# An Analytic Proof that the Hohmann-Type Transfer is the True Minimum Two-Impulse Transfer


By

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(With 8 Figures)


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**Abstract — Zusammenfassung — Résumé**

**An Analytic Proof that the Hohmann-Type Transfer is the True Minimum Two-Impulse Transfer.** Although it is commonly assumed that the Hohmann transfer is the optimum two-impulse transfer, an analytic proof has never been given. Ting [1] proved the conjecture for the "nearly Hohmann-type" orbit. An analytic proof of this conjecture in full generality is presented in this paper.

**Ein analytischer Beweis für den minimalen Zwei-Impuls-Übergang entlang der Hohmann-Ellipse.** Es wird gezeigt, daß die Hohmann-Ellipse der optimale Zwei-Impuls-Übergang ist. Ein analytischer Beweis wurde bisher noch nicht gegeben. Ting [1] untersuchte diese Vermutung für Hohmann-ähnliche Bahnen. Ein analytischer Beweis für diese Vermutung wird in dieser Arbeit in voller Allgemeinheit gegeben.

**Une preuve analytique du caractère minimal d'un transfert du type de Hohmann.** Aucune preuve analytique n'a été avancée jusqu'à présent pour établir le caractère minimal du transfert du type Hohmann pour le cas de deux impulsions. Ting [1] a donné une preuve pour une orbite quasi-Hohmannienne. Une preuve générale est présentée ici.


## Introduction

The present paper is motivated by the concluding paragraph in Ting [2], which reads: "The optimum solution of transfer discussed in the present paper will be the optimum among all possible trajectories instead of among the 'nearly Hohmann-type' trajectories, if it could be conjectured that the optimum two-impulse transfer of [1] is the true optimum among all the possible two-impulse transfers instead of only among the 'nearly Hohmann-type' as shown in [1]. All the available numerical results agree with this conjecture. Recently, Horner [5] extended the work by Munick, McGill, and Taylor [6] and verified this conjecture for non-intersecting orbits. Mathematical proof of the conjecture for intersecting orbits would bring the problem of optimum transfer between two elliptical orbits by impulse, to a close."

In this paper a mathematical proof will be given verifying this conjecture. The proof will cover both intersecting and non-intersecting orbits. By the Hohmann-type transfer between two orbits we will mean that the two orbits and

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the transfer trajectory are co-planar, with the same major axis and oriented in the same sense; and that the apogee of the transfer trajectory coincides with that of the orbit with larger apogee, while the perigee of the transfer trajectory coincides with that of the other orbit. By optimum transfer we will mean that the total velocity change (sum of the absolute values of the various velocity changes) will be a minimum.

### Background

We will use the results and methods of our paper [4] in this paper. In [4] we proved the result that even if orbits 1 and 2 are co-planar and can intersect, then the optimum one-impulse transfer at their point of intersection is greater than if one rotated the orbits and made a HOHMANN-type transfer between the two orbits.

In [4] we quoted a Theorem of WHITTAKER [3, p. 89].

### Whittaker Theorem

It can be shown that the velocity at any point on an elliptic orbit can be resolved into a component $y$ perpendicular to the radius vector, and a component $x$ (less than $y$) which is perpendicular to the axis of the conic, each of these components being constant.

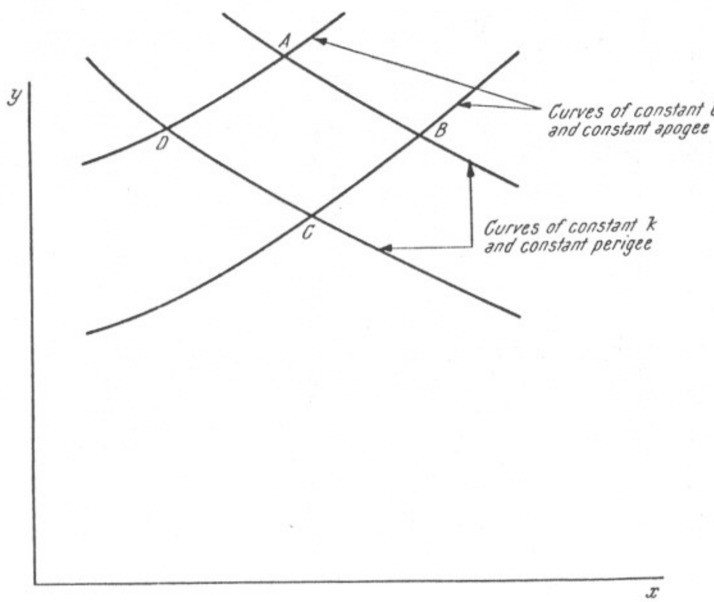


Fig. 1. The $x - y$ plane

The components $x$ and $y$ of velocity uniquely determine the orbit; in particular one can show

