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# OPTIMAL TRAJECTORIES FOR SPACE NAVIGATION

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# PREFACE

Problems of optimizing rocket trajectories have been extensively studied during the last decade and general techniques of both analytical and numerical types have been developed for their solution. This monograph is concerned exclusively with the analytical approach and makes no reference to numerical methods such as those of 'steepest descent' (see Kelley, H. J. (1960) 'Gradient Theory of Optimal Flight Paths.' *Journal of the American Rocket Society* **30**, No. 10, 947–954) and of 'dynamic programming' (see Bellman, R. and Dreyfus, S. (1959) 'An Application of Dynamic Programming to the Determination of Optimal Satellite Trajectories'. *J. Brit. Interplanet. Soc.* **17**, Nos. 3–4, 78–83). It is true that, on account of the complex nature of the majority of practical problems in this field, recourse to numerical techniques is usually inevitable before an acceptable solution is forthcoming, but the analytical approach is none the less valuable for the following reasons: Experience teaches us that the form of an optimal trajectory is rarely, if ever, very critically dependent upon the data of a problem and consequently if, by making suitable simplifying approximations, the actual problem can be transformed into an idealized problem whose solution is analytically tractable, then this latter solution will often provide an excellent substitute for the optimal motor thrust programme in the actual situation. All that then remains to be done is to recompute the trajectory employing this programme and taking account of the real circumstances. Further, it is only by adopting the analytical approach in any field of research, that those general principles, which lead to a real understanding of the nature of the solutions, are discovered. Lacking such an appreciation, our sense of direction for the numerical attack will be defective and, as a consequence, computations will become unnecessarily lengthy or even quite ineffective. As for almost every other application of mathematics to practical affairs, therefore, the analytical and numerical approaches are here complimentary rather than alternative.

The mathematical background assumed of the reader is described in the preliminary section. This is approximately of the standard normally reached by a modern honours mathematics student after two years at the university and the book is accordingly suitable as a text for a third-year course for this class of student. Such a course, supplemented by further material from the calculus of variations, could constitute an optional alternative to some other branch of applied mathematics and might appeal to students who have developed an interest in this old-established branch of analysis which, after a period of comparative neglect, is now attracting considerable attention for its practical applications. But my principal aim has been to provide an introduction to the mathematical theory of optimal trajectories for the rapidly growing body of young space scientists in private and public research establishments both in this country and abroad who are becoming involved with astronautical calculations.

It gives me great pleasure to acknowledge the debt I owe my fellow researchers in this field, the majority of whom are resident in the United States, for the generous way in which they have exchanged ideas with an outsider in a somewhat remote country. I am conscious that the mode of development of the theory I have chosen in this book owes much to the profit I have received from this traffic. The lists of references give some indication of the sources from which many of the subject's fundamental ideas derive and I hope all those who have contributed to my own approach to the subject will find their names at appropriate points in the text. To the Council of the University of Canterbury for granting me a period of absence from normal university duties during which this monograph was written, I offer my sincere thanks. In conclusion, I should also like to express my gratitude to Professor P. J. Hilton (Cornell University) who, during his occupancy of the Chair of Pure Mathematics at Birmingham University, offered me hospitality within his department and provided me with all necessary facilities.

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# PRELIMINARY REMARKS

The reader of this book is assumed to be familiar with the material normally included in a first honours mathematics course concerned with the analysis of functions of real variables and with the fundamentals of the theory of differential equations. Extensive use is made of techniques and results taken from the calculus of variations, but these are all developed *ab initio* in the first chapter, so that no preliminary acquaintance with this field of mathematics is necessary. However, some background knowledge of the type of problem to which this calculus may be applied and of the results to be expected will no doubt prove helpful in following the argument of the first chapter, and the reader who wishes to obtain some familiarity with this field before proceeding further is advised to consult one of the many texts dealing with the subject, two of the best known of which are Courant and Hilbert (1953) and Fox (1950).

In addition to standard techniques for the differentiation of functions of functions and of functions defined implicitly by means of sets of equations, a theorem relating to the conditions under which a set of implicit functions is properly defined by a set of equations, plays an important role in the argument of Chapter 1 (p. 13). For a proof of this theorem, the reader is referred to Goursat (1904).

The existence of certain sets of functions has been assumed at some points in the first chapter (p. 8, p. 10, p. 13, p. 22). For a rigorous demonstration of their existence, the reader should consult the standard work in the English language on the calculus of variations, *viz.* Bliss (1946). At appropriate points in the argument, this book is referred to by the author's name followed by a page number.

