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METHODS OF Celestial Mechanics

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Chapter XIII

PERTURBATIONS OF THE COORDINATES

1. Introduction. In Chapter XI has been developed a method of determining the motion of a point mass moving under the disturbing influence of other point masses, around a central point mass so massive as to dominate the system. In this method, variously known as the variation of elements, the variation of parameters, and the variation of arbitrary constants, the six continuously varying osculating elements of the disturbed body are expressed as sums of trigonometric series, the arguments of which are either linear functions of the time or linear functions of some other variables connected with the time by the formulas of elliptic motion. In the present chapter will be developed another method, in which the deviations of a body from a purely elliptic orbit are expressed as perturbations of the coordinates which have place in the ellipse. The method is analogous in many respects to Encke's method for special perturbations, discussed in Chapter V, but here we shall discuss the general perturbations of the polar coordinates as well as those of the rectangular coordinates.

Although the method of variation of arbitrary constants differs sharply in principle from the method of perturbations of coordinates, it is in fact possible to combine the two methods into one, in various ways. We shall discuss the method used by Newcomb for the four inner planets, where the eccentricity, perihelion, inclination, and node are conceived to vary strictly proportionally to the time, and the periodic perturbations of the longitude, latitude, and radius vector, being applied to the corresponding coordinates in this varying ellipse, give the actual position of the planet.

Finally we shall describe Brouwer's method, which is better adapted to the numerical calculation of disturbed rectangular coordinates than any other method of the classical planetary theory.

2. Differential equations. We begin with the differential equations of relative motion, discussed in earlier chapters, considering one disturbed planet and one disturbing planet, since it is easy to see what to do when there are more. Let it be proposed to find the perturbations

of a planet having the mass m moving about the sun having the mass unity and being disturbed by another planet having the mass m'. Writing μ for $k^2(1+m)$, the equations of motion of m relative to the sun are

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=\frac{\partial R}{\partial x},\quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=\frac{\partial R}{\partial y},\quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=\frac{\partial R}{\partial z}, \tag{1}$$

where the rectangular coordinates are referred to any fixed plane passing through the sun, $r^2=x^2+y^2+z^2$, and R, the disturbing function, has the expression

$$R=k^2m'\left\{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{-1/2}-\frac{x'x+y'y+z'z}{r'^3}\right\}. \tag{2}$$

If the longitude, reckoned in the reference plane, is denoted by v, the radius vector by r, and the latitude by B, then in terms of these three variables the differential equations of motion are

$$\begin{aligned}\frac{d^2r}{dt^2}-r\cos^2B\left(\frac{dv}{dt}\right)^2-r\left(\frac{dB}{dt}\right)^2+\frac{\mu}{r^2}&=\frac{\partial R}{\partial r},\\ \frac{d}{dt}\left(r^2\cos^2B\frac{dv}{dt}\right)&=\frac{\partial R}{\partial v},\\ \frac{d}{dt}\left(r^2\frac{dB}{dt}\right)+r^2\sin B\cos B\left(\frac{dv}{dt}\right)^2&=\frac{\partial R}{\partial B},\end{aligned} \tag{3}$$

and

$$R=k^2m'\left[(r'^2-2rr'\cos H+r^2)^{-1/2}-\frac{r\cos H}{r'^2}\right], \tag{4}$$

where

$$\cos H=\cos B\cos B'\cos(v-v')+\sin B\sin B'. \tag{5}$$

It is generally simpler to retain the rectangular coordinate z instead of the latitude B, and adopt r and v such that

$$x=\sqrt{r^2-z^2}\cos v,\qquad y=\sqrt{r^2-z^2}\sin v. \tag{6}$$

Let us now suppose that each coordinate of the disturbed planet is separated into two parts such that

$$x=x_0+\delta x,\qquad y=y_0+\delta y,\qquad z=z_0+\delta z, \tag{7}$$

the first of which satisfy the differential equations

$$\frac{d^2x_0}{dt^2}+\frac{\mu x_0}{r_0^3}=0,$$
$$\frac{d^2y_0}{dt^2}+\frac{\mu y_0}{r_0^3}=0, \tag{8}$$
$$\frac{d^2z_0}{dt^2}+\frac{\mu z_0}{r_0^3}=0,$$

where $r_0^2 = x_0^2 + y_0^2 + z_0^2$, and the parts δx, δy, δz are of the order of the disturbing forces.

