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AN INTRODUCTION
TO
CELESTIAL MECHANICS

BY

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preserve a symmetrical form. This method was employed by Jacobi in an important memoir entitled, *Sur l'élimination des nœuds dans le problème des trois corps* (*Journal de Math.* vol. IX., 1844), and by Radau in a memoir entitled, *Sur une transformation des équations différentielles de la Dynamique* (*Annales de l'École Normale*, 1st series, vol. V.).

XIX. PROBLEMS.

1. Make the transformation $x_i = x_i' + x_n$ in the integrals (12) and (15), and eliminate $x_n, y_n, z_n, \frac{dx_n}{dt}, \frac{dy_n}{dt}$, and $\frac{dz_n}{dt}$ by means of equations (4) and (5). Prove that the resulting expressions are four integrals of equations (23).

2. Derive equations (23) directly by taking the origin at m_n, without first making use of the fixed axes.

3. The equations (23) are not symmetrical, since each body requires a different perturbative function $R_{i,j}$ in the right members. Construct the corresponding system of differential equations where the motion of m_{n-1} is referred to a rectangular system of axes with the origin at m_n; the motion of m_{n-2} to a parallel system of axes with origin at the center of mass of m_n and m_{n-1}; the motion of m_{n-3} to a parallel system of axes with the origin at the center of mass of m_n, m_{n-1}, and m_{n-2}, and continue in this way. Show that the results are the symmetrical equations

$$\begin{aligned}
&\frac{\mu_n}{\mu_{n-1}} m_{n-1} \frac{d^2 x_{n-1}}{dt^2} = \frac{\partial U}{\partial x_{n-1}}, && \mu_n = m_n, \qquad \mu_{n-1} = m_{n-1} + m_n,\\
&\frac{\mu_{n-1}}{\mu_{n-2}} m_{n-2} \frac{d^2 x_{n-2}}{dt^2} = \frac{\partial U}{\partial x_{n-2}}, && \mu_{n-2} = m_{n-2} + m_{n-1} + m_n,\\
&\vdots\\
&\frac{\mu_2}{\mu_1} m_1 \frac{d^2 x_1}{dt^2} = \frac{\partial U}{\partial x_1}, && \mu_1 = m_1 + m_2 + \cdots + m_n,
\end{aligned}$$

and similar equations in y and z, where

$$\begin{aligned}
U = {} & k^2 m_n \left(\frac{m_{n-1}}{r_{n,\,n-1}} + \frac{m_{n-2}}{r_{n,\,n-2}} + \cdots + \frac{m_1}{r_{n,\,1}} \right)\\
& + k^2 m_{n-1} \left(\frac{m_{n-2}}{r_{n-1,\,n-2}} + \frac{m_{n-3}}{r_{n-1,\,n-3}} + \cdots + \frac{m_1}{r_{n-1,\,1}} \right)\\
& \vdots\\
& + k^2 m_2 \frac{m_1}{r_{1,\,2}}.
\end{aligned}$$

(These equations are the same as found by Radau from a different standpoint in the memoir cited in Art. 150. They have been employed by Tisserand in a very elegant demonstration of Poisson's theorem of the invariability of the major axes of the planets' orbits up to perturbations of the second order inclusive with respect to the masses. Poincaré has generally used this system in his researches in the Problem of Three Bodies.)

4. Derive the differential equations corresponding to (23) in polar coordinates.

$$\left\{\begin{aligned}
&\frac{d^2 r_{j,\,n}}{dt^2} - r_{j,\,n} \cos^2 \phi_j \left(\frac{d\theta_j}{dt}\right)^2 - r_{j,\,n} \left(\frac{d\phi_j}{dt}\right)^2 = -\frac{k^2(m_j + m_n)}{r^2_{j,\,n}} + \sum_{i=1}^{n-1} m_i \frac{\partial R_{j,\,i}}{\partial r_{i,\,n}},\\
&\frac{d}{dt}\left(r^2_{j,\,n} \cos^2 \phi_j \frac{d\theta_j}{dt} \right) = \sum_{i=1}^{n-1} m_i \frac{\partial R_{j,\,i}}{\partial \theta_j},\\
&\frac{d}{dt}\left(r^2_{j,\,n} \frac{d\phi_j}{dt} \right) + r^2_{j,\,n} \sin \phi_j \cos \phi_j \left(\frac{d\theta_j}{dt}\right)^2 = \sum_{i=1}^{n-1} m_i \frac{\partial R_{j,\,i}}{\partial \phi_j},\\
&\qquad\qquad\qquad (j = 1, \cdots, n-1), \quad (i \neq j).
\end{aligned}\right.$$

HISTORICAL SKETCH AND BIBLIOGRAPHY.

The investigations in the Problem of n Bodies are of two classes; first, those which lead to general theorems holding in every system; and second, those which give good approximations for a certain length of time in particular systems, such as the solar system. Investigations of the second class are known as theories of perturbations, the discussion of which will be given in another chapter.