A knowledge of the Newtonian mechanics of a particle, including the basic formulae relating to the motion of a particle in an inverse square law field of attractive force, is also assumed to be possessed by the reader. All such formulae will be found derived in the book by Lawden (1961). Wherever necessary, this book is referred to as 'Lawden', and a page number is stated.

The reader who requires further information bearing on the subject of this monograph should consult the comprehensive bibliography to be found at the close of the survey article by Leitmann, (1962).

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# 1

# THE PROBLEM OF MAYER

## 1.1. Introduction

This book concerns itself with the problem of finding the trajectory a rocket must follow if it is to accomplish some specified mission in an optimal manner as judged against some criterion of a quantitative nature. The mission may be a military one as, for example, when an intercontinental missile is launched with the object of transporting an atomic bomb payload to a given target area; or the mission may have a scientific object, as when a rocket probe, carrying a payload of observing instruments, is to be guided into an orbit about the Moon or a planet. In the not too distant future, it is to be expected that the mission will frequently have for its object the transport of a human cargo to another body in the solar system or beyond. In most cases, optimization of the trajectory with respect to propellent expenditure will be desired, since this will permit the largest payload to be delivered for a given size of vehicle and we shall accordingly pay particular attention to this case.

However, although economy of propellent expenditure will often be the most pressing requirement, circumstances in which allowance must also be made in the optimization criterion for other factors, are easily envisaged. For example, in the case of a spaceship transporting a human cargo, the quantities of food and supplies which have to be carried will be proportional to the time of transit between the two terminals for the journey and, for this reason, it may prove desirable to reduce this time at the cost of some additional expenditure of propellent, with the ultimate object of minimizing the overall weight of the vehicle. The criterion for optimization will then involve a combination of the two quantities, mass of propellent and time of transit. The techniques we shall develop will be applicable to optimization problems of this more complex nature, but we have thought it advisable, in an introductory work such as this monograph is intended to be, to confine our attention to problems whose mathematical statement is simply expressed, in order that the principal features of our methods should not be obscured by irrelevant details. It should, however, be mentioned here that our analysis will reveal that the equations governing the arcs from which the optimal trajectory must be constructed are quite independent of the optimization criterion. Much of the argument, therefore, is perfectly relevant to the issue, no matter what criterion is ultimately adopted.

Although the mathematical theory of this chapter has been constructed with the express object of applying it to rocket trajectory problems, it can also be employed in many other problem situations. In particular, it has received application to the problem of optimizing the response of control systems and servomechanisms and, in this connection, reference may be made to the work of the Russian mathematician Pontryagin who has based similar techniques upon an alternative mode of development of the mathematical theory. An account in English of his approach will be found in a paper due to R. E. Kopp (1962).

### 1.2. Synopsis of Results

As was first noticed by Cicala and Miele (1956) and Miele (1958), the problem of optimizing a rocket trajectory is a particular case of a general mathematical problem from the calculus of variations associated with the name of Mayer. In this chapter, we state the Mayer problem in a form which differs from that given by Bliss (p. 189) in that certain parameters, termed the control functions, are permitted to enter into the constraining eqs (1.1), (1.3), whilst their derivatives are absent from these conditions. We then obtain a number of sets of conditions which are necessarily satisfied by a solution to the problem, the line of argument being that due to Bliss, modified where necessary to make allowances for the intrusion of the control functions. The boundary conditions (eqs (1.2), (1.4)) are not in the very general form given by Bliss, but are sufficiently so for our present purpose. The sets of necessary conditions will be found stated on pages 16, 19, 25 and 26.

In Chapter 2, we make our first application of the general



theory to a number of problems relating to terrestrial rocket trajectories, *viz.* maximization of the range of a rocket missile, optimal launching of a satellite into orbit and optimization of the performance of an atmospheric sounding rocket.

The theory of optimal rocket trajectories in a general gravitational field, when it is assumed that there is no atmospheric resistance, is developed in Chapter 3. It is shown (p. 59) that the conditions to be satisfied by the required trajectory are conveniently expressed in terms of a *switching function* $\kappa$ and a *primer vector* $\mathbf{p}$, $\kappa$ determining the instants of transition from one motor thrust phase to the next and $\mathbf{p}$ the direction of the motor thrust. In the special, but practically important, case when the possible motor thrust is assumed to be of unlimited magnitude, it is demonstrated that the conditions can all be expressed in terms of the primer vector alone (p. 63). The cases when optimization is to be carried through with respect to propellent expenditure and the net final vehicle energy (the escape problem), are given particular consideration.

