarily develop the perturbative function in treatises on Celestial Mechanics.

We take as variables: /262

The major axes, the eccentricities, the inclinations, the mean longitudes and the longitudes of the perihelions and of the nodes.

/261 However, it is easy to see that this goes back to saying the same thing.

If we set

$$B_{m_1 m_2} = C_{m_1 m_2} e^{\sqrt{-1}(m_1 g + m_2 g' + m_1 \theta + m_2 \theta')},$$

it will follow that

$$F_1 = \Sigma C_{m_1 m_2} e^{\sqrt{-1}[m_1(l+g+\theta) + m_2(l'+g'+\theta')]}. \qquad (2)$$

the exponential factor depends only on the mean longitudes

$$l + g + \theta, \quad l' + g' + \theta'$$

and the factor $C_{m_1 m_2}$ depends only on the other variables, major axes, eccentricities, inclinations, longitudes of the perihelions and of the nodes. Thus we will in this way fall back to the usual development of the perturbative function.

Expressions (14) can then be written

$$B^{\lambda'}_{\lambda p, \lambda q} B^{-\lambda}_{\lambda' p, \lambda' q} = C^{\lambda'}_{\lambda p, \lambda q} C^{-\lambda}_{\lambda' p, \lambda' q}.$$

In order for there to be a uniform integral, it is therefore necessary that there be a relationship between an arbitrary 2n-4 (n=4 in the plane, n=6 in space) of expressions

$$C^{\lambda'}_{\lambda p, \lambda q} C^{-\lambda}_{\lambda' p, \lambda' q} \quad (\lambda, \lambda' = 0, \pm 1, \pm 2, \pm 3, \ldots, \text{ad inf.}) \qquad (14a)$$

formed by means of the coefficients of development (2).

Thus, in order to apply the principles of the present chapter, it is not necessary to make a new development of the perturbative function by means of new variables, as it would be in development (1). We can make use of the development already used by astronomers, that is, development (2).

The coefficients $C_{m_1 m_2}$ can be developed in increasing powers of the eccentricities and inclinations. Let us therefore consider the development of one of these coefficients in powers of the eccentricities and inclinations. We know (cf. article 12) that all terms of this development will be of the degree $|m_1+m_2|$ at least with respect to these quantities and, if their degree differs from $|m_1+m_2|$ the difference is an even number.

We will therefore be able to write

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$$C_{m_1 m_2} = C^0_{m_1 m_2} + C^1_{m_1 m_2} + \ldots + C^p_{m_1 m_2} + \ldots,$$

$C^p_{m_1 m_2}$ representing the total of the terms of the development which are of the degree

$$|m_1 + m_2| + 2p$$

with respect to the eccentricities and inclinations.

We will say that $C^0_{m_1 m_2}$ is the principal term of $C_{m_1 m_2}$ and that the other terms are its secondary terms.

There will be an exception for coefficient C_{00}; we have, in this case,

$$C_{00} = C^0_{00} + C^1_{00} + \dots .$$

C^0_{00} depends only on the major axes, if these major axes are momentarily regarded as constants, as we have done in previous articles (it is, in fact, in assuming the major axes constant that the existence of a uniform integral implies that of a relationship between 2n-4 expressions (14)); if, therefore, the major axes are constants C^0_{00} will also be a constant which will play no role whatsoever in the calculation.

It is therefore C^1_{00} which is of the second degree with respect to the eccentricities and to the inclinations, which we will agree to call the principal term of C_{00}.

If, then, we replace development (2) by the following

$$C^0_{00} + C^1_{00} + \Sigma C^0_{m_1 m_2} e^{\sqrt{-1}[m_1(l+g+\theta)+m_2(l'+g'+\theta')]}, \tag{3}$$

we will say that we have written the development of the perturbative function F_1 reduced to its principal terms.

This granted, what is the condition for which there be a relationship between an arbitrary 2n-4 of the expressions

$$C^{\lambda'}_{\lambda p.\lambda q} C^{-\lambda}_{\lambda' p.\lambda' q} \quad (\lambda, \lambda' = 0, \pm 1, \pm 2, \dots). \tag{14}$$

Let us form a table composed of an infinity of rows formed as follows:

The various lines will correspond to the various integral values of the index λ, positive, negative or zero.

The first element of the row with index λ will be /264

$$\lambda C_{\lambda p.\lambda q},$$

the others will be the derivatives of $C_{\lambda p.\lambda q}$ with respect to the different variables

$$e,\ e',\ \varpi,\ \varpi',\ i,\ i',\ \theta,\ \theta',$$

that is, with respect to the eccentricities, longitudes of the perihelions, to the inclinations and to the node longitudes.

Thus, the necessary and sufficient condition for there to be a relationship between 2n-4=8 (n=6 in space) of relations (14) is that all determinants formed by taking nine arbitrary lines in this table be zero.

Needless to add, in the simplest cases, for example when the three bodies move in a plane, the number of columns and rows of these determinants is smaller than 9.

We have seen that all terms of the development of $C_{m_1 m_2}$ are of the degree $|m_1+m_2|$ at least. Therefore, among the elements of row of index λ (which I assume developed in powers of the eccentricities and of the inclinations), the first λ $C_{\lambda p. \lambda q}$ begin with terms of the degree

$$|\lambda p + \lambda q|.$$

The case is the same for derivatives $C_{\lambda p, \lambda q}$ with respect to ϖ and to θ, while the derivatives of $C_{\lambda p, \lambda q}$ with respect to e and to i will begin with terms of the degree

$$|\lambda p + \lambda q| - 1.$$

For the row index 0, the first term reduces to 0; the developments of the derivatives of C_{00} with respect to ϖ and to θ will begin with terms of the second degree, and those of the derivatives of C_{00} with respect to e and to i will begin with terms of the first degree.

