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THE CALCULUS OF VARIATIONS APPROACH TO THE GENERAL OPTIMUM IMPULSIVE TRANSFER PROBLEM

By

G. W. Haynes[1]

Abstract—Zusammenfassung—Résumé

The Calculus of Variations Approach to the General Optimum Impulsive Transfer Problem. The purpose of this paper is to show how the existing theory of the calculus of variations may be used to optimize the general impulsive transfer problem. By introducing an artifice of quasi-time, the general impulsive transfer problem can be made to possess the required class properties so that the variational calculus can be applied directly. The addition of two new constraints to the problem eliminates variable or programmed thrust sub-arcs so that the extremal is comprised of impulsive sub-arcs and coasting sub-arcs. This method has been used to verify and enlarge the results that LAWDEN obtained.

Anwendung der Variationsrechnung auf Probleme der optimalen Impulsübertragung. Zweck der Arbeit ist es, zu zeigen, wie die Theorie der Variationsrechnung benützt werden kann, um für das Impulsübertragungsproblem optimale Lösungen zu finden. Mittels eines Kunstgriffes, nämlich der Einführung der "Quasizeit," lässt sich das allgemeine Impulsübertragungsproblem so umwandeln, dass es die für eine direkte Anwendung der Variationsrechnung notwendigen Eigenschaften besitzt. Die zusätzliche Einführung zweier neuer Begrenzungen in das Problem eliminiert Teilbögen der Flugbahn mit veränderlichem oder gesteuertem Schub, so dass die Extremalkurve aus Teilbögen mit Impuls und aus schwerelosen Teilbögen besteht. Diese Methode wurde benützt, un die von LAWDEN gefundenen Ergebnisse zu bestätigen und zu erweitern.

Une approche variationnelle du problème général de transfert impulsif optimum. Le but de ce papier est de montrer de quelle manière la théorie actuelle du calcul des variations peut être employée pour optimiser le problème général de transfert impulsif. En introduisant un artifice de quasi-temps on peut s'arranger pour que le problème général de transfert impulsif possède le genre de propriétés requises pour que le calcul des variations puisse y être appliqué directement. L'addition de deux contraintes nouvelles au problème élimine les sub-arcs d'impulsion programmés ou variables, en sorte que l'extremum se compose de sub-arcs impulsifs et de sub-arcs de raccordement. On a employé cette méthode pour vérifier et étendre les résultats obtenus par LAWDEN.

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[1]The Martin Company, Denver, Colorado, U. S. A.

Symbols

s = quasi-time
t = real time
x_α = Cartesian coordinates of vehicle
y_α = velocity components of vehicle
V = gravitational potential
l_α = directional cosines of the thrust vector
m = mass of the vehicle
B = hypothetical mass flow rate in quasi-time
c = hypothetical exhaust velocity
$\lambda_1 \ldots \lambda_9$ = Lagrangian multipliers
C = first integral constant
ξ = Weierstrass Function
z'_j = used to denote the derivatives $\frac{dy_\alpha}{ds}$, $\frac{dx_\alpha}{ds}$, $\frac{dm}{ds}$, and $\frac{dt}{ds}$
μ = gravitational parameter of inverse square law field
r_{ni} = initial radius of the nth coasting subarc
r_{nf} = final radius of the nth coasting subarc
E = energy per unit mass of the vehicle
h = angular momentum per unit mass of the vehicle

Subscripts

i, f = denote initial and final values, respectively

I. Introduction

In the past, the major mathematical tool used to optimize impulsive transfers has been the theory of maxima and minima. One of the earliest attempts to optimize an impulsive transfer by a calculus of variations approach was that of LAWDEN [1] who considered the problem of minimum mass loss impulsive transfer as a parameterized problem of LAGRANGE in the calculus of variations. Recently, LAWDEN [2] and LEITMANN [3] have investigated the problem of optimum impulsive transfers by a calculus of variations approach, considering the impulsive transfer as a limiting case of the finite thrust transfers as the thrust becomes unbounded. LEITMANN's approach has been to discuss the switching function as the thrust becomes unbounded, and in this manner verifies LAWDEN's results. LAWDEN evaluated the limit as the thrust becomes unbounded, and his results in part depend on the nonvanishing of a particular integral. LAWDEN's conclusions are still correct, even though in the limit as the thrust becomes unbounded this particular integral does indeed vanish. However, the approach taken by LAWDEN and LEITMANN does not rule out the existence of variable or programed thrust subarcs, so that taking the limit as the thrust becomes unbounded does not assure a completely impulsive transfer.

To apply the existing theory of the calculus of variations [4], it is necessary that the state variables of the problem under consideration possess certain continuity properties. The state variables for the impulsive transfer problem do not possess the required continuity properties. However, by introducing a concept of quasi-time, together with certain constraints in addition to the kinematical con-

straints, it is shown that the impulsive transfer problem thus formulated possesses the required continuity properties.

The purposes of this report are:

1. to present the concept of quasi-time;

2. to use the concept of quasi-time to verify and enlarge LAWDEN's results; and

3. to illustrate the method, the problem of minimum velocity transfer between coplanar circular orbits in an inverse square law field is considered.