It is evident that certain functions of the time t might be added to x_0, y_0, z_0 without their ceasing to satisfy the differential equations (8), and that then, in order to represent the position of the planet, the same functions would have to be subtracted from δx, δy, δz; thus the separation of the coordinates into two parts is, to a certain extent, arbitrary. The indetermination must be removed by taking for the constants of integration values accordant with the sort of elements from which x_0, y_0, z_0 have been obtained. If these elements are osculating at a particular epoch, then it is evident that the constants of integration must be determined so as to cause δx, δy, δz and their first derivatives with respect to the time to vanish at the same epoch. On the other hand it is often advantageous to use mean elements so defined that certain terms of the perturbations vanish identically, and in this case the constants of integration must be determined in accordance with the definitions of the mean elements. It will be seen later that it is possible to associate each of the six independent constants of integration with one of the six elements in such a way that the perturbations of the coordinates are reduced to their smallest possible numerical values.

We now write

$$r = r_0 + \delta r, \tag{9}$$

$$\begin{aligned} dR &= \frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz \\ &= \frac{\partial R}{\partial r}\,dr + \frac{\partial R}{\partial v}\,dv + \frac{\partial R}{\partial B}\,dB. \end{aligned} \tag{10}$$

The last expression is the differential of R when the coordinates of the disturbed planet alone vary. We also have

$$r\frac{\partial R}{\partial r} = x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}, \tag{11}$$

which is evidently correct if, in the first member, R is expressed in terms of r and any two other coordinates which make x/r, y/r, z/r independent of r.

Multiplying (1) severally by $2dx$, $2dy$, $2dz$, and adding the products and integrating, we have

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2-\frac{2\mu}{r}+\frac{\mu}{a}=2\int dR, \tag{12}$$

where μ/a is the arbitrary constant of integration; it is plain that we are at liberty to suppose that it is such that the equation of two-body motion,

$$\left(\frac{dx_0}{dt}\right)^2+\left(\frac{dy_0}{dt}\right)^2+\left(\frac{dz_0}{dt}\right)^2-\frac{2\mu}{r_0}+\frac{\mu}{a}=0, \tag{13}$$

is satisfied; if there is any residual constant it must be supposed to be contained in $2\int dR$. In terms of polar coordinates (12) becomes

$$\left(\frac{dr}{dt}\right)^2+r^2\cos^2 B\left(\frac{dv}{dt}\right)^2+r^2\left(\frac{dB}{dt}\right)^2-\frac{2\mu}{r}+\frac{\mu}{a}=2\int dR. \tag{14}$$

Again, multiplying (1) severally by x, y, z, and adding the products, we obtain, with the use of (11),

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}+\frac{\mu}{r}=r\frac{\partial R}{\partial r}, \tag{15}$$

or in polar coordinates

$$r\frac{d^2r}{dt^2}-r^2\cos^2 B\left(\frac{dv}{dt}\right)^2-r^2\left(\frac{dB}{dt}\right)^2+\frac{\mu}{r}=r\frac{\partial R}{\partial r}. \tag{16}$$

Adding (16) to (14) gives

$$\frac{1}{2}\frac{d^2r^2}{dt^2}-\frac{\mu}{r}+\frac{\mu}{a}=2\int dR+r\frac{\partial R}{\partial r}. \tag{17}$$

If from this is subtracted the equation

$$\frac{1}{2}\frac{d^2r_0^2}{dt^2}-\frac{\mu}{r_0}+\frac{\mu}{a}=0,$$

then, making use of (9),

$$\frac{d^2(r_0\delta r)}{dt^2}+\frac{\mu}{r_0^3}r_0\delta r=2\int dR+r\frac{\partial R}{\partial r}-\frac{1}{2}\frac{d^2(\delta r)^2}{dt^2}+\frac{\mu(\delta r)^2}{r_0^2 r}. \tag{18}$$

The two final terms of the right-hand member of this equation are of the order of the square of the disturbing force; if this is to be neglected they can be omitted.

By a similar process, (1) can be transformed into

$$\begin{aligned}\frac{d^2\delta x}{dt^2} + \frac{\mu}{r_0^3}\,\delta x &= \frac{\partial R}{\partial x} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu x,\\ \frac{d^2\delta y}{dt^2} + \frac{\mu}{r_0^3}\,\delta y &= \frac{\partial R}{\partial y} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu y,\\ \frac{d^2\delta z}{dt^2} + \frac{\mu}{r_0^3}\,\delta z &= \frac{\partial R}{\partial z} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu z.\end{aligned} \tag{19}$$