$$(\mu/r) = y^2 + y\,x\cos f \tag{1}$$

where

$\mu$ = universal gravitational constant times mass of earth
$r$ = distance from center of earth
$f$ = true anomaly.



We further will use the notation

$$k = \mu/r_{\text{perigee}} = y^2 + x\,y \tag{2}$$

$$l = \mu/r_{\text{apogee}} = y^2 - x\,y \tag{3}$$

Hence:

$$y = \sqrt{(k+l)/2} \tag{4}$$

$$x = (k-l)/\sqrt{2(k+l)} \tag{5}$$

The curves of constant $k$ and $l$ are hyperbolas in the $x$, $y$ plane (see discussion in [4]), oriented as in Fig. 1. When we speak of one-impulse perigee to perigee transfers between orbits 1 and 2, then $(x_1, y_1)$ and $(x_2, y_2)$ must be on the same constant $k$ hyperbola; otherwise they would not have the same perigee distance. Similar remarks apply for apogee to apogee transfers and constant $l$ hyperbolas.

From the definition of $y$ and $x$ in WHITTAKER's Theorem one has velocity at perigee $= y + x$, velocity at apogee $= y - x$.

### Proof of Basic Assertion

We will now begin our discussion of transferring between orbits 1 and 2 by means of a third orbit—4. (See Fig. 2.) We assume all orbits have a common focus, and are co-planar. Without any loss of generality, we will assume that of orbits 1 and 2, orbit 2 is the one with the largest apogee, i.e., $(r_a)_2 > (r_a)_1$ or $l_1 > l_2$.

*Case 1.* $(r_a)_4 \leqslant (r_a)_2$ *or* $l_4 \geqslant l_2$

We show in this section that the HOHMANN transfer orbit is minimum in this case.

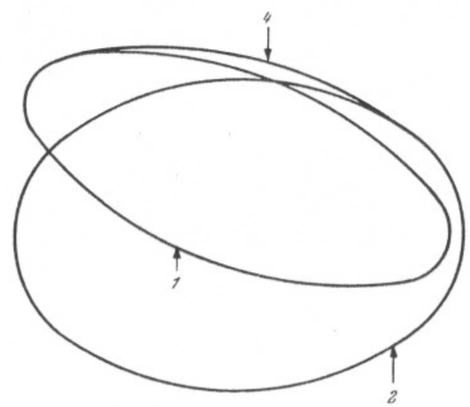


Fig. 2. Intersecting ellipses with common focus

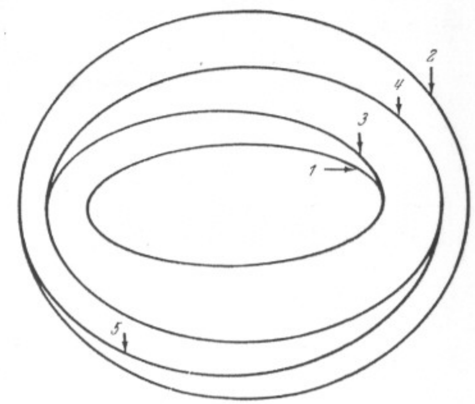


Fig. 3. Intersecting orbits lined up

Here we can proceed, partially using the logic of the arguments in TING [2], Sections II and III but not completely, since TING actually assumes the HOHMANN-type transfer is a minimum, and this is just what we want to prove.

Since orbits 4 and 2 intersect, we can apply the results of [4] to say that the one-impulse transfer between them will be minimized by a HOHMANN-type transfer with orbit 5 between apogee of 2 and perigee of 4. (See Fig. 3.)