The first general theorems are regarding the motion of the center of mass, and were given by Newton in the *Principia*. The ten integrals and the theorems to which they lead were known by Euler. The next general result was the proof of the existence and the discussion of the properties of the invariable plane by Laplace in 1784. In the winter semester of 1842–43 Jacobi gave a course of lectures in the University of Königsberg on Dynamics. In this course he gave the results of some very important investigations on the integration of the differential equations which arise in Mechanics. In all cases where the forces depend upon the coördinates alone, and where a potential function exists, conditions which are fulfilled in the Problem of n Bodies, he proved that if all the integrals except two have been found the last two can always be found. He also showed, in extending some investigations of Sir William Rowan Hamilton, that the problem is reducible to that of solving a partial differential equation whose order is one-half as great as that of the original system. Jacobi's lectures are published in the supplementary volume to his collected works. They are of great importance in themselves, as well as being an absolutely necessary prerequisite to the reading of the epoch-making memoirs of Poincaré, and they should be accessible to every student of Celestial Mechanics.

It is a question of the highest interest whether the motions of the members of such a system as the sun and planets are purely periodic. Newcomb has shown in an important memoir published in the *Smithsonian Contributions to Knowledge*, December 1874, that the differential equations can be formally satisfied by purely periodic series. He did not, however, prove the convergence of these series; and, indeed, Poincaré has shown in *Les Méthodes Nouvelles*, chaps. IX. and XII., that they are in general divergent.

As was stated in Art. 147, Bruns has proved in the *Acta Mathematica*, vol. xi., that, using rectangular coördinates, there are no new algebraic integrals; and Poincaré, in the *Acta Mathematica*, vol. xiii., that, using the elements as variables, there are no new uniform transcendental integrals, even when the masses of all the bodies except one are very small.

For further reading regarding the general differential equations in different sets of variables the student will do well to consult Tisserand's *Mécanique Céleste*, vol. i. chapters iii., iv., and v.

CHAPTER VIII.

THE PROBLEM OF THREE BODIES.

151. Problem Considered. There are a number of important results in the Problem of Three Bodies which have been established with mathematical rigor if the initial coördinates and the components of velocity fulfill certain special conditions. While these special cases have not been found in nature, there are nevertheless some applications of the results obtained, and the processes employed are mathematically elegant and lead to most interesting conclusions. This chapter will contain such of these results as fall within the scope of this work, reserving the theories of perturbations, by means of which the positions of the heavenly bodies are predicted, to subsequent chapters.

The first part of the chapter will be devoted to a discussion of some of the properties of motion of an infinitesimal body when it is attracted by two finite bodies which revolve in circles around their center of mass, and will include the proof of the existence of certain particular solutions in which the distances of the infinitesimal body from the finite bodies are constants. The second part of the chapter will be devoted to an exposition of a method of finding particular solutions of the motion of three finite bodies such that the ratios of their mutual distances are constants. These solutions include the former, but the discoverable properties of motion are so much fewer, and are obtained with so much more difficulty, that it is advisable to divide the discussion into two parts.

The particular solutions of the Problem of Three Bodies which will be discussed here were given for the first time by Lagrange in a prize memoir in 1772. The method adopted here is radically different from that employed by him, and lends itself much more readily to a generalization to the case where a larger number of bodies is involved. But, on the other hand, the reduction of the order of the problem by one unit, which was a very interesting feature of Lagrange's memoir, is not accomplished by this method. However, as it has not been possible to make any use of this reduction, it has not been of any practical importance.

Mathematically speaking, an *infinitesimal body* is one that is attracted by finite masses but does not attract them. Physically speaking, it is a body of such a small mass that it will disturb the motion of finite bodies less than an arbitrarily assigned amount, however small, during any arbitrarily assigned time, however long. To actually determine a small mass fulfilling these conditions it is only necessary to make it so small that its whole attraction, which is always greater than its disturbing force, on one of the large bodies, if placed at the minimum distance possible, would move the large body less than the assigned small distance in the assigned time.

Motion of the Infinitesimal Body.

152. The Differential Equations of Motion. Suppose the system consists of two finite bodies revolving in circles around their common center of mass, and of an infinitesimal body subject to their attraction. Let the unit of mass be so chosen that the sum of the masses of the finite bodies shall be unity; then they can be represented by $1 - \mu$ and μ, where the notation is so chosen that $\mu \leqq \frac{1}{2}$. Let the unit of distance be so chosen that the constant distance between the finite bodies shall be unity. Let the unit of time be so chosen that k^2 shall equal unity. Let the origin of coördinates be taken at the center of mass of the finite bodies, and let the direction of the axes be so chosen that the $\xi\eta$-plane is the plane of their motion. Let the coördinates of $1-\mu$, μ, and the infinitesimal body be $\xi_1, \eta_1, 0$; $\xi_2, \eta_2, 0$; and ξ, η, ζ respectively, and

$$r_1 = \sqrt{(\xi - \xi_1)^2 + (\eta - \eta_1)^2 + \zeta^2},$$
$$r_2 = \sqrt{(\xi - \xi_2)^2 + (\eta - \eta_2)^2 + \zeta^2}.$$