The simplification which occurs when the gravitational field is uniform is indicated in Chapter 4, and it is demonstrated that a fairly complete theory can be offered to cover this case. It is shown (p. 72) that, in general, there can be at most three distinct thrust phases and that these occur in the sequence, maximum thrust, null thrust, maximum thrust. In special circumstances, when the boundary conditions are favourable, a phase during which the motor thrust is not a maximum can occur but, in this case, the solution is not unique and a solution involving null and maximum thrust phases alone is always available (p. 71).

The extremal arcs, from which any optimal trajectory must be constructed (irrespective of the optimization criterion) when the gravitational field obeys an inverse square law of attraction to a point, are discussed in Chapter 5, and the form taken by the primer vector on such arcs is calculated. The results obtained are employed in the final chapter to analyse a number of orbital transfer problems in a plane. It is proved (p. 106) that, if the optimal trajectory contains a circular arc and if the motor thrust is unlimited, then the trajectory is formed from conic arcs with their axes all aligned and tangential to one another at their apses; the thrusts are then all impulsive and are applied at these apses in a tangential direction. In particular, the case of transfer between two circular orbits is discussed in detail and the Hohmann mode of transfer via a single cotangential ellipse is proved to be optimal provided the ratio of their radii is not too great (p. 110). Optimal escape from any orbit is proved (p. 111) to be effected by the application of impulsive thrusts alone and the escape trajectory to consist of conic arcs with their axes aligned and tangential to one another at their apses. Finally, the optimal two-impulse transfer manœuvre between any two coplanar orbits is studied in some detail and a set of equations derived (p. 118) from which the elements of the transfer orbit can be calculated in any particular case.

### 1.3. Statement of the Mayer Problem

The calculus of variations is concerned with the problem of minimizing or maximizing functionals, a *functional* being a quantity whose value depends upon the sets of values taken by certain associated functions over domains of their variables for which they are defined. Thus, the quantity $I$ defined by the equation

$$I=\int_0^1 f(x)\,\mathrm{d}x$$

is a functional, for its value depends upon the values assumed by the function $f(x)$ over the interval $0\leqslant x\leqslant 1$. Clearly, a functional is a mathematical entity which has a more complex nature than a function, since it depends for its value, not on the values taken by a finite set of variables, but on the, in general, infinite set of values assumed by a function over its domain of definition.

The problem of Mayer which we are about to formulate relates to the minimization of a quantity which depends upon a number of functionals, the values of which are related to the forms taken by certain unknown functions occurring as parameters in a given set of differential equations. Specifically, given $m$ functions $\alpha_j(t)$ $(j=1, 2, \ldots, m)$, $n$ further functions $x_i(t)$ $(i=1, 2, \ldots, n)$ are to satisfy differential equations of the form

$$\dot{x}_i=f_i(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m, t), \tag{1.1}$$

where, as usual, $\dot{x}_i$ denotes $\mathrm{d}x_i/\mathrm{d}t$. These equations are to be



valid for $t_0\leqslant t\leqslant t_1$ and the $\alpha_j$ are defined throughout this interval as continuous functions apart from a finite number of finite discontinuities. The functions $f_i$ are continuous in all their arguments and possess continuous partial derivatives of an order sufficient to validate all our subsequent arguments and these derivatives are defined over a region sufficiently extensive to include all values of the $x_i$, $\alpha_j$ and $t$ to be encountered, as interior points. The initial values of the $x_i$ at $t=t_0$ are specified by the equations

$$x_i=x_{i0} \tag{1.2}$$

and it follows that the eqs (1.1) determine the functions $x_i(t)$ uniquely over the interval $(t_0, t_1)$ when the $\alpha_j(t)$ are given. The $x_i$ so determined will be continuous functions of $t$, but their derivatives may be discontinuous at points of discontinuity of the $\alpha_j$.