Our determinants are in turn capable of being developed in powers of e and i. If a determinant Δ is formed by the rows of indices

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\ \lambda_9,$$

all the terms of its development will then be at least of the degree /265

$$|p+q|\ (|\lambda_1| + |\lambda_2| + \ldots + |\lambda_8| + |\lambda_9|) - 4.$$

/264 I set this quantity equal to α.

There is an exception in the case where λ_0=0; all terms are then at least of the degree

$$|p+q|\ (|\lambda_1|+|\lambda_2|+\ldots+|\lambda_8|)-2.$$

I will still set this quantity equal to α.

The determinants Δ having to be identically zero, the total of the terms of degree α will also have to be identically zero. Now we will obtain these terms of degree α, in replacing in the determinant Δ each of the coefficients $C_{\lambda p.\lambda q}$ by its principal $C^0_{\lambda p,\lambda q}$ (or $C^1_{0.0}$ if $\lambda=0$).

The determinant Δ_0 thus obtained will therefore have to be identically zero; now what does this condition

$$\Delta_0=0$$

signify?

Let us form the expressions

$$(C^0_{\lambda p.\lambda q})^{\lambda'}(C^0_{\lambda' p.\lambda' q})^{-\lambda}\quad (\lambda,\lambda'=\pm 1,\pm 2,\ldots), \tag{14a}$$

obtained by replacing, in expressions (14), each coefficient C by its principal term.

If in expression (14) we make $\lambda=0$, this expression reduces to

$$C_{0.0}.$$

We will adjoin to the table of expressions (14a) the expression C^1_{00} which is a polynomial integral of the second degree with respect to e and to i.

Thus, the condition $\Delta_0=0$ signifies that there is a relationship among an arbitrary eight of the expressions (14a) contained in the table thus completed.

Thus, in order for there to be a uniform integral, it is necessary that there be an integral relationship among an arbitrary of these expressions (14a).

The coefficients C were infinite series, and expressions (14) were presented in the form of the quotient of such series.

On the contrary, expressions (14a) are rational with respect to e, i, the sine and cosine of the ϖ and of the θ.

Verification is therefore facilitated by substitution in the coefficients of their principal terms.

It even becomes easy for small values of the two integers p and q.

When we thus have proved that the determinants corresponding to small values of the integers p and q are not zero, it becomes difficult to retain the illusion that the determinants corresponding to the large values of the same integers can vanish and thus permit the existence of a uniform integral.

A doubt might, nevertheless, still remain.

We could assume, however improbable it may seem, that among the classes (to use the language of article 84), there is a finite number of them which are ordinary and it is precisely these on which verification is based; but there are an infinity of them which are singular.

In order to completely erase this final doubt, it would be necessary to have a general expression of functions (14) or (14a) for all values of the integers λ, λ', p and q and this expression could only be extremely complicated.

Happily, Flamme, in a recent thesis,[1] has given the approximate expression of the terms of increasing rank in the development of the perturbative function, and this approximate expression, much simpler than the complete expression, can suffice for our purpose.

Nevertheless, the form which Flamme has given it is not useful for the problem which concerns us; we will be obliged to complete his results and to transform them considerably.

I will, therefore, return to this topic in the next chapter, after having treated the approximate calculus of the various terms of the perturbative function form; although the preceding considerations are of a nature to convince the most skeptical, they do not, nevertheless, constitute a rigorous mathematical demonstration.

89. One last remark can, to a certain measure, facilitate verification.

Let us again take relations (13), from article 84, which is written

$$-B_{m_1,m_2}\left(m_1\frac{d\Phi_0}{dx_1}+m_2\frac{d\Phi_0}{dx_2}\right)+\sum\left(\frac{dB_{m_1,m_2}}{dz_i}\frac{d\Phi_0}{du_i}-\frac{dB_{m_1,m_2}}{du_i}\frac{d\Phi_0}{dz_i}\right)=0.$$

In setting $m_1=\lambda p$, $m_2=\lambda q$ in this equation, I will obtain a particular relationship which I will call (13a); in setting $m_1=\lambda' p$, $m_2=\lambda' q$ in it, I will obtain another particular relation which I will call (13b).

Then let

$$M_{\lambda,\lambda'}=B^{\lambda'}_{\lambda p,\lambda q}\,B^{-\lambda}_{\lambda' p,\lambda' q};$$

[1] Paris, Gauthier-Villars, 1887.

$M_{\lambda.\lambda'}$ will be one of expressions (14) which have played such a large role in the preceding articles.

Let us multiply (13a) and (13b), respectively, by

$$\frac{\lambda'}{B_{\lambda p.\lambda q}} \text{ and } \frac{-\lambda}{B_{\lambda' p.\lambda' q}}$$

and add; it will follow that

$$\sum\left(\frac{d\log M_{\lambda.\lambda'}}{dz_i}\frac{d\Phi_0}{du_i}-\frac{d\log M_{\lambda.\lambda'}}{du_i}\frac{d\Phi_0}{dz_i}\right)=0,$$

or, in adopting the notation of the brackets of Jacobi,

$$[\log M_{\lambda.\lambda'},\ \Phi_0]=0,$$

or

$$[M_{\lambda.\lambda'},\ \Phi_0]=0.$$

If therefore M and M' are two expressions (14) belonging to the same class, we will have to have

$$[M,\ \Phi_0]=[M',\ \Phi_0]=0,$$

or, by virtue of the theorem of Poisson,

$$[[M,\ M'],\ \Phi_0]=0,$$

from which we can conclude that (M,M') is a function of 2n-4 of expressions (14)./268

It must not be forgotten that the brackets must be calculated while considering x_1 and x_2 (that is to say in the case of the Problem of Three Bodies, βL and $\beta' L'$) as constants.