Likewise since 4 and 1 intersect, their one-impulse transfer will be minimized by a HOHMANN-type transfer orbit 3 between them (see Fig. 3), but here we do not know which of 4 and 1 has the larger apogee, only that both their apogees are less than that of 2. Similarly the perigee of orbit 4 may be larger or smaller than that of orbits 2 and 1. An $x-y$ plot of the various possibilities for orbit 4 are shown in Fig. 4. [The possibilities are designated by $4$, $4^{I}$, $4^{II}$, $4^{III}$, $4^{IV}$, $4^{V}$, in the Figure.] Point 6 represents the HOHMANN-type transfer orbit between orbits 1 and 2; note that our discussion does not depend on whether orbits 1 and 2 intersect or not. Further, note that since we have reduced the total velocity change by introducing the orbits 3 and 5, if we can prove that even so the resulting total change in velocity is greater than that of the HOHMANN-type transfer between orbits 1 and 2 we have proved our assertion. We will do this below.

Before proceeding we will need some estimates about velocity changes. Thus, in Fig. 1, let orbits $A$, $B$, $C$, $D$, be on hyperbolas of constant $k$ and $l$ as shown in the Figure, and let $V_{AD}(V_{BC})$ be the absolute velocity change in making an apogee to apogee change between orbits $A$ and $D$, ($B$ and $C$); and $V_{AB}(V_{DC})$ be the absolute velocity change in making a perigee to perigee change between orbits $A$ and $B$ ($D$ and $C$).

Then we have the estimates

$$V_{AD} + V_{DC} > V_{AB} + V_{BC} \tag{6}$$
$$V_{BA} + V_{AD} > V_{BC} + V_{CD} \tag{7}$$

Physically these two inequalities say that in a perigee-apogee transfer it is always better to go to the apogee of the orbit with the largest apogee, rather than its perigee. Thus, they follow from results in TING [1]. The method of proving them in the present framework is as follows:

Writing (6) out in terms of $y$'s and $x$'s it is equivalent to

$$(x_D - x_C) > (x_A - x_B)$$

but this is just Estimate (2) of [4].

Inequality (7) is equivalent to

$$(y_D - y_C) > (y_A - y_B)$$

which in turn is equivalent to showing that $H'(k) < 0$ for $H(k) = \sqrt{(k + l_1)/2} - \sqrt{(k + l_2)/2}$ for $l_1 > l_2$, which in turn is obvious.

From (6) and (7) it follows immediately that

$$V_{AD} + V_{DC} + V_{CB} > V_{AB} \tag{8}$$
$$V_{BA} + V_{AD} + V_{DC} > V_{BC} \tag{9}$$
$$V_{CB} + V_{BA} + V_{AD} > V_{CD}. \tag{10}$$

Returning to our main line of thought, we wish to show that any intermediate transfer to an orbit 4, $4^{I}$, $4^{II}$, etc., always takes more impulse change than a HOHMANN-type transfer between orbits 1 and 2. We will consider three possibilities; the others can be proven by some obvious modifications of our reasoning.

*Subcase 1*

Show

$$V_{13} + V_{34} + V_{45} + V_{52} > V_{16} + V_{62}$$

Proof: Since $V_{16} = V_{13} + V_{36}$ and $V_{52} = V_{56} + V_{62}$, as follows by writing out these expressions in terms of $y$'s and $x$'s; this inequality is equivalent to

$$V_{34} + V_{45} + V_{56} > V_{36}$$

which is just (10).



*Subcase 2*

Show

$$V_{13} + V_{34^I} + V_{4^I5^I} + V_{5^I2} > V_{16} + V_{62}$$

Proof: Arguing as above this is equivalent to

$$V_{34^I} + V_{4^I5^I} > V_{36} + V_{65^I},$$

which is just (6).

*Subcase 3*

Show

$$V_{13^{II}} + V_{3^{II}4^{II}} + V_{4^{II}5} + V_{52} > V_{16} + V_{62}$$

Proof: Since

$$V_{13^{II}} + V_{3^{II}4^{II}} + V_{4^{II}5} + V_{52} = V_{13^{II}} + 2\,V_{3^{II}4^{II}} + V_{3^{II}5} + V_{52} > V_{13^{II}} + V_{3^{II}5} + V_{52}$$

we need only prove:

$$V_{13^{II}} + V_{3^{II}5} + V_{56} > V_{16}$$

which is just inequality (6) again.