The functions $\alpha_j(t)$ will be termed the *control functions* and the $x_i(t)$ will be referred to as the *state functions*; they are required to satisfy certain constraining equations, *viz.*

$$g_k(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m, t)=0, \tag{1.3}$$

where $k=1, 2, \ldots, p<m$ and the $g_k$ are continuous and possess continuous partial derivatives of sufficiently high order in all their arguments. The control functions are also required to be such that the functions $x_1, x_2, \ldots, x_q$ take prescribed values at $t=t_1$. Thus, for $t=t_1$,

$$x_l=x_{l1}, \quad l=1, 2, \ldots, q. \tag{1.4}$$

and $q\leqslant n$. The control functions can otherwise be chosen arbitrarily. The existence of control functions satisfying the constraints (1.3), (1.4) will be assumed.

Let $x_{q+1,\,1}, x_{q+2,\,1}, \ldots, x_{n1}$ be the values of the $x_i$ at $t=t_1$ not fixed by the constraints (1.4). Then our problem is to find control functions $\alpha_j$ determining the $x_i$ so that the constraints (1.3), (1.4) are satisfied and also so that a given function

$$J(x_{q+1,\,1}, x_{q+2,\,1}, \ldots, x_{n1}) \tag{1.5}$$

is minimized.

The problem can be generalized by permitting $t_1$ to be variable.

In this case, we shall permit $J$ to depend upon this quantity also and write

$$J = J(x_{q+1,\,1}, \ldots, x_{n1}, t_1). \tag{1.6}$$

$J$ is then to be minimized with respect to $t_1$ also.

$J$ will be supposed continuous in all its variables and to possess continuous partial derivatives of sufficiently high order.

### 1.4. Admissible Variations

In general, there will exist infinitely many sets of functions $x_i(t)$, $a_j(t)$ satisfying the eqs (1.1)–(1.4) and to each set there will correspond a value of $J$. Among these sets, we shall suppose that there is one which generates a minimum value for $J$. This *minimal set* will henceforward be denoted by $x_i(t)$, $a_j(t)$.

However it will be convenient first to consider a wider class of sets of functions for which $J$ is defined and including the minimal set as a member. This is the class of *admissible sets* comprising those functions which satisfy the constraints (1.1), (1.3), but do not necessarily satisfy the end conditions (1.2), (1.4). It will be assumed that a one-parameter family of such admissible sets can be found including the minimal set as one of its members. This sub-class of admissible sets will be denoted by $x_i(t, \epsilon)$, $a_j(t, \epsilon)$, where $\epsilon$ is the parameter and $\epsilon = 0$ corresponds to the minimal set. Thus

$$x_i(t, 0) = x_i(t), \quad a_j(t, 0) = a_j(t). \tag{1.7}$$

The functions $x_i(t, \epsilon)$, $\dot{x}_i(t, \epsilon)$ and $a_j(t, \epsilon)$ will be supposed to possess continuous first derivatives with respect to $\epsilon$, for $\epsilon$ satisfying $|\epsilon| < \epsilon_0$ and for $t$ in the interval $(t_0, t_1)$; these functions will also be supposed to possess, for the same values of $\epsilon$ and $t$, the continuity properties relative to $t$ enunciated in section 1.3. In the case when $t_1$ is variable, we shall also make this quantity dependent upon $\epsilon$ and take $t_1(0) = t_1$ to be the end value appropriate to the minimal set.

Substituting

$$x_i = x_i(t, \epsilon), \quad a_j = a_j(t, \epsilon) \tag{1.8}$$

into eqs (1.1), these must be satisfied identically with respect to both $t$ and $\epsilon$. Hence, differentiating both sides with respect to $\epsilon$, it is found that



$$\frac{\partial^2 x_i}{\partial \epsilon\, \partial t} = \frac{\partial f_i}{\partial x_r}\frac{\partial x_r}{\partial \epsilon} + \frac{\partial f_i}{\partial a_j}\frac{\partial a_j}{\partial \epsilon}, \tag{1.9}$$

it being understood that the two terms of the right-hand member of this equation are to be summed with respect to $r$ $(1, 2, \ldots, n)$ and $j$ $(1, 2, \ldots, m)$ respectively, according to the well-known repeated index summation convention. (This summation convention will be operative throughout the book and it is convenient here to state the integral values over which the various literal subscripts employed in this chapter are supposed to range. These are as follows:

$$\left.\begin{array}{lll} i = 1, 2, \ldots, n; & j = 1, 2, \ldots, m; & k = 1, 2, \ldots, p; \\ l = 1, 2, \ldots, q; & r = 1, 2, \ldots, n; & s = q+1, q+2, \ldots, n; \\ & t = p+1, p+2, \ldots, m.) & \end{array}\right\} \tag{1.10}$$