For brevity we now put

$$\begin{aligned}Q_r &= 2\int dR + r\,\frac{\partial R}{\partial r} - \frac{1}{2}\,\frac{d^2(\delta r)^2}{dt^2} + \frac{\mu(\delta r)^2}{r_0^2 r},\\ Q_x &= \frac{\partial R}{\partial x} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu x,\\ Q_y &= \frac{\partial R}{\partial y} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu y,\\ Q_z &= \frac{\partial R}{\partial z} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu z.\end{aligned} \tag{20}$$

Then the differential equations take the form

$$\begin{aligned}\frac{d^2(r_0\delta r)}{dt^2} + \frac{\mu}{r_0^3}\,r_0\delta r &= Q_r,\\ \frac{d^2\delta x}{dt^2} + \frac{\mu}{r_0^3}\,\delta x &= Q_x,\\ \frac{d^2\delta y}{dt^2} + \frac{\mu}{r_0^3}\,\delta y &= Q_y,\\ \frac{d^2\delta z}{dt^2} + \frac{\mu}{r_0^3}\,\delta z &= Q_z.\end{aligned} \tag{21}$$

Since μ/r_0^3 is a known function of t, and the Q are all of the order of the disturbing force and consequently in the first approximation are known in terms of t, these equations are linear with known final terms.

3. Integration. In order to integrate (21) let us consider the linear differential equation without final term

$$\frac{d^2q}{dt^2} + \frac{\mu}{r_0^3}\,q = 0. \tag{22}$$

According to the theory of this class of differential equations, the value of q has the form

$$q = K_1q_1 + K_2q_2, \tag{23}$$

K_1 and K_2 being the arbitrary constants, and q_1 and q_2 any two particular solutions which are independent of each other. Then there must necessarily exist the two equations

$$\begin{aligned}\frac{d^2q_1}{dt^2} + \frac{\mu}{r_0^3}\,q_1 &= 0,\\ \frac{d^2q_2}{dt^2} + \frac{\mu}{r_0^3}\,q_2 &= 0.\end{aligned} \tag{24}$$

By the elimination of μ/r_0^3 from these is obtained

$$q_1\frac{d^2q_2}{dt^2} - q_2\frac{d^2q_1}{dt^2} = 0. \tag{25}$$

This is an exact differential; integrating,

$$q_1\frac{dq_2}{dt} - q_2\frac{dq_1}{dt} = \text{const.} \tag{26}$$

The constant is arbitrary and, the value zero excepted, may be taken at will; for simplicity we take it equal to unity.

Taking now the more general equation, with a final term,

$$\frac{d^2q}{dt^2} + \frac{\mu}{r_0^3}\,q = Q, \tag{27}$$

let us eliminate μ/r_0^3 from this and (24). We get

$$\begin{aligned}q_1\frac{d^2q}{dt^2} - q\,\frac{d^2q_1}{dt^2} &= Qq_1,\\ q_2\frac{d^2q}{dt^2} - q\,\frac{d^2q_2}{dt^2} &= Qq_2,\end{aligned}$$

and, integrating,

$$\begin{aligned}q_1\frac{dq}{dt} - q\,\frac{dq_1}{dt} &= K_2 + \int q_1Q\,dt,\\ q_2\frac{dq}{dt} - q\,\frac{dq_2}{dt} &= -K_1 + \int q_2Q\,dt.\end{aligned} \tag{28}$$

From these we obtain, in view of (26),

$$q = K_1 q_1 + K_2 q_2 + q_2 \int q_1 Q\, dt - q_1 \int q_2 Q\, dt,$$
$$\frac{dq}{dt} = K_1 \frac{dq_1}{dt} + K_2 \frac{dq_2}{dt} + \frac{dq_2}{dt} \int q_1 Q\, dt - \frac{dq_1}{dt} \int q_2 Q\, dt. \tag{29}$$

The second of these equations may also be derived by differentiating the first. The arbitrary constants K_1 and K_2 may be regarded as contained in the integrals $\int q_2 Q dt$ and $\int q_1 Q dt$; hereafter we shall write the equations in this way.