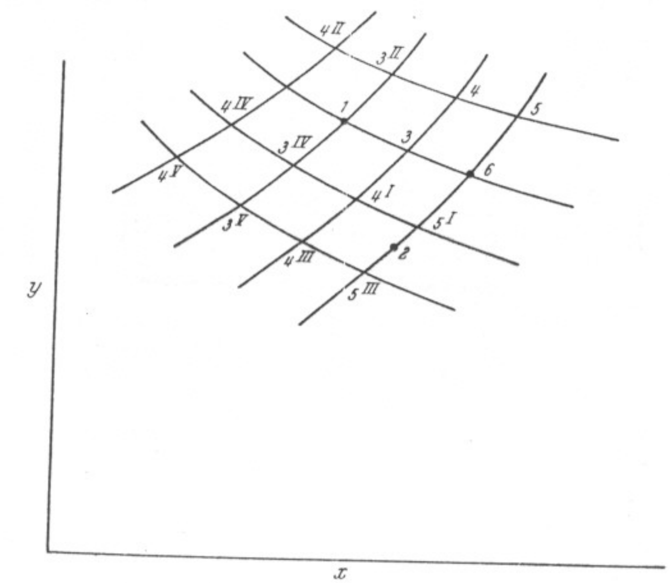


Fig. 4. The possible locations of orbit 4

The other possibilities follow similarly. Hence, this part of the proof is complete. Although we have not considered it specifically, it is clear from Fig. 4, that the same method of proof even works if $(r_a)_4 = (r_a)_2$.

## Case 2. $(r_a)_4 > (r_a)_2$ or $l_4 < l_2$

In this section we will prove that there are no minima in the whole region $(l_4 < l_2)$; that is, given any orbit 4 in this region, there is a neighboring orbit 4′ or a neighboring configuration of the three orbits 1, 2, 4 (i.e., a rotation of the orbits about their common focus, with each orbit rotated differently), that will accomplish the transfer from orbit 1 to orbit 2, with less total velocity change. (Actually the orbits 4' tend toward orbits of Case 1, but we will not have need for this fact.)

### Method of Procedure

The total change of velocity in going from orbit 4 to orbit 2, with one impulse at the point where they intersect is:

$$T_2 = \sqrt{((y_4 + x_4 \cos f) - (y_2 + x_2 \cos g))^2 + (x_4 \sin f - x_2 \sin g)^2} \qquad (11)$$

$$= \sqrt{((y_4 + x_4 \cos f) - (y_2 + x_2 \cos g))^2 + (\sqrt{x_4^2 - (x_4 \cos f)^2} - x_2 \sin g)^2} \qquad (11')$$

where $f$ and $g$ are the true anomalies of orbits 4 and 1 respectively. This equation follows directly from the definitions in Whittaker's Theorem when one resolves the velocity perpendicular and parallel to the radius vector. We restrict $f$ and $g$ to be between 0° and 180°. Then the last two terms in (11) always subtract. This minimizes the $T_2$ that satisfies (13) below.

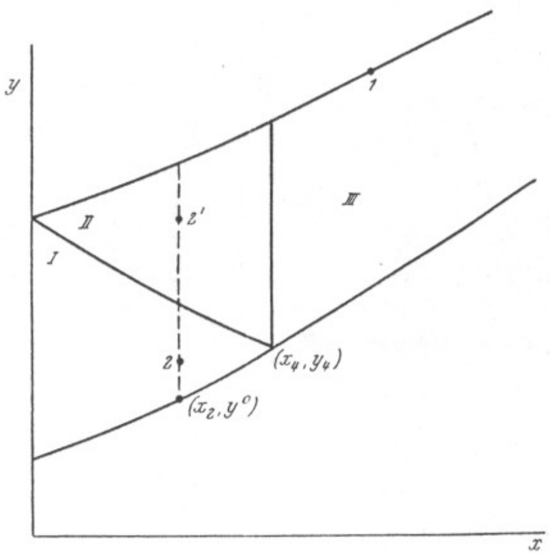


Fig. 5. Subcases I, II, III

It is of course obvious, but well to remember, that both one-impulse transfers $T_2$ and $T_1$ (defined analogously to $T_2$), must be optimum one-impulse transfers with respect to orbit 4. Otherwise one could find a neighboring configuration that would decrease the total velocity change.