II. Concept of Quasi-time

In the analytical foundation of the calculus of variations, an admissible arc $y(x)$ is stated to possess certain properties. The arcs of the type considered are supposed to be continuous and to consist of a finite number of subarcs on each of which the functions $y(x)$ have continuous derivatives. In view of these continuity restrictions, consider the motion of a rocket maneuvering in a given gravitational field where the problem is to minimize some function

$$J = J(x_{\alpha i}, y_{\alpha i}, m_i, t_i, x_{\alpha f}, y_{\alpha f}, m_f, t_f)$$

subject to the kinematical constraints

$$\left.\begin{aligned} &\dot{y}_\alpha + V_{x\alpha} + \frac{mc}{m} l_\alpha = 0 \\ &\dot{x}_\alpha - y_\alpha = 0 \end{aligned}\right\} \alpha = 1,2,3 \tag{1}$$

$$l_1^2 + l_2^2 + l_3^2 = 1. \tag{2}$$

If at any point the motion is impulsive, then the velocity components of the rocket, y_α, will be discontinuous at that point. Therefore, this will violate the foundations on which the existing theory of the calculus of variations is based. Undoubtedly, the theory could be modified to include discontinuous arcs, but this is not the intent of this report.

By parameterizing, the state variables of the impulsive transfer problem can be made to possess the required continuity properties. Therefore, Eqs. (1) are expressed in terms of a new independent variable s, which for convenience shall be termed quasi-time.

Eqs. (1) become

$$\left.\begin{aligned} &\frac{dy_\alpha}{ds} + V_{x\alpha}\frac{dt}{ds} + \frac{dm}{ds}\frac{c}{m} l_\alpha = 0 \\ &\frac{dx_\alpha}{ds} - y_\alpha \frac{dt}{ds} = 0 \end{aligned}\right\} \alpha = 1,2,3. \tag{3}$$

To rule out variable thrust subarcs so that the transfer comprises only coasting and impulsive subarcs, the following conditions are required:

1. When the quasi-time assumes the character of real time, then the coasting or thrust-off equations of motion must be obtained. This condition requires

$$\frac{dm}{ds} = 0, \quad \text{when} \quad \frac{dt}{ds} = 1;$$

2. When the real time is stationary with respect to the quasi-time, the impulsive or field-free equations of motion are obtained, in which case the thrust must be non-zero. This condition requires

$$\frac{dm}{ds} = -B, \quad \text{when} \quad \frac{dt}{ds} = 0$$

where B is some positive constant.

The above conditions can be simply included in the constraints

$$\left(\frac{dm}{ds} + B\right) \cdot \left(\frac{dt}{ds} - 1\right) = 0 \tag{4}$$

and

$$\frac{dm}{ds} \cdot \frac{dt}{ds} = 0. \tag{5}$$

The state variables (x_α, y_α) of the impulsive transfer defined by Eqs. (2), (3), (4), and (5) are now everywhere continuous, and the existing theory of the calculus of variations may be applied directly to find the optimum.

III. Lawden's Problem

The purpose of this chapter is to use the concept of quasi-time to verify and enlarge the results obtained by LAWDEN.

A. Statement of the Problem

Treating the optimum impulsive transfer as a MAYER Problem in the Calculus of Variations, the problem is to minimize

$$J = J\,(x_{\alpha i}, x_{\alpha f}, y_{\alpha i}, y_{\alpha f}, m_i, m_f, t_i, t_f, s_i, s_f)$$

subject to the constraints

$$\Phi_\alpha = \frac{dy_\alpha}{ds} + V_{x\alpha}\frac{dt}{ds} + \frac{dm}{ds}\frac{c}{m}\,l_\alpha = 0 \tag{6}$$

$$\Phi_{\alpha+3} = \frac{dx_\alpha}{ds} - y_\alpha \frac{dt}{ds} = 0 \tag{7}$$

$$\Phi_7 = l_1^2 + l_2^2 + l_3^2 - 1 = 0 \quad \alpha = 1,2,3 \tag{8}$$

$$\Phi_8 = \left(\frac{dm}{ds} + B\right) \cdot \left(\frac{dt}{ds} - 1\right) = 0 \tag{9}$$

$$\Phi_9 = \frac{dm}{ds} \cdot \frac{dt}{ds} = 0. \tag{10}$$

B. Euler-Lagrange Equations

For J to possess a minimum subject to the constraints (Eqs. (6), (7), (8), (9), and (10)), it is necessary that the first variation of the constrained function van-

ishes. This leads to the following EULER-LAGRANGE equations derived from the function

$$F = \lambda_i \Phi_i \quad i = 1, \ldots, 9, \tag{11}$$

where the λ_i's are undetermined multipliers. The usual summation convention applies for repeated subscripts.