Then the differential equations of motion for the infinitesimal body are

$$(1) \qquad \begin{cases} \dfrac{d^2\xi}{dt^2} = -(1-\mu)\dfrac{(\xi - \xi_1)}{r_1^3} - \mu\dfrac{(\xi - \xi_2)}{r_2^3}, \\[2ex] \dfrac{d^2\eta}{dt^2} = -(1-\mu)\dfrac{(\eta - \eta_1)}{r_1^3} - \mu\dfrac{(\eta - \eta_2)}{r_2^3}, \\[2ex] \dfrac{d^2\zeta}{dt^2} = -(1-\mu)\dfrac{\zeta}{r_1^3} - \mu\dfrac{\zeta}{r_2^3}. \end{cases}$$

As a consequence of the way the units have been chosen the mean angular motion of the finite bodies is

$$n = k\frac{\sqrt{(1-\mu) + \mu}}{a^{\frac{3}{2}}} = 1.$$

Let the motion of the bodies be referred to a new system of axes having the same origin as the old, and rotating in the $\xi\eta$-plane in the direction in which the finite bodies move with the uniform angular velocity unity. The coördinates in the new system are defined by the equations

$$(2) \qquad \begin{cases} \xi = x\cos t - y\sin t, \\ \eta = x\sin t + y\cos t, \\ \zeta = z, \end{cases}$$

and similar equations for the letters with subscripts 1 and 2. On computing the second derivatives of (2) and substituting in (1), it is found that

$$(3) \qquad \begin{cases} \left\{\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} - x\right\}\cos t - \left\{\dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} - y\right\}\sin t \\[2ex] \qquad = -\left\{(1-\mu)\dfrac{(x - x_1)}{r_1^3} + \mu\dfrac{(x - x_2)}{r_2^3}\right\}\cos t \\[2ex] \qquad\qquad + \left\{(1-\mu)\dfrac{(y - y_1)}{r_1^3} + \mu\dfrac{(y - y_2)}{r_2^3}\right\}\sin t, \\[2ex] \left\{\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} - x\right\}\sin t + \left\{\dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} - y\right\}\cos t \\[2ex] \qquad = -\left\{(1-\mu)\dfrac{(x - x_1)}{r_1^3} + \mu\dfrac{(x - x_2)}{r_2^3}\right\}\sin t \\[2ex] \qquad\qquad - \left\{(1-\mu)\dfrac{(y - y_1)}{r_1^3} + \mu\dfrac{(y - y_2)}{r_2^3}\right\}\cos t, \\[2ex] \dfrac{d^2z}{dt^2} = -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}. \end{cases}$$

Multiply the first two equations by $\cos t$ and $\sin t$ respectively, then by $-\sin t$ and $\cos t$, and add; the results are

$$\begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1-\mu)\dfrac{(x - x_1)}{r_1^3} - \mu\dfrac{(x - x_2)}{r_2^3}, \\[2ex] \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1-\mu)\dfrac{(y - y_1)}{r_1^3} - \mu\dfrac{(y - y_2)}{r_2^3}, \\[2ex] \dfrac{d^2z}{dt^2} \qquad = \quad -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}. \end{cases}$$

The position of the axes can be so taken at the origin of time that the x-axis will continually pass through the centers of the finite bodies; then $y_1 = 0$, $y_2 = 0$, and the equations become

$$(4)\quad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1-\mu)\dfrac{(x-x_1)}{r_1^3} - \mu\dfrac{(x-x_2)}{r_2^3}, \\ \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1-\mu)\dfrac{y}{r_1^3} - \mu\dfrac{y}{r_2^3}, \\ \dfrac{d^2z}{dt^2} \qquad\quad = \;-(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}. \end{cases}$$

These are the differential equations of motion of the infinitesimal body referred to axes rotating so that the finite bodies always lie on the x-axis. They have the important property that they do not involve explicitly the independent variable t because the coördinates of the finite bodies have become constants as a consequence of the particular manner in which the axes are rotated. On the other hand, in equations (1) the quantities ξ_1, ξ_2, η_1, and η_2 are functions of t.

The general problem of determining the motion of the infinitesimal body is of the sixth order; if it moves in the plane of motion of the finite bodies, the problem is of the fourth order.