Applying these results to (21) we have

$$\begin{aligned} r_0\, \delta r &= q_2 \int q_1 Q_r\, dt - q_1 \int q_2 Q_r\, dt,\\ \delta x &= q_2 \int q_1 Q_x\, dt - q_1 \int q_2 Q_x\, dt,\\ \delta y &= q_2 \int q_1 Q_y\, dt - q_1 \int q_2 Q_y\, dt,\\ \delta z &= q_2 \int q_1 Q_z\, dt - q_1 \int q_2 Q_z\, dt. \end{aligned} \tag{30}$$

Since $r^2 = x^2 + y^2 + z^2$, these equations must satisfy the relation

$$r_0 \delta r = x_0 \delta x + y_0 \delta y + z_0 \delta z + \frac{1}{2}(\delta x^2 + \delta y^2 + \delta z^2) - \frac{1}{2r_0^2}(r_0 \delta r)^2. \tag{31}$$

It is, however, necessary to employ all the equations of (30), since in proceeding by successive approximations, as we are obliged to do, we cannot get the values of Q_x, Q_y, Q_z until δr is known. These equations contain, in the general case, nine arbitrary constants: the one added to the term $2 \int dR$ in Q_r and the eight introduced by the eight integrals of (30). But these eight will be reduced to six constants independent of each other, by the condition (31); and the constant added to $2 \int dR$ will be determined as a function of these six by the condition derived from (12),

$$\frac{dx_0}{dt} \cdot \frac{d\delta x}{dt} + \frac{dy_0}{dt} \cdot \frac{d\delta y}{dt} + \frac{dz_0}{dt} \cdot \frac{d\delta z}{dt} + \frac{\mu}{r_0^2 r} r_0 \delta r$$
$$= \int dR - \frac{1}{2}\left[\left(\frac{d\delta x}{dt}\right)^2 + \left(\frac{d\delta y}{dt}\right)^2 + \left(\frac{d\delta z}{dt}\right)^2\right]. \tag{32}$$

In the case of osculating elements all the constants are determined by making each integral expression vanish with t.

4. Hansen's device. There is a remarkable device, due to P. A. Hansen, for reducing the right-hand members of (30) to a single integral expression, thus avoiding the difficulty, otherwise encountered in calculation, that the perturbations come out as the small differences of large numbers. The factors q_1 and q_2 outside the signs of integration may be moved within the signs, if it is agreed to regard the t that they contain as constant in the integration. As this t must then be kept distinct from the t of the quantities already under the sign of integration, we write it τ; and to denote that any quantity that is a function of t has its t changed to τ, we enclose it in parentheses. Thus making

$$N = (q_2)\, q_1 - (q_1)\, q_2, \tag{33}$$

we have the simple expressions

$$\begin{aligned} r_0 \delta r &= \int N Q_r\, dt, & \delta y &= \int N Q_y\, dt,\\ \delta x &= \int N Q_x\, dt, & \delta z &= \int N Q_z\, dt. \end{aligned} \tag{34}$$

After the integration is accomplished τ will be replaced by t. Since τ is regarded as constant in the integration, an arbitrary function of τ must be added to each of these expressions, which, after τ is changed to t, becomes an arbitrary function of t. These functions must in each case be determined so that the expressions (34) coincide with (30). However, it will not be necessary to consider these arbitrary functions if it is agreed to take the integrations between the upper limit t itself and a lower limit that may be any constant. In the general case, then, an arbitrary expression of the form

$$K_1 q_1 + K_2 q_2$$

must be added to each equation. In the case of osculating elements, if the lower limit is taken at zero, this arbitrary expression vanishes.

Equations (34) may be exhibited in the form of definite integrals; thus, since N is a symmetrical function of q and (q),

$$\begin{aligned} r_0 \delta r &= -\int_0^t N(Q_r)\, d\tau, & \delta y &= -\int_0^t N(Q_y)\, d\tau,\\ \delta x &= -\int_0^t N(Q_x)\, d\tau, & \delta z &= -\int_0^t N(Q_z)\, d\tau, \end{aligned} \tag{35}$$

where N may be regarded as a factor whose value is virtually zero, but a part of the time involved in its expression is regarded as constant in the integration.

5. The factors q_1 and q_2. We now determine the values of q_1 and q_2. If we put

$$n = \sqrt{\mu/a^3}, \qquad nt + c = l = u - e \sin u, \tag{36}$$

where n is the mean motion, a is half the major axis, e is the eccentricity, c the mean anomaly at the epoch, l and u the mean and eccentric anomalies of m in the ellipse that is adopted as the first approximation, then

$$\frac{r_0}{a} = 1 - e \cos u, \qquad dl = \frac{r_0}{a}\, du. \tag{37}$$

Equation (22) may then be transformed into

$$\frac{d^2 q}{dl^2} + \frac{a^3}{r_0^3}\, q = 0.$$

Further, if u is made the independent variable, it becomes

$$(1 - e \cos u) \frac{d^2 q}{du^2} - e \sin u \frac{dq}{du} + q = 0. \tag{38}$$

Differentiating this, and afterward removing the useless factor $1 - e \cos u$, we get