The EULER-LAGRANGE equations are

$$\frac{d\lambda_\alpha}{ds} + \lambda_{\alpha+3} \frac{dt}{ds} = 0 \tag{12}$$

$$\frac{d\lambda_{\alpha+3}}{ds} - \frac{dt}{ds} \lambda_\gamma V_{x_\gamma x_\alpha} = 0 \tag{13}$$

$$\lambda_\alpha \frac{c}{m} \frac{dm}{ds} + 2\lambda_7 l_\alpha = 0 \tag{14}$$

$$\frac{d}{ds}\left\{\frac{c}{m}\lambda_\gamma l_\gamma + \lambda_8\left(\frac{dt}{ds} - 1\right) + \lambda_9 \frac{dt}{ds}\right\} + \frac{c}{m^2}\frac{dm}{ds}\lambda_\gamma l_\gamma = 0 \tag{15}$$

$$\frac{d}{ds}\left\{\lambda_\gamma V_{x_\gamma} - \lambda_{\gamma+3} y_\gamma + \lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9 \frac{dm}{ds}\right\} = 0, \tag{16}$$

where γ ranges over the values 1, 2, 3.

C. First Integral

A consequence of the EULER-LAGRANGE equations is that

$$\frac{d}{ds}\left\{F - z_j' F_{z_j'}\right\} = F_s \quad j = 1, \ldots, 8, \tag{17}$$

where z_j' denotes the derivatives $\left(\frac{dy_\alpha}{ds}, \frac{dx_\alpha}{ds}, \frac{dm}{ds}, \frac{dt}{ds}\right)$, respectively. Evaluating Eq. (17) and using the constraints (Eqs. (6), (7), (8), (9), and (10)) to simplify the final expression yields

$$\frac{d}{ds}\left\{\lambda_8\left(B\frac{dt}{ds} - \frac{dm}{ds}\right)\right\} + \lambda_\gamma V_{x_\gamma t}\left(\frac{dt}{ds}\right)^2 = 0. \tag{18}$$

If the gravitational potential V is only dependent on the displacement coordinates x_α and does not contain t explicitly, then a first integral exists:

$$\lambda_8\left(B\frac{dt}{ds} - \frac{dm}{ds}\right) = C, \text{ a constant.} \tag{19}$$

During a coasting subarc where

$$\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dm}{ds} = 0,$$

the first integral becomes

$$\lambda_8 B = C. \tag{20}$$

Similarly, during an impulsive subarc where

$$\frac{dt}{ds} = 0, \quad \text{and} \quad \frac{dm}{ds} = -B,$$

the first integral becomes

$$\lambda_8 B = C. \tag{21}$$

Therefore, in this case λ_8 is everywhere a constant. If the gravitational potential does contain t explicitly, then a first integral exists for the impulsive subarcs only. Then Eq. (18) becomes

$$\frac{d}{ds}\{\lambda_8 B\} = 0,$$

which integrates to the same form as Eq. (21).

D. Weierstrass Test

For J to be a minimum, it is necessary that $\xi \geq 0$ for all admissible variations

$$\left(\frac{dy_\alpha^*}{ds}, \frac{dx_\alpha^*}{ds}, \frac{dm^*}{ds}, \frac{dt^*}{ds}, l_\alpha^*\right) \neq \left(\frac{dy_\alpha}{ds}, \frac{dx_\alpha}{ds}, \frac{dm}{ds}, \frac{dt}{ds}, l_\alpha\right)$$

consistent with the constraints (Eqs. (6), (7), (8), (9), and (10)). Since $F \equiv 0$, ξ becomes simply

$$\xi = -(z_j'^* - z_j')F_{z_j'} \quad j = 1, \ldots, 8 \tag{22}$$

where z_j' denotes the derivatives $\left(\frac{dy_\alpha}{ds}, \frac{dx_\alpha}{ds}, \frac{dm}{ds}, \frac{dt}{ds}\right)$, respectively. Evaluating ξ by Eq. (22) and using the constraints (Eq. (6), (7), (8), (9), and (10)) to simplify, yields

$$\begin{aligned}\xi = \lambda_\gamma\left[\frac{c}{m}\frac{dm^*}{ds}l_\gamma^* - \frac{c}{m}\frac{dm}{ds}l_\gamma\right] - \\ \left(\frac{dm^*}{ds} - \frac{dm}{ds}\right)\left[\frac{c}{m}\lambda_\gamma l_\gamma + \lambda_8\left(\frac{dt}{ds} - 1\right) + \lambda_9\frac{dt}{ds}\right] - \\ \left(\frac{dt^*}{ds} - \frac{dt}{ds}\right)\left[\lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9\frac{dm}{ds}\right] \geq 0,\end{aligned} \tag{23}$$

which has to be satisfied for all admissible variations

$$\left(\frac{dm^*}{ds}, \frac{dt^*}{ds}, l_\alpha^*\right) \neq \left(\frac{dm}{ds}, \frac{dt}{ds}, l_\alpha\right)$$

consistent with the constraints (Eqs. (8), (9), and (10)).