Our method of procedure below will be the following: First one observes from eq. (11') that we may consider $T_2 = T_2(x_4, x_4 \cos f)$ with $x_2$ and $g$ fixed, so that one finds

$$(\partial T_2/\partial x_4)\, x_4 \cos f =$$
$$= ((x_4 \sin f - x_2 \sin g)/T_2) \cdot$$
$$\cdot (x_4/(x_4 \sin f)) \qquad (12)$$

(When $x_4 \cos f$ is constant, eq. (13) below is easily satisfied.)

Below we will prove that except for a few exceptional points where the difference is equal to zero (Points $A, B, C$ of Figs. 7 and 8), that $x_4 \sin f - x_2 \sin g > 0$, in the following instances: (Subcases I, II, III are defined below).

Assertion A. In Subcase I, $x_4 \sin f - x_2 \sin g > 0$.

Assertion B. In Subcases II and III at the optimum one-impulse transfer point between orbits 2 and 4, $x_4 \sin f - x_2 \sin g > 0$.

Thus, except for the few exceptional points (which will also be discussed), it follows from (12), and the analogous equation for orbit 1, that one can always decrease $x_4$ slightly and this will decrease the total velocity change. This will prove the stated result for Case 2, with $x_4' = x_4 - \varepsilon$, $y_4' = y_4$.

We will have to treat three subcases as shown symbolically in Fig. 5. We will only discuss the relation between $(x_4, y_4)$ and $(x_2, y_2)$, but the same discussion



will apply to the relation between $(x_4, y_1)$ and $(x_1, y_1)$. In Subcase III $x_2 > x_4$; in Subcase I and II $x_2 < x_4$, but in Subcase I $k_2 = y_2^2 + y_2 x_2 < y_4^2 + y_4 x_4 = k_4$, while in Subcase II $k_2 > k_4$.

*Subcase I*

In Fig. 6, a plot of $x_2 \cos g$ vs $x_4 \cos f$ is shown, where $g$ is the true anomaly of ellipse 2, and $f$ of ellipse 4. For these two ellipses to meet at a distance $r$, one has

$$\mu/r = y_2^2 + y_2(x_2 \cos g) = y_4^2 + y_4(x_4 \cos f) = k \qquad (13)$$

which means that for fixed $x_2, y_2, x_4, y_4$ the plot is one of a straight line as shown in Fig. 6. [Eq. (13) defines $k$ as a generalization of the $k$ of eq. (2). The $k$ of eq. (2) from now on will have a subscript.]

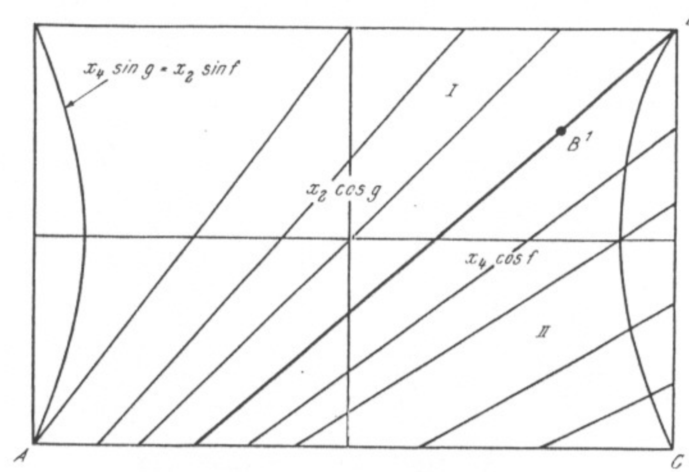


Fig. 6. $x_2 \cos g$ versus $x_4 \cos f$ in Subcases I and II

For Subcase I, we have

$$y_4^2 - y_4 x_4 < y_2^2 - y_2 x_2 < y_2^2 + y_2 x_2 < y_4^2 + y_4 x_4 \qquad (14)$$

hence, there exist $f_1$ and $f_2$ such that

$$\begin{aligned} y_2^2 + y_2 x_2 &= y_4^2 + y_4 x_4 \cos f_1 \\ y_2^2 - y_2 x_2 &= y_4^2 + y_4 x_4 \cos f_2 \end{aligned} \qquad (15)$$