$$\frac{d^3 q}{du^3} + \frac{dq}{du} = 0,$$

the integral of which is

$$q = K_1 \cos u + K_2 \sin u + K_3.$$

In order that this may satisfy (38) we put $K_3 = -K_1 e$. Hence the complete integral of (38) is

$$q = K_1(\cos u - e) + K_2 \sin u. \tag{39}$$

It is evident now that we may take

$$q_1 = k(\cos u - e), \qquad q_2 = k \sin u,$$

k (not to be confused with the Gaussian constant) being adopted so as to satisfy (26). When these values are substituted it is found that $k^2 = 1/n$. Consequently,

$$\begin{aligned} q_1 &= \sqrt{a^3 n/\mu}\,(\cos u - e) = \sqrt{an/\mu}\; r_0 \cos f, \\ q_2 &= \sqrt{a^3 n/\mu}\, \sin u \qquad = \sqrt{an/\mu(1 - e^2)}\; r_0 \sin f, \end{aligned} \tag{40}$$

where f denotes the true anomaly of the disturbed planet in its elliptic orbit. Substituting these values in (33) we get the two expressions for N,

$$\begin{aligned} N &= \frac{a^3 n}{\mu} \{\sin [(u) - u] - e \sin (u) + e \sin u\} \\ &= \frac{an}{\mu \sqrt{1 - e^2}}\, (r_0)\, r_0 \sin [(f) - f]. \end{aligned} \tag{41}$$

If t is retained as the independent variable, either of these values may be used in (34). But in some cases it may be desired to integrate with reference to u or f, and since

$$n dt = \frac{r_0}{a}\, du = \frac{r_0^2}{a^2 \sqrt{1 - e^2}}\, df, \tag{42}$$

we should have

$$\begin{aligned} N\, dt &= \frac{a^2 r_0}{\mu} \{\sin [(u) - u] - e \sin (u) + e \sin u\}\, du \\ &= \frac{1}{\mu a(1 - e^2)}\, (r_0)\, r_0^3 \sin [(f) - f]\, df. \end{aligned} \tag{43}$$

In the latter case the expressions for the perturbations become

$$\begin{aligned} \delta r &= \frac{1}{\mu a(1 - e^2)} \int Q_r\, r_0^3 \sin [(f) - f]\, df, \\ \delta x &= \frac{r_0}{\mu a(1 - e^2)} \int Q_x\, r_0^3 \sin [(f) - f]\, df, \\ \delta y &= \frac{r_0}{\mu a(1 - e^2)} \int Q_y\, r_0^3 \sin [(f) - f]\, df, \\ \delta z &= \frac{r_0}{\mu a(1 - e^2)} \int Q_z\, r_0^3 \sin [(f) - f]\, df. \end{aligned} \tag{44}$$

These equations are entirely rigorous, as no terms have been omitted in their derivation; but in the practical application of them one is subject to the necessity of deriving the values of the Q in terms of the independent variable by a series of approximations. In the first of these the Q will be affected with errors which are of two dimensions with respect to the disturbing forces; in the second with errors of three dimensions; and so on. One advantage these equations possess is that the factors by which the Q must be multiplied prior to the integration are pure functions of the coordinates in the ellipse of the first approximation; they remain identically the same, however far the approximations may be carried out. A similar advantage is possessed by the equations of Brouwer's method, discussed later in this chapter.

The set of equations (44), although very symmetrical, present the inconvenience of containing one more relationship than is necessary. Hence for the second and third we shall substitute a single equation. From (1) is deduced

$$x\frac{dy}{dt} - y\frac{dx}{dt} = H + \int\left(x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x}\right)dt, \tag{45}$$

where the constant H is not to be confused with the H of (4), and in terms of polar coordinates

$$r^2\cos^2 B\frac{dv}{dt} = H + \int\frac{\partial R}{\partial v}\,dt. \tag{46}$$

If we prefer the rectangular coordinate z to the variable B, (46) may be written

$$(r^2 - z^2)\frac{dv}{dt} = H + \int\frac{\partial R}{\partial v}\,dt. \tag{47}$$

H is a constant such that

$$x_0\frac{dy_0}{dt} - y_0\frac{dx_0}{dt} = (r_0^2 - z_0^2)\frac{dv_0}{dt} = H. \tag{48}$$

It can be shown that the numerical value of H is given by

$$H = \sqrt{\mu a(1 - e^2)}\cos i,$$

where i is the inclination of the plane of the elliptic orbit to the plane of xy.