E. Impulse Direction

Solving Eq. (14) for l_α gives us

$$l_\alpha = -\frac{\lambda_\alpha}{2\lambda_7}\frac{dm}{ds}\frac{c}{m}. \tag{24}$$

Substituting this expression for l_α into Eq. (8), we find λ_7 to be

$$\frac{1}{2\lambda_7}\frac{dm}{ds}\frac{c}{m} = \pm\frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}. \tag{25}$$

From Eqs. (24) and (25), the direction cosines defining the impulse direction are

$$l_\alpha = \mp\frac{\lambda_\alpha}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}. \tag{26}$$

To resolve the ambiguity in sign, the WEIERSTRASS Test is applied for the admissible variation

$$\frac{dt^*}{ds} = \frac{dt}{ds},\ \frac{dm^*}{ds} = \frac{dm}{ds},\ l_\alpha \neq l_\alpha.$$

From Inequality (23), this gives

$$\frac{c}{m}\frac{dm}{ds}\left[\lambda_\gamma\, l_\gamma^* - \lambda_\gamma\, l_\gamma\right] \geq 0.$$

Since $\frac{dm}{ds} = -B$ during an impulsive subarc, then the WEIERSTRASS Test implies

$$\lambda_\gamma l_\gamma \geq \lambda_\gamma l_\gamma^*. \tag{27}$$

Therefore, l_γ must be chosen so as to maximize $\lambda_\gamma l_\gamma$. This rules out the negative sign [3], so that

$$l_\alpha = \frac{\lambda_\alpha}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}. \tag{28}$$

It should be noted that during an impulsive subarc the λ_α's are constants, so that the directional cosines are also constants, which is to be expected.

F. Switching Function

To determine a switching function for the coasting and impulsive subarcs, the WEIERSTRASS Test is now applied for the admissible variations

$$\frac{dt^*}{ds} \neq \frac{dt}{ds},\ \frac{dm^*}{ds} \neq \frac{dm}{ds},\ l_\alpha^* = l_\alpha.$$

The WEIERSTRASS Test becomes

$$\xi = -\left(\frac{dm^*}{ds} - \frac{dm}{ds}\right)\left[\lambda_8\left(\frac{dt}{ds} - 1\right) + \lambda_9\frac{dt}{ds}\left[-\left(\frac{dt^*}{ds} - \frac{dt}{ds}\right)\left[\lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9\frac{dm}{ds}\right] \geq 0.$$

There are two cases to be evaluated with their junction and the behavior of the switching function during the subarcs:

1. Coasting Subarc

In this case,

$$\frac{dt}{ds} = 1,\quad \frac{dm}{ds} = 0,$$

so that the admissible variations satisfying the constraints (Eqs. (9) and (10)) are

$$\frac{dt^*}{ds} = 0, \quad \frac{dm^*}{ds} = -B.$$

This gives

$$\xi = B(\lambda_8 + \lambda_9) \geq 0.$$

Since B is a positive constant, then

$$\lambda_8 + \lambda_9 \geq 0 \tag{29}$$

must be satisfied during a coasting subarc.

2. Impulsive Subarc

In this case,

$$\frac{dt}{ds} = 0, \quad \frac{dm}{ds} = -B$$

so that the admissible variations satisfying the constraints (Eqs. (9) and (10)) are

$$\frac{dt^*}{ds} = 1, \quad \frac{dm^*}{ds} = 0.$$

This gives

$$\xi = B(\lambda_8 + \lambda_9) \geq 0.$$

Therefore, as in the coasting subarc,

$$\lambda_8 + \lambda_9 \geq 0 \tag{30}$$

must be satisfied during an impulsive subarc.

3. Corner Conditions

At a junction between a coasting subarc and an impulsive subarc, the WEIERSTRASS-ERDMANN corner conditions require

$$\lambda_\alpha;\ \lambda_{\alpha+3};\ \frac{c}{m}\lambda_\gamma l_\gamma + \lambda_8\left(\frac{dt}{ds} - 1\right) + \lambda_9\frac{dt}{ds} \quad \text{and}$$

$$\lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma + \lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9\frac{dm}{ds}$$

to be continuous across a corner.

Of these terms λ_α, $\lambda_{\alpha+3}$, y_α, Vx_α, and l_α are in fact continuous. Therefore,

$$\lambda_8\left(\frac{dt}{ds} - 1\right) + \lambda_9\frac{dt}{ds}$$

and

$$\lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9\frac{dm}{ds}$$

must be continuous across a corner.

Evaluating these quantities at a corner where $\left(\frac{dt}{ds} = 1, \frac{dm}{ds} = 0\right)$ changes to

$\left(\frac{dt}{ds} = 0, \frac{dm}{ds} = -B\right)$, or conversely, implies

$$\lambda_{9(-)} = -\lambda_{8(+)}$$
$$\lambda_{8(-)} = -\lambda_{9(+)}.$$

Adding these two results together gives

$$(\lambda_8 + \lambda_9)_{(-)} + (\lambda_8 + \lambda_9)_{(+)} = 0.$$

This condition can be satisfied only when

$$\lambda_8 + \lambda_9 = 0$$

since the WEIERSTRASS Test demands that for both subarcs

$$\lambda_8 + \lambda_9 \geq 0.$$

Hence, at a junction between subarcs,

$$\lambda_8 + \lambda_9 = 0. \tag{31}$$

Therefore, as a consequence of the results of Sections 1, 2, and 3 of this chapter, the switching function can be defined by the quantity $(\lambda_8 + \lambda_9)$.