Thus the lines in Subcase I hit $x_2 \cos g = + x_2$ and $x_2 \cos g = - x_2$, as shown in Fig. 6. Moreover from (13) one has

$$x_2 \cos g = (y_4/y_2)\, x_4 \cos f + y_4^2 - y_2^2 \qquad (16)$$

hence the lines have positive slope that decreases with increasing $y_2$. This is also shown in Fig. 6.

The hyperbolas in Fig. 6 are the curves $x_4 \sin f = x_2 \sin g$, or $x_4^2 - (x_4 \cos f)^2 = x_2^2 - (x_2 \cos g)^2$. As is easily checked for these hyperbolas: when $x_2 \cos g = 0$, $(x_4 \cos f) = \pm \sqrt{x_4^2 - x_2^2}$, and the slope of the tangent at the points $(x_4, x_2)$ and $(-x_4, -x_2)$ is $x_4/x_2$.

We now wish to show that each line of the family of lines in Subcase I only hits the hyperbola $(x_4 \sin f = x_2 \sin g)$ once in the region $-x_4 \leqslant x_4 \cos f \leqslant x_4$,

$-x_2 \leqslant x_2 \cos g \leqslant x_2$ as shown in Fig. 6. To prove this we choose the smallest $y_2$ possible (this is the point $y_2^0$ in Fig. 5). This will make the slope of the line i.e., $(y_4/y_2^0)$ maximum; however, below we prove

$$(x_4/x_2) > (y_4/y_2^0) > (y_4/y_2) \tag{17}$$

thus the slope of the tangent to the hyperbola at $(x_4, x_2)$ and $(-x_4, -x_2)$ is greater than that of any line (13) and hence, the lines (13) only hit the hyperbola once as shown.

*Proof of* (17): From Fig. 5, and eqs. (13) and (14):

$$(y_2^0)^2 - (y_2^0)\,x_2 - [y_4^2 - y_4\,x_4] = 0$$

Thus,

$$y_2^0 = (x_2 + \sqrt{4\,(y_4^2 - y_4\,x_4) + x_2^2})/2$$
$$y_2^0/x_2 = (1 + \sqrt{(4\,(y_4^2 - y_4\,x_4)/x_2^2) + 1})/2$$

But since $x_4 > x_2$

$$y_2^0/x_2 > (1 + \sqrt{4\,(y_4/x_4)^2 - 4\,(y_4/x_4) + 1})/2 = (1 + (2\,(y_4/x_4) - 1))/2 = y_4/x_4.$$

q.e.d.

In particular this shows that in Subcase I, one has $x_4 \sin f \geqslant x_2 \sin g$, with equality only at points $A$, $B$, Fig. 6. This completes the discussion of this subcase, and proves Assertion A.

*Subcase II*

From our previous discussion, it is clear that the lines for constant $y_2$ are as shown in Fig. 6. In particular, since now

$$y_4^2 - y_4\,x_4 < y_2^2 - y_2\,x_2 < y_4^2 + y_4\,x_4 < y_2^2 + y_2\,x_2 \tag{14'}$$

there exists a $\cos f_2$, and $\cos g_1$ such that

$$y_4^2 + y_4\,x_4 \cos f_2 = y_2^2 - y_2\,x_2 \tag{15'}$$
$$y_2^2 + y_2\,x_2 \cos g_1 = y_4^2 + y_4\,x_4$$

hence the lines (13) now cross the lines $x_4 \cos f_2 = x_4$, $x_2 \cos g = -x_2$ as shown in Fig. 6.

To complete this case, we easily compute

$$(\partial T_2/\partial x_4 \cos f)^* = (y_4 + x_4 \cos f - y_2 - x_2 \cos g)/T_2 \tag{18}$$
$$= (k/T_2)\,((1/y_4) - (1/y_2)) > 0 \text{ (since } y_2 > y_4)$$

Where there is an * it means that the partial derivative is evaluated at $x_4 \sin f = x_2 \sin g$.