153. Jacobi's Integral. Equations (4) admit an integral which was first given by Jacobi in *Comptes Rendus de l'Académie des Sciences de Paris*, vol. III., p. 59, and which has been discussed by Hill in the first of his celebrated papers on the Lunar Theory, *The American Journal of Mathematics*, vol. I., p. 18, and again by Darwin in his memoir on Periodic Orbits in *Acta Mathematica*, vol. XXI., p. 102. Let

$$(5)\qquad U = \tfrac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2};$$

then equations (4) can be written in the form

$$(6)\quad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = \dfrac{\partial U}{\partial x}, \\ \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = \dfrac{\partial U}{\partial y}, \\ \dfrac{d^2z}{dt^2} \qquad\quad = \dfrac{\partial U}{\partial z}. \end{cases}$$

If these equations are multiplied by $2\dfrac{dx}{dt}$, $2\dfrac{dy}{dt}$, and $2\dfrac{dz}{dt}$ respectively, and added, the resulting equation can be integrated, since U is a function of x, y, and z alone, and give

$$(7)\quad \begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= V^2 = 2U - C \\ &= x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C. \end{aligned}$$

Five integrals more are required in order completely to solve the problem. If the infinitesimal body moved in the xy-plane only three would remain to be found, the last two of which could be obtained by Jacobi's *last multiplier*,* if the first one were found. Thus it appears that only one new integral is needed for the complete solution of this special problem in the plane.† But Bruns has proved in *Acta Mathematica*, vol. XI., that, when rectangular coördinates are used, no new algebraic integrals exist; and Poincaré has proved in *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I., chap. V., that when the elements of the orbits are used as variables, there are no new uniform transcendental integrals, even when the mass of one of the finite bodies is very small compared to that of the other (see Art. 147). These demonstrations are entirely outside the scope of this work and cannot be reproduced here.

154. The Surfaces of Zero Relative Velocity.‡ Equation (7) is a relation between the square of the velocity and the coördinates of the infinitesimal body referred to the rotating axes. Therefore, when the constant of integration C has been determined numerically by the initial conditions, equation (7) determines the velocity with which the infinitesimal body will move, if at all, at all points of the rotating space; and conversely, for a given velocity, equation (7) gives the locus of those points of relative space where alone the infinitesimal body can be. In particular, if V is put equal to zero in this equation it will define the surfaces at which the velocity will be zero. On one side of these surfaces the velocity will be real and on the other side imaginary; or, in other words, it is

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* Developed in *Vorlesungen über Dynamik*, supplementary volume to Jacobi's collected works.

† Hill put his special equations in such a form that they would be reduced to quadratures if a single variable were expressed in terms of the time, *American Journal of Mathematics*, vol. I., p. 16.

‡ First discussed by Hill in his *Lunar Theory*, *The American Journal of Mathematics*, vol. I.; and again, for motion in the xy-plane, by Darwin in his *Periodic Orbits*, in *Acta Mathematica*, vol. XXI.

possible for the body to move on one side, and impossible for it to move on the other. The general proposition that a function changes sign as the surface at which it is zero is crossed (at least at a regular point of the surface) was proved in Art. 120. While it will not be possible to say in any except very particular cases what the orbit will be, yet this partition of relative space will show in what portions the infinitesimal body can move and in what portions it can not.

The equation of the surfaces of zero relative velocity is

$$(8)\qquad \begin{cases} x^2 + y^2 + \dfrac{2(1-\mu)}{r_1} + \dfrac{2\mu}{r_2} = C, \\ r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}, \\ r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}. \end{cases}$$

Since only the squares of y and z occur the surfaces defined by (8) are symmetrical with respect to the xy and xz-planes, and, when $\mu = \frac{1}{2}$, with respect to the yz-plane also. The surfaces for $\mu \neq \frac{1}{2}$ can be regarded as being deformations of those for $\mu = \frac{1}{2}$. It follows from the way in which z enters that a line parallel to the z-axis pierces the surfaces in two (or no) real points. Moreover, the surfaces are contained within a cylinder whose axis is the z-axis and whose radius is $\sqrt{C}$, to which certain of the folds are asymptotic at $z^2 = \infty$; for, as z^2 increases the equation approaches as a limit

$$x^2 + y^2 = C.$$

155. Approximate Forms of the Surfaces. From the properties of the surfaces given in the preceding article and from the shapes of the curves in which the surfaces intersect the reference planes, a general idea of their form can be obtained. The equation of the curves of intersection of the surfaces with the xy-plane is obtained by putting z equal to zero in the first of (8), and is

$$(9)\quad x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x - x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x - x_2)^2 + y^2}} = C.$$

For large values of x and y which satisfy this equation the third and fourth terms are relatively unimportant, and the equation may be written

$$x^2 + y^2 = C - \frac{2(1-\mu)}{\sqrt{(x - x_1)^2 + y^2}} - \frac{2\mu}{\sqrt{(x - x_2)^2 + y^2}} = C - \epsilon,$$

where ϵ is a small quantity. This is the equation of a circle whose radius is $\sqrt{C - \epsilon}$; therefore, one branch of the curve in the xy-plane is an approximately circular oval within the asymptotic cylinder. It is also to be noted that the larger C is, the larger are the values of x and y which satisfy the equation, the smaller is ϵ, the more nearly circular is the curve, and the more nearly does it approach its asymptotic cylinder.