4. The Behavior of the Switching Function during an Impulsive Subarc

During an impulsive subarc,

$$\frac{dt}{ds} = 0, \quad \frac{dm}{ds} = -B. \tag{32}$$

For this condition, Eqs. (6), (7), (12), (13), and (16) become

$$\frac{dy_\alpha}{ds} + \frac{dm}{ds}\frac{c}{m} l_\alpha = 0 \tag{33}$$

$$\frac{dx_\alpha}{ds} = 0 \tag{34}$$

$$\frac{d\lambda_\alpha}{ds} = 0 \tag{35}$$

$$\frac{d\lambda_{\alpha+3}}{ds} = 0 \tag{36}$$

$$\frac{d}{ds}\{\lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma - \lambda_9 B\} = 0. \tag{37}$$

Therefore, t, x_α, λ_α, and $\lambda_{\alpha+3}$ are constants, in view of Eqs. (32), (34), (35), and (36). Since V is a function of x_α and t, then Vx_γ is a constant. Furthermore, l_α is a constant, so that Eq. (33) may be integrated directly to give y_α.

Integrating Eq. (37) and substituting for y_α gives us

$$\lambda_9 = \lambda_9^\circ + \lambda_{\gamma+3} l_\gamma \frac{c}{B} \log m, \tag{38}$$

where all the constant terms have been grouped into λ_9°.

Since m is a monotonic function of s as given by

$$m = m_0 - Bs, \tag{39}$$

then from Eq. (38), λ_9 will be a monotonic function of s. Furthermore, $(\lambda_8 + \lambda_9)$ will be a monotonic function of s, since from the first integral, λ_8 is a constant over an impulsive subarc. The switching function demands that

$$\lambda_8 + \lambda_9 = 0$$

at the end points of the impulsive subarc, and $\lambda_8 + \lambda_9 \geq 0$ everywhere else.

If during an impulsive subarc, $(\lambda_8 + \lambda_9)$ is a monotonic function of s, there will be no other values of s apart from the initial value, which makes

$$\lambda_8 + \lambda_9 = 0.$$

Accordingly, the only possible solution satisfying the switching function requirements is

$$\lambda_8 + \lambda_9 \equiv 0 \tag{40}$$

over an impulsive subarc. This implies from Eq. (38) that

$$\lambda_{\gamma+3} l_\gamma \equiv 0$$

or, on elimination of l_γ,

$$\lambda_{\gamma+3} \lambda_\gamma \equiv 0 \tag{41}$$

over an impulsive subarc.

5. The Behavior of the Switching Function during a Coasting Subarc

During a coasting subarc,

$$\frac{dt}{ds} = 1, \quad \frac{dm}{ds} = 0. \tag{42}$$

Eqs. (15) and (16) now become

$$\frac{d}{ds}\left\{\frac{c}{m}\lambda_\gamma l_\gamma + \lambda_9\right\} = 0 \tag{43}$$

$$\frac{d}{ds}\{\lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma + \lambda_8 B\} = 0. \tag{44}$$

These equations can be integrated directly to give expressions for λ_8 and λ_9, from which the inequality

$$\lambda_8 + \lambda_9 \geq 0$$

can be tested to see if the coasting subarc is permissible.

Differentiating Eq. (43) yields

$$\frac{c}{m} \frac{\lambda_\gamma}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \frac{d\lambda_\gamma}{ds} + \frac{d\lambda_9}{ds} = 0, \tag{45}$$

where l_γ has been eliminated from the original expression by use of Eq. (28).

For Condition (42), Eq. (12) becomes

$$\frac{d\lambda_\alpha}{ds} + \lambda_{\alpha+3} = 0.$$

Using this equation to eliminate $\frac{d\lambda_\alpha}{ds}$ from Eq. (45) results in

$$\frac{d\lambda_9}{ds} = \frac{c}{m} \frac{\lambda_\gamma \lambda_{\gamma+3}}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}. \tag{46}$$

Therefore,

$$\frac{d\lambda_9}{ds} = 0 \tag{47}$$

at the end points of a coasting subarc, since

$$\lambda_\gamma \lambda_{\gamma+3} \equiv 0$$

during an impulsive subarc, and λ_γ, $\lambda_{\gamma+3}$ are continuous across the corners.

If the gravitational potential V does not contain t explicitly, then

$$\frac{d\lambda_8}{ds} \equiv 0,$$

in which case

$$\frac{d}{ds}(\lambda_8 + \lambda_9) = 0 \tag{48}$$

at the end points of a coasting subarc. Furthermore, since

$$\lambda_9 + \frac{c}{m}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \text{constant},$$

and

$$\lambda_8 = \frac{C}{B} = \text{constant},$$

it can be concluded from the switching function requirements that a coasting arc is permissible, if the function

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

has the same value at the end points of the coasting arc, and has a value everywhere else along the coasting arc less than or equal to the endpoint value.

G. Transversality Condition

The transversality condition,

$$dJ + \left[\lambda_\gamma dy_\gamma + \lambda_{\gamma+3} dx_\gamma + dm \left\{ \frac{c}{m} \lambda_\gamma l_\gamma + \lambda_8 \left(\frac{dt}{ds} - 1 \right) + \lambda_9 \frac{dt}{ds} \right\} + \right.$$

$$dt \left\{ \lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma + \lambda_8 \left(\frac{dm}{ds} + B \right) + \lambda_9 \frac{dm}{ds} \right\} -$$

$$ds \left\{ \lambda_8 \left(B \frac{dt}{ds} - \frac{dm}{ds} \right) \right\} \Bigg]_i^f \equiv 0, \quad (49)$$

must be satisfied for all variations consistent with the boundary conditions.