Hence, at $x_4 \sin f = x_2 \sin g$, $T_2$ is decreasing with $x_4 \cos f$; thus, the optimum one-impulse transfer point will occur to the left of this point or where $x_4 \sin f > x_2 \sin g$ in Fig. 6. This discussion takes care of Subcase II, Assertion B except at the point $C$ of Fig. 6.

*Subcase III*

Here the Figure looks like Fig. 7; that is, since $x_2 > x_4$, the hyperbolas have changed their orientation, and inequality (17) now reads $y_2^0/x_2 < y_4/x_4$.

Now when $(\partial T_2/\partial x_4 \cos f)^* > 0$, and the optimum one-impulse transfer point is to the left of the point $x_4 \sin f = x_2 \sin g$ you still end up in the region where $x_4 \sin f > x_2 \sin g$. This completes the discussion of Subcase III, Assertion B.

Thus, except for the discussion of the exceptional points $A$, $B$, $C$, we have carried out our program as stated at the beginning of Case 2, and proved Assertions A and B.



**Exceptional Points**

We now consider the three corner points labeled $A$, $B$, $C$ in Figs. 6 and 7. Actually we also treat the lines going through those points.

*Point A*

Here $y_2^2 - y_2\,x_2 = y_4^2 - y_4\,x_4$, so that orbit 4 has the same apogee as orbit 2, so that we are really in Case 1. Hence, this case has already been treated. (See discussion at end of Case 1.)

*Point B*

Point $B$ when $x_2 < x_4$ can be treated as an example of Subcase II, (in the limit as $\sin f \to 0$). (See eq. (18).) Thus, by the reasoning of that Subcase, the minimum transfer point is at a point such as $B'$ in Fig. 6 and offers no difficulty. In Subcase III, with $x_2 > x_4$ point $B$ does not occur. (See Fig. 7.)

The other possibility for the occurrence of a point $B$ is when $x_2 = x_4$. That example is shown in Fig. 8. Actually in this example points $A$ and $B$ occur on the same line. On this line $(x_2, y_2) = (x_4, y_4)$, so both orbits have the same apogee. Hence, the reasoning of Case 1 applies.

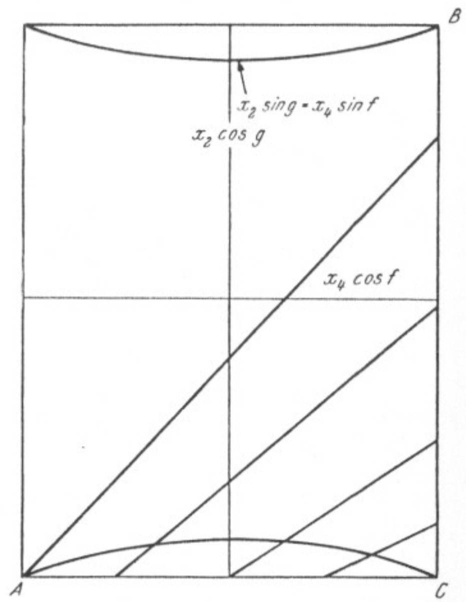


Fig. 7. $x_2 \cos g$ versus $x_4 \cos f$ in Subcase III

*Point C*

This case is the most difficult of the three exceptional points to treat. Here we discuss the relation of orbit 4 with orbit 1 and orbit 2 at the same time. We begin with the relation of orbit 4 and 1.

Thus, at a point $C$ with respect to orbits 1 and 4 one has $y_4^2 + y_4\,x_4 = y_1^2 - y_1\,x_1$, which means that the perigee of orbit 4 is equal to the apogee of orbit 1, (see point designated 1