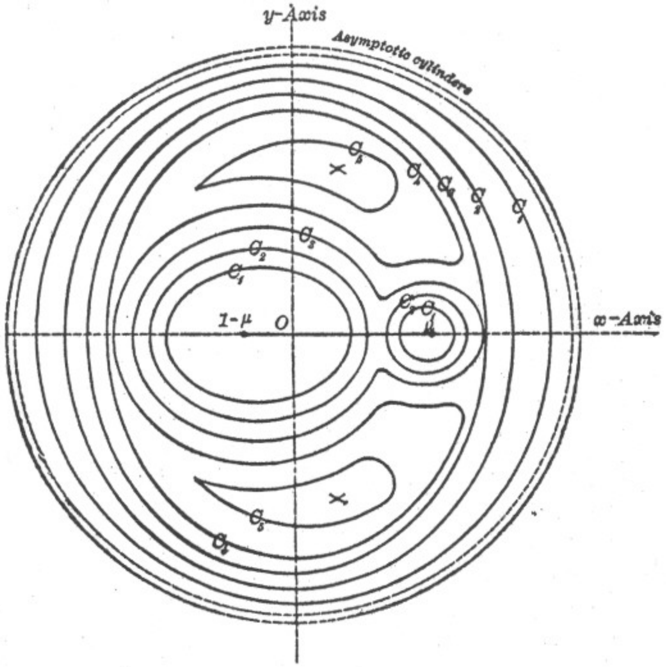

Fig. 38.

For small values of x and y satisfying (9) the first and second terms are relatively unimportant, and the equation may be written

$$\frac{1-\mu}{r_1} + \frac{\mu}{r_2} = \frac{C}{2} - \frac{x^2 + y^2}{2} = \frac{C}{2} - \epsilon.$$

This is the equation of the *equipotential curves** for the two centers of force, $1 - \mu$ and μ. For large values of C they consist of closed ovals around each of the bodies $1 - \mu$ and μ; for smaller values of C these ovals unite between the bodies forming a dumb-

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*Thomson and Tait's *Natural Philosophy*, Part II., Art. 508.

bell shaped figure in which the ends are of different size except when $\mu = \frac{1}{2}$; and for still smaller values of C the handle of the dumb-bell enlarges until the figure becomes an oval enclosing both of the bodies.

From the foregoing considerations it follows that the approximate forms of the curves in which the surfaces intersect the xy-plane are as given in Fig. 38. The curves C_1, C_2, C_3, C_4, C_5 are in the order of decreasing values of the constant C. They were not drawn from numerical calculations and are intended to show only qualitatively the character of the curves.

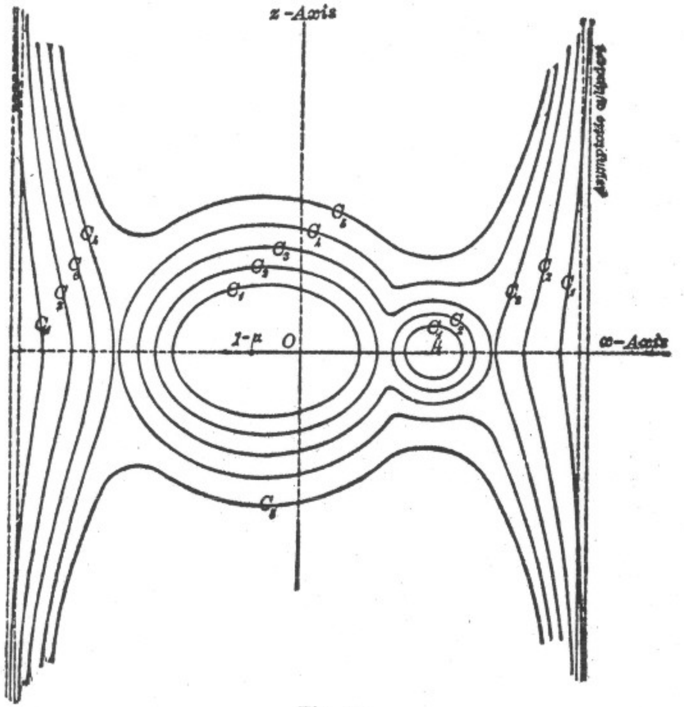

Fig. 39.

The equation of the curves of intersection of the surfaces and the xz-plane is obtained by putting y equal to zero in equation (8), and is

$$x^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2+z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2+z^2}} = C. \tag{10}$$

For large values of x and z satisfying this equation the second and third terms are relatively unimportant, and it may be written

$$x^2 = C - \epsilon,$$

which is the equation of a symmetrical pair of straight lines parallel to the z-axis. The larger C is, the larger is the value of x which, for a given value of z, satisfies the equation, and, therefore, the smaller is ϵ. Hence, the larger C the closer the lines are to the asymptotic cylinder.