On the initial and final boundaries,

$$\frac{dt}{ds} = 1, \quad \frac{dm}{ds} = 0. \quad (50)$$

However, the coefficients of the differentials appearing in Eq. (49) must be evaluated along the extremals. If the initial and final subarcs are impulsive, the coefficients have to be evaluated for the impulsive subarc condition

$$\frac{dt}{ds} = 0, \quad \frac{dm}{ds} = -B. \quad (51)$$

From Conditions (50) and (51), the transversality condition becomes

$$dJ + \Bigg[\lambda_\gamma dy_\gamma + \lambda_{\gamma+3} dx_\gamma + dm \left\{ \frac{c}{m} \lambda_\gamma l_\gamma - \lambda_8 \right\} + dt \left\{ \lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma - B(\lambda_8 + \lambda_9) \right\} \Bigg]_i^f \equiv 0. \quad (52)$$

At the junction between the initial or final impulsive subarc and the boundaries,

$$\lambda_8 + \lambda_9 = 0,$$

which reduces the transversality condition to

$$dJ + \Bigg[\lambda_\gamma dy_\gamma + \lambda_{\gamma+3} dx_\gamma + dm \left\{ \frac{c}{m} \lambda_\gamma l_\gamma + \lambda_9 \right\} + dt \left\{ \lambda_\gamma V x_\gamma - \lambda_{\gamma+3} y_\gamma \right\} \Bigg]_i^f \equiv 0, \quad (53)$$

and this must be satisfied for all variations dy_γ, dx_γ, dm, and dt consistent with the boundary conditions. If the initial and final subarcs are coasting, the same transversality condition is obtained. Therefore, the transversality condition as given by Eq. (53) holds for any initial and final subarcs.

H. Verification of Lawden's Results

Each of LAWDEN's results can be verified as follows:

1. λ_α, $\dot{\lambda}_\alpha$ must be continuous at each junction point. From Eq. (35), λ_α is a constant across an impulsive subarc. Hence, λ_α has the same value at those endpoints on the coasting subarcs that are adjacent to the impulsive subarc. This verifies the first part of result (1). Since $\lambda_{\alpha+3}$ is a constant across an impulsive subarc, then from Eq. (12), $\frac{d\lambda_\alpha}{ds}$ or $\frac{d\lambda_\alpha}{dt}$ has the same value at those endpoints on the coasting subarcs that are adjacent to the impulsive subarc. This verifies the second part of result [1].

2. $\lambda_\gamma \dot{\lambda}_\gamma = 0$ at each junction point. This result follows immediately from Eq. (41)

$$\lambda_{\gamma+3} \lambda_\gamma = 0$$

and the above, evaluated at the appropriate endpoints on the coasting subarcs adjacent to the impulsive subarc.

3. λ_α is proportional to the directional cosines of the thrust vector. This is evident from the fact that

$$l_\alpha = \frac{\lambda_\alpha}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

and

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

is a particular constant during an impulsive subarc.

IV. Conclusions

The conclusions from this report are:

1. The method of quasi-time has been used to verify LAWDEN's results.

2. In addition to LAWDEN's results, by applying Condition II of WEIERSTRASS, a switching function has been found that has to satisfy the following requirements:

During an impulsive subarc, $\lambda_8 + \lambda_9 \equiv 0$;

During a coasting subarc, $\lambda_8 + \lambda_9 \geq 0$.

3. If the gravitational potential does not contain time explicitly, then $\frac{d}{dt}(\lambda_8 + \lambda_9) = 0$ at the endpoints of a coasting subarc; and a coasting arc is permissible, if the function

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

has the same value at the endpoints of the coasting subarc, and has a value everywhere else along the coasting subarc less than or equal to the endpoint value.

4. An obvious advantage of this method is that the two additional constraints rule out variable thrust subarcs, so that the transfer will be completely impulsive.

V. Minimum Mass Loss Transfer Between Two Coplanar Circular Orbits

A. Statement of the Problem

As an example, consider a minimum mass loss transfer between two coplanar circular orbits in an inverse square law field. Letting the circular orbits be contained in the $(x_1, x_2, 0)$ plane so that

$$x_3 = y_3 = l_3 = 0,$$

then the kinematical constraints (6) and (7) are reduced in number by two, and become for an inverse square law field

$$\left.\begin{aligned} \Phi_1 &= \frac{dy_1}{ds} + \frac{\mu x_1}{r^3}\frac{dt}{ds} + \frac{dm}{ds}\frac{c}{m} l_1 = 0 \\ \Phi_2 &= \frac{dy_2}{ds} + \frac{\mu x_2}{r^3}\frac{dt}{ds} + \frac{dm}{ds}\frac{c}{m} l_2 = 0 \end{aligned}\right\}, \tag{54}$$

where

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\Phi_4 = \frac{dx_1}{ds} - y_2 \frac{dt}{ds} = 0$$