Supposing $v = v_0 + \delta v$, the following equation is obtained for the determination of δv from (47), putting Q_v for $\partial R/\partial v$.

$$\delta v = \int\left\{\int\frac{an}{\mu}Q_v\,dt - \sqrt{1 - e^2}\cos i\,\frac{(r + r_0)\,\delta r - (z + z_0)\,\delta z}{r_0^2 - z_0^2}\right\}\frac{a^2 n}{r^2 - z^2}\,dt. \tag{49}$$

By substituting for ndt either of its values from (42), we can make u or f the independent variable. With the latter procedure (49) becomes

$$\delta v = \int\left\{\int\frac{r_0^2}{\mu p}Q_v\,dv - \cos i\,\frac{(r + r_0)\,\delta r - (z + z_0)\,\delta z}{r_0^2 - z_0^2}\right\}\frac{r_0^2}{r^2 - z^2}\,dv. \tag{50}$$

Like the equations (44) this is an entirely rigorous equation, no terms having been neglected. Together with the first and last equations of (44) it suffices for the complete solution of the problem. The equation is also perfectly general since no restrictions have been put upon the position of the plane of xy, from which the coordinate z is measured. In the case where the plane of the elliptic orbit of reference is adopted

as the plane of xy, (50) is somewhat simplified. In that case $i = 0$, $z_0 = 0$ and $z = \delta z$; thus

$$\delta v = \int\left\{\int\frac{r_0^2}{\mu p}Q_v\,dv - \frac{(r + r_0)\,\delta r - \delta z^2}{r_0^2}\right\}\frac{r_0^2}{r^2 - \delta z^2}\,dv. \tag{51}$$

6. The superfluous constant. In using the first and last equations of (44), and (50) or (51), seven arbitrary constants will be introduced, three in the equation that determines δr and two in each of the others. One of these is superfluous and must be determined as a function of the rest. When we are deriving perturbations to be applied to coordinates given by osculating elements the difficulty is readily overcome; we have only to add to each integral a constant that will make it vanish at the epoch. But when the perturbations are to be added to coordinates derived from mean elements, the readiest method is to suppose that the constant added to $\int dR$ is the superfluous one. Then, as stated before, Eq. (32) will determine this constant in terms of the remaining six. In developing both members of this equation in periodic series it will be necessary to retain only the nonperiodic terms. The nonperiodic term of $\int dR$ must be the same as that of

$$\frac{dx_0}{dt}\cdot\frac{d}{dt}\delta x + \frac{dy_0}{dt}\cdot\frac{d}{dt}\delta y + \frac{dz_0}{dt}\cdot\frac{d}{dt}\delta z + \frac{1}{2}\left[\left(\frac{d}{dt}\delta x\right)^2 + \left(\frac{d}{dt}\delta y\right)^2 + \left(\frac{d}{dt}\delta z\right)^2\right] + \frac{\mu}{r_0 r}\delta r.$$

If the plane of the reference ellipse is taken as the plane of xy, we shall have $dz_0/dt = 0$, and one of the terms of this equation will disappear. Suppose that this is done, and that the coordinates chosen for defining the position of m are r, v, and δz; then (32) can be made to take the form

$$\frac{1}{2}\frac{d}{dt}(r + r_0)\cdot\frac{d}{dt}\delta r + \frac{1}{2}\left(r\frac{dv}{dt} + r_0\frac{dv_0}{dt}\right)\left(r_0\frac{d}{dt}\delta v + \frac{dv}{dt}\delta r\right) + \frac{\mu}{r_0 r}\delta r$$
$$= \int dR + \frac{1}{2}\left(\frac{dv}{dt}\right)^2\delta z^2 - \frac{1}{2}\frac{r^4}{r^2 - \delta z^2}\left(\frac{d}{dt}\frac{\delta z}{r}\right)^2. \tag{52}$$

We can make v_0 the independent variable in place of t by substituting $dt = r_0^2/\sqrt{\mu p}\,df$, which gives

$$\frac{1}{2}\frac{d}{df}(r + r_0)\frac{d}{df}\delta r + \frac{1}{2}\left(r\frac{dv}{df} + r_0\right)\left(r_0\frac{d}{df}\delta v + \frac{dv}{df}\delta r\right) + \frac{r_0^3}{pr}\delta r$$
$$= \frac{r_0^4}{\mu p}\int dR + \frac{1}{2}\left(\frac{dv}{df}\right)^2\delta z^2 - \frac{1}{2}\frac{r^4}{r^2 - \delta z^2}\left(\frac{d}{df}\frac{\delta z}{r}\right)^2. \tag{53}$$