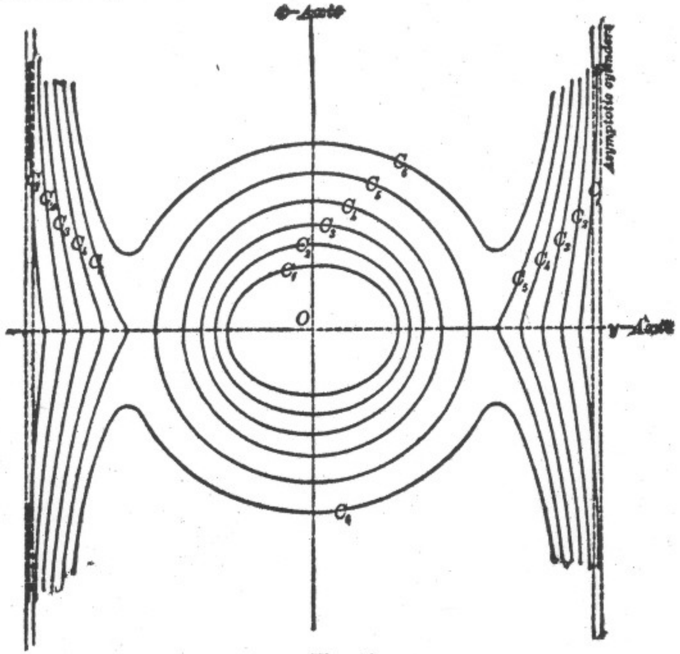

Fig. 40.

For small values of x and z satisfying equation (10) the first term is relatively unimportant, and the equation may be written

$$\frac{1-\mu}{r_1} + \frac{\mu}{r_2} = \frac{C}{2} - \epsilon.$$

This is again the equation of the equipotential curves and has the same properties as before. Hence, the forms of the curves in the xz-plane are qualitatively like those given in Fig. 39. Again, the curves $C_1, \cdots, C_5$ are in the order of decreasing values of the constant C, and were not drawn from numerical calculations.

The equation of the curves of intersection of the surfaces and the yz-plane is obtained by putting x equal to zero in equation (8), and is

$$(11) \qquad y^2 + \frac{2(1-\mu)}{\sqrt{x_1^2 + y^2 + z^2}} + \frac{2\mu}{\sqrt{x_2^2 + y^2 + z^2}} = C.$$

For large values of y and z satisfying this equation the second and third terms are relatively unimportant, and it may be written

$$y^2 = C - \epsilon,$$

which is the equation of a pair of lines near the asymptotic cylinder, approaching it as C increases.

If $1 - \mu$ is much greater than μ, the numerical value of x_2 is much greater than that of x_1; hence, for small values of y and z satisfying (11), this equation may be written

$$\frac{2(1-\mu)}{r_1} = C - \epsilon,$$

which is the equation of a circle which becomes larger as C decreases. Hence, the forms of the curves in the yz-plane are qualitatively as given in Fig. 40. Again, the curves $C_1, \cdots, C_5$ are in the order of decreasing values of the constant C.

From these three sections of the surfaces it is easy to infer their forms for the different values of C. They may be roughly described as consisting of, for large values of C, a closed fold approximately spherical in form around each of the finite bodies, and of curtains hanging from the asymptotic cylinder symmetrically with respect to the xy-plane; for smaller values of C, the folds expand and coalesce (Fig. 38, curve C_3); for still smaller values of C the united folds coalesce with the curtains, the first points of contact being in every case in the xy-plane; and for sufficiently small values of C the surfaces consist of two parts symmetrical with respect to the xy-plane but not intersecting it (Figs. 39, curve C_5, and 40, curve C_6).

156. The Regions of Real and Imaginary Velocity. Having determined the forms of the surfaces, it remains to find in what regions of relative space the motion is real and in what it is imaginary. The equation for the square of the velocity is

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C.$$

Suppose C is so large that the ovals and curtains are all separate. The motion will be real in those portions of relative space for which the right member of this equation is positive. If it is positive in one point in a closed fold it will be positive in every other point within it, for the function changes sign only at a surface of zero relative velocity.

It is evident from the equation that x and y can be taken so large that the right member will be positive, however great C may be; therefore, *the motion is real outside of the curtains.* It is also clear that a point can be chosen so near to either $1 - \mu$ or μ, that is, either r_1 or r_2 may be taken so small, that the right member will be positive, however great C may be; therefore, *the motion is real within the folds around the finite bodies.*

If the value of C were so large that the folds around the finite bodies were closed, and if the infinitesimal body should be within one of these folds at the origin of time, it would always remain there since it could not cross a surface of zero velocity. If the earth's orbit is supposed to be circular and the mass of the moon infinitesimal, it is found that the constant C, determined by the motion of the moon, is so large that the fold around the earth is closed with the moon within it. Therefore the moon cannot recede indefinitely from the earth. It was in this manner, and with these approximations, that Hill proved that the moon's distance from the earth has a superior limit.*

157. Method of Computing the Surfaces. Actual points on the surfaces can be found most readily by first determining the curves in the xy-plane, and then finding by methods of approximation the values of z which satisfy (7). Besides, the curves in the xy-plane are of most interest because the first points of contact as the various folds coalesce occur in this plane, and, indeed, on the x-axis, as can be seen from the symmetries of the surfaces.