$$\Phi_5 = \frac{dx_2}{ds} - y_2 \frac{dt}{ds} = 0. \tag{55}$$

Since it is a planar problem, the Lagrangian multipliers λ_3 and λ_6 are no longer present, and the EULER-LAGRANGE Eq. (13) becomes

$$\frac{d\lambda_4}{ds} - \frac{dt}{ds}\left[\lambda_1\left(\frac{\mu}{r^3} - \frac{3\mu x_1^2}{r^5}\right) - 3\lambda_2 \frac{\mu}{r^5} x_1 x_2\right] = 0$$

$$\frac{d\lambda_5}{ds} - \frac{dt}{ds}\left[-3\lambda_1 \frac{\mu x_1 x_2}{r^5} + \lambda_2\left(\frac{\mu}{r^3} - 3\frac{\mu x_2^2}{r^5}\right)\right] = 0. \tag{56}$$

Similarly, Eq. (16) becomes

$$\frac{d}{ds}\left\{\lambda_1 \frac{\mu x_1}{r^3} + \lambda_2 \frac{\mu x_2}{r^3} - \lambda_4 y_1 - \lambda_5 y_2 + \lambda_8\left(\frac{dm}{ds} + B\right) + \lambda_9 \frac{dm}{ds}\right\} = 0. \tag{57}$$

B. Transversality Condition

The problem is to minimize the mass loss, which is equivalent to minimizing the initial mass for a specified final mass. Therefore, $J = m_i$.

On the initial and final orbits, the points of departure and approach can be considered to be free. Since both the initial and final orbits are circular, the transversality condition demands that

$$-\lambda_1 y_2 + \lambda_2 y_1 = 0$$

and

$$-\lambda_4 x_2 + \lambda_5 x_1 = 0 \tag{58}$$

be satisfied initially and finally. However, the first and last subarcs are impulsive, and on all impulsive subarcs,

$$-\lambda_1 y_2 + \lambda_2 y_1 = \text{constant}$$

$$-\lambda_4 x_2 + \lambda_5 x_1 = \text{constant}$$

so that conditions (58) hold over the first and last impulsive subarc.

The initial and final time is free, so that

$$\lambda_1 \frac{\mu x_1}{r^3} + \lambda_2 \frac{\mu x_2}{r^3} - \lambda_4 y_1 - \lambda_5 y_2 = 0 \tag{59}$$

must be satisfied initially and finally.

C. Solution of λ's

To determine which coasting subarcs are permissible, it is required to solve for λ_1 and λ_2 during coasting. Since the gravitational potential does not contain s ex-

plicitly, then a first integral exists

$$\lambda_8 B = C. \tag{60}$$

Eq. (57) integrates to

$$\lambda_1 \frac{\mu x_1}{r^3} + \lambda_2 \frac{\mu x_2}{r^3} - \lambda_4 y_1 - \lambda_5 y_2 + \frac{dm}{ds}(\lambda_8 + \lambda_9) + \lambda_8 B = \text{constant}. \tag{61}$$

The transfer starts and finishes with an impulsive subarc, so that at the initial and final points,

$$\lambda_8 + \lambda_9 = 0.$$

Also, Eq. (59) must hold at these points, so that in view of these conditions, Eq. (61) becomes

$$\lambda_1 \frac{\mu x_1}{r^3} + \lambda_2 \frac{\mu x_2}{r^3} - \lambda_4 y_1 - \lambda_5 y_2 + \frac{dm}{ds}(\lambda_8 + \lambda_9) = 0. \tag{62}$$

However, the last term is always zero by virtue of the switching function so that

$$\lambda_1 \frac{\mu x_1}{r^3} + \lambda_2 \frac{\mu x_2}{r^3} - \lambda_4 y_1 - \lambda_5 y_2 = 0 \tag{63}$$

holds throughout the entire transfer.

A second integral of Eqs. (54), (55), and (56) is

$$\lambda_4 x_2 - \lambda_5 x_1 + \lambda_1 y_2 - \lambda_2 y_1 = \text{constant} = 0, \tag{64}$$

where the constant of integration is zero by virtue of Conditions (58).

A third integral of Eqs. (54), (55), and (56) during coasting is

$$\frac{\lambda_1 y_1}{2} + \frac{\lambda_2 y_2}{2} - \lambda_4 x_1 - \lambda_5 x_2 + D = 0, \tag{65}$$

where D is a constant of integration which will be different for each coasting subarc.

Eliminating λ_2 and λ_5 from Eqs. (63), (64), and (65) yields

$$\lambda_1 \left\{ E y_2 - \frac{\mu}{r^3} \frac{h x_1}{2} \right\} + \lambda_4 \left\{ E x_2 + \frac{y_1 h}{2} \right\} + D \left\{ y_1 y_2 + \frac{\mu x_1 x_2}{r^3} \right\} = 0, \tag{66}$$

where

E = energy/unit mass,
h = angular momentum/unit mass.

During coasting,

$$\lambda_4 = -\frac{d\lambda_1}{dt}$$

so that Eq. (66) can be integrated to give

$$\lambda_1 = \frac{2Dx_2}{h} + K\left(E x_2 + \frac{h y_1}{2}\right), \tag{67}$$

where K is a constant of integration.