If we retain only terms of the first order with respect to disturbing forces, this reduces to

$$p\frac{d}{df}\,\delta v + (2 + e\cos f)\,\delta r + e\sin f\frac{d}{df}\,\delta r = \frac{r^2}{\mu}\int dR. \tag{54}$$

The difficulty with the superfluous constant can be avoided by employing $\int dR$ instead of the function Q_v. Subtracting twice (18) from (52), and employing the equation given by elliptic theory

$$\frac{d^2r_0}{dt^2} - r_0\left(\frac{dv_0}{dt}\right)^2 + \frac{\mu}{r_0^2} = 0$$

for reducing the result, we get

$$r_0^2\frac{dv_0}{dt}\cdot\frac{d}{dt}\,\delta v = \frac{d}{dt}\left(\frac{dr_0}{dt}\,\delta r + 2r_0\frac{d}{dt}\,\delta r\right) - 3\int dR - 2r\frac{\partial R}{\partial r} + \frac{1}{2}\frac{d^2}{dt^2}\,\delta r^2 - \frac{\mu\delta r^2}{r_0^2 r}$$
$$- \frac{1}{2}\left(\frac{dv}{dt}\right)^2(\delta r^2 - \delta z^2) - \frac{1}{2}r_0\delta r\frac{d}{dt}\,\delta v\left(\frac{dv}{dt} + 2\frac{dv_0}{dt}\right) - \frac{1}{2}\frac{r^4}{r^2 - \delta z^2}\left(\frac{d}{dt}\frac{\delta z}{r}\right)^2. \tag{55}$$

No terms have been neglected in this equation; if we limit ourselves to the first order of disturbing forces it takes the simpler form

$$\delta v = \frac{\dfrac{dr}{dt}\,\delta r + 2r\dfrac{d}{dt}\,\delta r - \displaystyle\int\left(3\int dR + 2r\frac{\partial R}{\partial r}\right)dt}{a^2n\sqrt{1 - e^2}}. \tag{56}$$

If we make f the independent variable, this equation becomes

$$\delta v = \frac{2}{p}(1 + e\cos f)\frac{d}{df}\,\delta r + \frac{e}{p}\sin f\cdot\delta r - \frac{1}{\mu}\int\left(3\int dR + 2r\frac{\partial R}{\partial r}\right)\frac{p\,df}{(1 + e\cos f)^2}. \tag{57}$$

If (55), (56), or (57) is used in place of (51), the number of arbitrary constants involved in the expressions for the perturbations will be six, the proper number. The first-mentioned equations also have the advantage that δr, as it appears in the expression for δv, is free from the sign of integration, which is not the case with (51). Moreover, in expanding the quantities to be integrated in periodic series, the determination of the coefficients of terms of long period to additional accuracy is limited to the single quantity dR.

7. Perturbations of the first order. We give here the formulas for the case where the true anomaly f is employed as the independent variable, because of the simplicity of the analytical expressions. It is not to be understood, however, that this independent variable is

necessarily the one to be preferred in numerical applications. The use of the eccentric anomaly renders the series more rapidly convergent, an important advantage when the eccentricity of the disturbed object is large; on the other hand, use of the mean anomaly facilitates the calculation of positions of the disturbed object, and simplifies the process of integration. However, the modifications to be made in the formulas when either of the last two variables named is to be employed are readily perceived.

Since, when we limit ourselves to the first order of disturbing forces, elliptic values are to be substituted for the coordinates in the functions Q, there is no need to make a distinction between the elliptic and the rigorous values, we shall omit the zero subscripts. If we put

$$\begin{aligned} T &= \frac{r^3}{\mu p}Q_r = \frac{r^3}{\mu p}\left[2\int dR + r\frac{\partial R}{\partial r}\right],\\ Y &= \frac{r^2}{\mu p}\frac{\partial R}{\partial v},\\ Z &= \frac{r^3}{\mu p}\frac{\partial R}{\partial z}, \end{aligned} \tag{58}$$

the first and last of (44) and (51) reduce to

$$\begin{aligned} \delta r &= \int T\sin[(f) - f]\,df,\\ \delta v &= \int\left[\int Y\,df - 2\frac{\delta r}{r}\right]df,\\ \delta B &= \int Z\sin[(f) - f]\,df. \end{aligned} \tag{59}$$

By referring to the value of R in (2) the expressions for Y and Z are readily seen. In order to find T, put

$$X = \frac{r^4}{\mu p}\frac{\partial R}{\partial r}. \tag{60}$$

Then it is found that

$$\frac{1}{\mu p}\,dR = r^{-2}\left[\frac{e}{p}\sin f\cdot X + Y\right]df. \tag{61}$$