The equation of the curves in the xy-plane is

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C.$$

If this equation is rationalized and cleared of fractions the result is a polynomial of the sixteenth degree in x and y. When the value of one of the variables is taken arbitrarily the corresponding values of the other can be found by solving this rationalized equation. This problem presents great practical difficulties

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* *Lunar Theory, Am. Jour. Math.*, vol. I., p. 23.

bodies are at the vertices of an equilateral triangle. It is easy to show that, unless they are collinear, there is no other solution. In the case of the equilateral triangle solution equations (67) and (68) also reduce to (72), and the orbits are similar conic sections of arbitrary eccentricity.

XXII. PROBLEMS.

1. Take as an hypothesis that a solution exists in which the three bodies are always collinear. Prove that the law of areas holds for each body separately with respect to the center of mass of the system, with respect to either of the other bodies, and with respect to the center of mass of any two of the bodies.

2. Write the conditions that the accelerations to which the bodies are subject shall be directed toward their common center of mass and proportional to their respective distances.

Ans. Equations (55).

3. The resultant of the forces acting on each body always passes through a fixed point. Prove that the equilateral triangle configuration is the only solution of equations (55) unless the bodies lie in a straight line.

4. Suppose $m_1 = m_2 = m_3 = 1$, and that the bodies move according to the equilateral triangular solution. Find the radius of the circle in which a particle would revolve around one of them in the period in which they revolve around their center of mass.

Ans. $R = 3^{\frac{1}{3}}$.

5. Prove that the equilateral triangular circular solutions hold when the mutual attractions of the bodies vary as any power of the distance.

6. Find the number of collinear solutions when the force varies as any power of the distance.

7. Prove that when the force varies inversely as the fifth power one solution is that each of the bodies moves in a circle through their center of mass in such a way that the three bodies are always at the vertices of an equilateral triangle.

8. Prove that if the three bodies are placed at rest in any one of the configurations admitting circular solutions, they will fall to their center of mass in the same time in straight lines.

9. Find the distribution of mass among the three bodies for which the time of falling to their center of mass will be the least; the greatest.

10. Prove that if any four masses are placed at the vertices of a regular tetrahedron, the resultant of all the forces acting on each body passes through the center of mass of the four, and that the magnitudes of the accelerations are proportional to the respective distances of the bodies from their center of mass.

11. Prove that there are no circular solutions in the Problem of Four Bodies in which the bodies do not all move in the same plane.

12. Investigate the stability of the triangle and straight line solutions of the Problem of Three Bodies when all of the masses are finite.

HISTORICAL SKETCH AND BIBLIOGRAPHY.

The first particular solutions of the Problem of Three Bodies were found by Lagrange in his prize memoir, *Essai sur le Problème des Trois Corps*, which was submitted to the Paris Academy in 1772 (*Coll. Works*, vol. VI., p. 229, Tisserand's *Méc. Cél.* vol. I., chap. VIII.). The solutions which he found are precisely those given in the last part of this chapter. His method was to divide the problem into two parts; (*a*) the determination of the mutual distances of the bodies, (*b*) having solved (*a*), the determination of the plane of the triangle in space and the orientation of the triangle in the plane. He proved that if the part (*a*) were solved the part (*b*) could also be solved. To solve (*a*) it was necessary to derive three differential equations involving the three mutual distances alone as dependent variables. He found three equations, one of which was of the third order, and the remaining two of the second order each, making the whole problem of the *seventh* order. The reduction of the general problem of three bodies by the ten integrals leaves it of the *eighth* order; hence Lagrange's analysis reduced the problem by one unit. He found that he could integrate the differential equations completely by assuming that the ratios of the mutual distances were constants. The demonstration was repeated by Laplace in the *Mécanique Céleste*, vol. V., p. 310. In *l'Exposition du Système du Monde* he remarked that if the moon had been given to the earth by Providence to illuminate the night, as some have maintained, the end sought has been only imperfectly attained; for, if the moon were properly started in opposition to the sun it would always remain there relatively, and the whole earth would have either the full moon or the sun always in view. The demonstration upon which he based his remark was made under the assumption that there was no disturbing force. If there were disturbing forces the configuration would not be preserved unless the solution were stable, which it is not, as was proved by Liouville, *Journal de Mathématiques*, vol. VII., 1845.