The expressions for λ_2, λ_4, λ_5 are

$$\lambda_2 = -\frac{2Dx_1}{h} - K\left(Ex_1 - \frac{hy_2}{2}\right) \tag{68}$$

$$\lambda_4 = -\frac{2Dy_2}{h} - K\left(Ey_2 - \frac{h}{2}\frac{\mu x_1}{r^3}\right) \tag{69}$$

$$\lambda_5 = \frac{2Dy_1}{h} + K\left(Ey_1 + \frac{h\mu x_2}{2r^3}\right). \tag{70}$$

These are the solutions for λ_1, λ_2, λ_4, and λ_5 during a coasting subarc, whereas during an impulsive subarc, λ_1, λ_2, λ_4, and λ_5 are constants.

D. Permissible Coasting Subarcs

A coasting subarc is permissible if $\lambda_1^2 + \lambda_2^2$ has the same value at the endpoints of the coasting subarc and a value everywhere else less than or equal to the endpoint value. Also, at the endpoints of the coasting subarc

$$\lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 = 0.$$

Hence, from Eqs. (67) and 68),

$$\lambda_1^2 + \lambda_2^2 = \left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r^2 + \frac{K^2h^2E^2}{2} + \frac{4DKh}{2} + K^2Eh^2 + \frac{K^2h^2\mu}{2r}. \tag{71}$$

On differentiating,

$$\lambda_1\dot{\lambda}_1 + \lambda_2\dot{\lambda}_2 = 2\left[\left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r - \frac{K^2h^2\mu}{4r^2}\right]\dot{r} \tag{72}$$

and is equal to zero at the endpoints of the coasting subarc. At the initial point of the first coasting subarc, identified by the subscript 1_i, $\dot{r}_{1i} = 0$ as can be verified from Conditions (58) and the fact that

$$\lambda_1\lambda_4 + \lambda_2\lambda_5 = 0$$

at this endpoint.

At the final point of the first coasting subarc, there are two possibilities from Eq. (72);

$$\dot{r}_{1f} \neq 0; \left[\left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r_{1f} - \frac{K^2h^2\mu}{4r_{1f}^2}\right] = 0$$

and

$$\dot{r}_{1f} = 0; \left[\left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r_{1f} - \frac{K^2h^2\mu}{4r_{1f}^2}\right] \neq 0.$$

Considering $\dot{r}_{1f} \neq 0$, the value of the radius (r_{1f}) of the final point of the first coasting subarc is, from Eq. (72),

$$\left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r_{1f} - \frac{K^2h^2\mu}{4r_{1f}^2} = 0. \tag{73}$$

However, since $\lambda_1^2 + \lambda_2^2$ must have the same value at both endpoints (r_{1i}, r_{1f}), then from Eq. (71),

$$\left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r_{1i}^2 + \frac{K^2h^2\mu}{2r_{1i}} = \left(\frac{4D^2}{h^2} + \frac{4DKE}{h} + K^2E^2\right)r_{1f}^2 + \frac{K^2h^2\mu}{2r_{1f}}. \tag{74}$$

By use of Eq. (73), Eq. (74) reduces to

$$\frac{r_{1i}^2}{2r_{1f}^3} + \frac{1}{r_{1i}} = \frac{1}{2r_{1f}} + \frac{1}{r_{1f}},$$

which can be written as

$$\left(\frac{r_{1i}}{r_{1f}}\right)^3 - 3\left(\frac{r_{1i}}{r_{1f}}\right) + 2 = 0$$

and factors to

$$\left\{\left(\frac{r_{1i}}{r_{1f}}\right) - 1\right\}^2 \cdot \left\{\left(\frac{r_{1i}}{r_{1f}}\right) + 2\right\} = 0.$$

Hence, the only solution possible is

$$r_{1i} = r_{1f}. \tag{75}$$

Since $\dot{r}_{1i} = 0$, r_{1i} is at an apsis, and Eq. (75) cannot be satisfied unless the first coasting subarc is a part of the initial circular orbit, which would be a trivial solution. Therefore, this case can be ruled out. The only solution possible is the case where

$$\dot{r}_{1f} = 0$$

and r_{1f} is determined from the condition that $\lambda_1^2 + \lambda_2^2$ has the same value at both ends of the coasting subarc.

During the second impulsive subarc,

$$\lambda_1 y_2 - \lambda_2 y_1 = \text{constant}.$$

Evaluating the constant from the conditions at the end of the first coasting subarc yields, from Eqs. (67) and (68)

$$\lambda_1 y_2 - \lambda_2 y_1 = \left(\frac{2D}{h} + KE\right) r_{1f}\dot{r}_{1f} = 0,$$

so that $\lambda_1 y_2 - \lambda_2 y_1 = 0$ during the second impulsive subarc, which implies that the second impulse is applied tangentially at the apsis r_{1f}.

By the same argument, it follows that

$$\dot{r}_{2f} = 0, \quad \text{and in general,} \quad \dot{r}_{nf} = 0,$$

so that on each impulsive subarc,

$$\lambda_1 y_2 - \lambda_2 y_1 = 0,$$

which implies that the impulses are applied tangentially at the apsis of each transfer ellipse.

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