Thus the equations for the perturbations of the first order are

$$\begin{aligned} \delta r &= \int\left[X + 2r^3\int r^{-2}\left(\frac{e}{p}\sin f\cdot X + Y\right)df\right]\sin[(f) - f]\,df,\\ \delta v &= \int\left[\int Y\,df - 2\frac{\delta r}{r}\right]df,\\ \delta B &= \int Z\sin[(f) - f]\,df. \end{aligned} \tag{62}$$

The equation (57) can be substituted for the second of these, but if this is done it will be well to use the function $\int dR$ instead of Y. If the coordinates r, r', z, z' are eliminated from R by means of their expressions in terms of the true anomalies f, f', then R becomes a function of f and f' only, and

$$\int dR = \int \frac{\partial R}{\partial f}\, df. \tag{63}$$

This equation is also true if f' is replaced by u' or l'. But in order to make the various expressions integrable with f as the independent variable it will be necessary to eliminate l' by means of the identity

$$l' = \frac{n'}{n} f + c' - \frac{n'}{n} c - \frac{n'}{n}(f - l). \tag{64}$$

We put then

$$\theta' = \frac{n'}{n} f + c' - \frac{n'}{n} c, \tag{65}$$

and thus

$$l' = \theta' - \frac{n'}{n}(f - l), \qquad \frac{d\theta'}{df} = \frac{n'}{n}. \tag{66}$$

With R in this form we shall have

$$\int dR = R - \frac{n'}{n} \int \frac{\partial R}{\partial \theta'} \frac{r^2}{a^2\sqrt{1-e^2}}\, df, \tag{67}$$

and

$$r \frac{\partial R}{\partial r} = a \frac{\partial R}{\partial a}. \tag{68}$$

If, however, Eqs. (62) are employed, Eqs. (2), (4), and (5) show that the expressions for X, Y, Z have the forms

$$\begin{aligned} X &= \frac{k^2 m'}{\mu p} r^4 \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) r' \cos B' \cos (v' - v) - \frac{k^2 m'}{\mu p} \frac{r^5}{\Delta^3}, \\ Y &= \frac{k^2 m'}{\mu p} r^4 \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) r' \cos B' \sin (v' - v), \\ Z &= \frac{k^2 m'}{\mu p} r^4 \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) r' \sin B', \end{aligned} \tag{69}$$

in which

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos B' \cos (v' - v). \tag{70}$$

When the variable θ' of Eq. (65) is employed in treating perturbations of the first order, the periodic development of each of the functions to be integrated, either with or without previous multiplication by the factor $\sin [(f) - f]$, will have the form

$$\sum_{j,j'} \left[K^{(c)}_{j,j'} \cos (jf + j'\theta') + K^{(s)}_{j,j'} \sin (jf + j'\theta')\right], \tag{71}$$

where j and j' are positive or negative integers (including zero), but one of them may always be restricted to positive values only (including zero) without loss of generality. When this expression is integrated with f as the independent variable, the result is

$$\sum_{j,j'} (j,j') \left[K^{(c)}_{j,j'} \sin (jf + j'\theta') - K^{(s)}_{j,j'} \cos (jf + j'\theta')\right], \tag{72}$$

where we adopt the notation

$$(j,j') = \frac{1}{j + j'(n'/n)}, \tag{73}$$

except for the absolute term, which being integrated gives $K^{(c)}_{0,0} f$. If the expression is multiplied by $\sin [(f) - f]$ before integration, the result of the integration is

$$\sum_{j,j'} \left\{-(j-1,j')(j+1,j')\left[K^{(c)}_{j,j'} \cos (jf + j'\theta') + K^{(s)}_{j,j'} \sin (jf + j'\theta')\right]\right\} \tag{74}$$

except for the terms having $j = 1$, $j' = 0$, for which we have

$$\frac{f}{2}\left(K^{(c)}_{1,0} \sin f - K^{(s)}_{1,0} \cos f\right). \tag{75}$$

The remaining terms that are proportional to $\cos f$ and $\sin f$ may be omitted as they combine with the arbitrary expression that completes the integral.

At this stage we might proceed to the analytic elaboration of perturbations of the second order, which are increments to the perturbations of the first order and are obtained by calculating the increments of the disturbing forces on the supposition that the planets move not in ellipses but in ellipses augmented by perturbations of the first order. The resulting developments would not, however, be well adapted to numerical calculation, and we prefer to postpone them until we treat a method not subject to this disability. In the meantime we proceed to further consideration of (75).