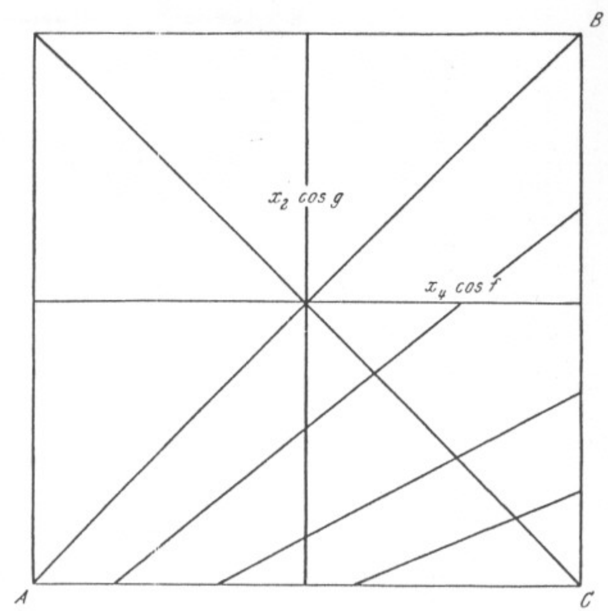


Fig. 8. $x_2 \cos g$ versus $x_4 \cos f$ when $x_2 = x_4$

in Fig. 5). Hence, orbit 4 cannot increase its perigee any more or it will not intersect orbit 1. Since $(r_a)_2 \geqslant (r_a)_1$, the combined relationship is as shown in Fig. 5 with 2, 2′ examples of where orbit 2 can be located with respect to orbits 1 and 4.

As one easily computes

$$V_{14} = V_{10} + V_{04} = (y_4 + x_4) - (y_1 - x_1) = (k_4/y_4) - (y_1 - x_1) = \\ = (\sqrt{2}\,k_4/\sqrt{k_4 + l_4}) - (y_1 - x_1) \quad (19)$$

On the other hand, one also easily computes, for arbitrary $\cos f$:

$$(y_4 + x_4 \cos f,\ x_4 \sin f) = (k/y_4,\ x_4 \sin f) = \quad (20) \\ = (\sqrt{2}\,k/\sqrt{k_4 + l_4},\ \sqrt{2(k_4 - k)\,(k - l_4)/(k_4 + l_4)})$$

with

$$k_4 \geqslant k \geqslant l_2 > l_4$$

Thus if

$$k/y_4 = y_4 + x_4 \cos f > y_2 + x_2 \cos g = k/y_2 \quad (21)$$

and

$$x_4 \sin f > x_2 \sin g \quad (22)$$

then as $l_4$ increases, first $y_4 + x_4$, $y_4 + x_4 \cos f$, and $x_4 \sin f$ will decrease, and hence both $V_{14}$ and $T_2$ will decrease with them. So even in exceptional Point $C$ there is a neighboring orbit 4′ which will decrease the total velocity change between orbits 1 and 2, namely one with $k_4' = k_4$ and $l_4' > l_4$.

However, although condition (22) is always satisfied at an optimum transfer point, (21) depends on $y_2 > y_4$ which may or may not be the case. (See Fig. 5.) Even so the sum of $V_{14} + T_2$ will decrease with increasing $l_4$ (hence the orbit 4′ described above will still suffice). This is shown as follows:

$$\partial(V_{14} + T_2)/\partial l_4 = \\ = -\,k_4/\sqrt{2(k_4 + l_4)^3} - ((y_4 + x_4 \cos f - y_2 - x_2 \cos g)/T_2)\,(k/\sqrt{2(k_4 + l_4)^3}) + \\ + ((x_4 \sin f - x_2 \sin g)/T_2)\,(\partial(x_4 \sin f)/\partial l_4) = \quad (23) \\ = -\,(k_4 + \alpha\,k)/\sqrt{2(k_4 + l_4)^3} - \sqrt{(1 - \alpha^2)\,(k_4 + k)^2\,(k_4 - k)/(2(k_4 + l_4)^3\,(k - l_4))} < 0$$

where

$$\alpha = (y_4 + x_4 \cos f - y_2 - x_2 \cos g)/T_2$$
$$|\alpha| \leqslant 1$$

This completes the discussion of Case 2 including all subcases and exceptional points. Hence, there is no relative minimum in this whole region and our proof is complete.

## Conclusion

In this paper we have proved that the HOHMANN-type transfer between two co-planar elliptical orbits is the optimum among all two-impulse transfers. Since TING [1] proved that the optimum orbital transfer can always be reduced to a planar transfer, our result carries over to non co-planar transfer as well.

## References

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