A number of memoirs have appeared following more or less closely along the lines marked out by Lagrange. Among them may be mentioned one by Radau in the *Bulletin Astronomique*, vol. III., p. 113; by Lindstedt in the *Annales de l'École Normale*, 3rd series, vol. I., p. 85; by Allegret in the *Journal de Mathématiques*, 1875, p. 277; by Bour in the *Journal de l'École Polytechnique*, vol. XXXVI.; and by Mathieu in the *Journal de Mathématiques*, 1876, p. 345.

Jacobi, without a knowledge of the work of Lagrange, reduced the general Problem of Three Bodies to the seventh order in *Crelle's Journal*, 1843, p. 115 (*Coll. Works*, vol. IV., p. 478). It has never been reduced further.

Concerning the solutions of the problem of more than three bodies in which the ratios of the mutual distances are constants a number of papers have appeared, among which are one by Lehmann-Filhes in the *Astronomische Nachrichten*, vol. CXXVII., p. 137, one by F. R. Moulton in *The Transactions of the American Mathematical Society*, vol. I., p. 17, and one by W. R. Longley in *Bulletin of the American Mathematical Society*, vol. XIII., p. 324.

No new periodic solutions of the problem of three bodies were discovered after those of Lagrange until Hill developed his Lunar Theory, *The American Journal of Mathematics*, vol. I. (1878). These solutions of Hill are of immensely greater practical value than those of the Lagrangian type. It should be stated, however, that they are not strictly periodic solutions of any actual case, because a small part of the perturbing action of the sun was neglected.

The next important advance was made by Poincaré in a memoir in the *Bulletin Astronomique*, vol. I., in which he proved that when the masses of two of the bodies are small compared to that of the third, there is an infinite number of sets of initial conditions for which the motion is periodic. These ideas were elaborated and the results extended in a memoir crowned with the prize offered by the late King Oscar of Sweden. This memoir appeared in *Acta Mathematica*, vol. XIII. The methods employed by Poincaré are incomparably more profound and powerful than any previously used in Celestial Mechanics, and mark an epoch in the development of the science. The work of Poincaré was recast and extended in many directions, and published in three volumes entitled, *Les Méthodes Nouvelles de la Mécanique Céleste*. It is written with admirable directness and clearness, and is given in sufficient detail to make so profound a work as easily read as possible.

An important memoir on Periodic Orbits by Sir George Darwin appeared in *Acta Mathematica*, vol. XXI. (1899). In this investigation it was assumed that one of the three masses is infinitesimal and that the finite masses, having the ratio of ten to one, revolve in circles. A large number of periodic orbits, belonging to a number of families, were discovered by numerical experiments. The question of their stability was answered by using essentially the method employed by Hill in his discussion of the motion of the lunar perigee.

A considerable number of investigations in the domain of periodic orbits, employing analytical processes based on the methods of Poincaré, have been published by F. R. Moulton and his former students Daniel Buchanan, Thomas Buck, F. L. Griffin, Wm. R. Longley, and W. D. MacMillan. These papers have appeared in the *Transactions of the American Mathematical Society*, the *Proceedings of the London Mathematical Society*, the *Mathematische Annalen*, and the *Proceedings of the Fifth International Congress of Mathematicians*. Besides containing the analysis for a great variety of periodic orbits, they show the existence of infinite sets of closed orbits of ejection which form the boundaries between different classes of periodic orbits. These investigations are published under the title "Periodic Orbits" as *Publication* 161 of the Carnegie Institution of Washington.

CHAPTER IX.

PERTURBATIONS—GEOMETRICAL CONSIDERATIONS.

170. Meaning of Perturbations. It was shown in chapter V. that if two spherical bodies move under the influence of their mutual attractions each describes a conic section with respect to their center of mass as a focus, and that the path of each body with respect to the other is a conic. The converse theorem is also true; that is, if the law of areas holds and if the orbit of one body is a conic with respect to the other as a focus, then if the force depends only on the distance it varies inversely as the square of the distance (see also Art. 58). If there is a resisting medium, or if either of the bodies is oblate, or if there is a third body attracting the two under consideration, or if there is any force acting upon the bodies other than that of the mutual attractions of the two spheres, their orbits will cease to be exact conic sections. Suppose the coördinates and components of velocity are given at a definite instant t_0; then, if the conditions of the two-body problem were precisely fulfilled, the orbits would be definite conics in which the bodies would move so as to fulfill the law of areas. The differences between the coördinates and the components of velocity in the actual orbits and those which the bodies would have had if the motion had been undisturbed are the *perturbations*. It is necessary to include the changes in the components of velocity as perturbations, for the paths described depend not only upon the relative positions of the bodies and the forces to which they are subject, but also upon the relative velocities with which they are moving.

Several methods of computing perturbations have been devised depending upon the somewhat different points of view which may be taken. Of these the two following are the ones most frequently used.

171. Variation of Coördinates. The simplest conception of perturbations is that the coördinates are directly perturbed. For example, if a planet is subject to the attraction of another planet the coördinates and components of velocity of the former at any time t differ by definite amounts from what they would have been