Putting $\epsilon = 0$ in eq (1.9), this can be written

$$\dot{y}_i = \frac{\partial f_i}{\partial x_r} y_r + \frac{\partial f_i}{\partial a_j}\beta_j, \tag{1.11}$$

where

$$y_i(t) = \left(\frac{\partial x_i}{\partial \epsilon}\right)_{\epsilon=0}, \quad \beta_j(t) = \left(\frac{\partial a_j}{\partial \epsilon}\right)_{\epsilon=0}, \tag{1.12}$$

and the arguments of $\partial f_i/\partial x_r$, $\partial f_i/\partial a_j$ are the functions $x_i(t)$, $a_j(t)$ of the minimal set. The functions $y_i$, $\beta_j$ possess the continuity properties of the functions $x_i$, $a_j$ respectively and will be termed the *variations* of the minimal set with respect to the family.

Again, substituting from eqs (1.8) into the constraints (1.3) and differentiating with respect to $\epsilon$, it is found that

$$\frac{\partial g_k}{\partial x_i}\frac{\partial x_i}{\partial \epsilon} + \frac{\partial g_k}{\partial a_j}\frac{\partial a_j}{\partial \epsilon} = 0. \tag{1.13}$$

Putting $\epsilon = 0$, this reduces to

$$\frac{\partial g_k}{\partial x_i} y_i + \frac{\partial g_k}{\partial a_j}\beta_j = 0, \tag{1.14}$$

where the arguments of $\partial g_k/\partial x_i$, $\partial g_k/\partial a_j$ again refer to the minimal set.

It has been shown, therefore, that the one-parameter family of admissible sets of functions must be such that eqs (1.11) and (1.14) are satisfied by its variations identically with respect to $t$. We shall assume, conversely, that if any set of functions $y_i(t)$, $\beta_j(t)$ is found to satisfy the eqs (1.11), (1.14), then a one-parameter family of admissible sets, including the minimal set, can be found, whose variations are these functions. A proof of this *imbedding theorem* will be found in Bliss (p. 196). Such a set of functions $y_i(t)$, $\beta_j(t)$ will be termed a set of *admissible variations* for the minimal set $x_i(t)$, $\alpha_j(t)$.

### 1.5. The First Variation of J

Constructing $J$ (eq (1.6)) for the members of the family of admissible sets defined in the previous section, it becomes a function of the parameter $\epsilon$. The value of $dJ/d\epsilon$ at $\epsilon=0$ will be termed the first variation of $J$ with respect to the family and we shall write

$$\left(\frac{\mathrm{d}J}{\mathrm{d}\epsilon}\right)_{\epsilon=0}=J_1. \tag{1.15}$$

Substituting

$$x_{s1}=x_s[t_1(\epsilon),\epsilon],\quad t_1=t_1(\epsilon), \tag{1.16}$$

in eq (1.6) and differentiating with respect to $\epsilon$, it follows that

$$\frac{\mathrm{d}J}{\mathrm{d}\epsilon}=\frac{\partial J}{\partial x_{s1}}\left(\frac{\partial x_s}{\partial t_1}\frac{\mathrm{d}t_1}{\mathrm{d}\epsilon}+\frac{\partial x_s}{\partial \epsilon}\right)+\frac{\partial J}{\partial t_1}\frac{\mathrm{d}t_1}{\mathrm{d}\epsilon}. \tag{1.17}$$

Putting $\epsilon=0$, this reduces to

$$J_1=\frac{\partial J}{\partial x_{s1}}(\dot{x}_{s1}u_1+y_{s1})+\frac{\partial J}{\partial t_1}u_1, \tag{1.18}$$

where

$$u_1=\left(\frac{\mathrm{d}t_1}{\mathrm{d}\epsilon}\right)_{\epsilon=0}, \tag{1.19}$$

and the arguments of the partial derivatives of $J$ refer to the minimal set. $u_1$ will be called the *variation of the end point.*

We shall now transform the expression (1.18) for $J_1$ into a form more convenient for the subsequent argument. This is carried out by introducing certain auxiliary functions $\lambda_i(t)$, $\mu_k(t)$ to be specified later and a quantity $F$ defined by the equation

$$F=-\lambda_i f_i+\mu_k g_k. \tag{1.20}$$

This device was first employed by Lagrange to assist with the analogous calculation of the maxima and minima of a function of a finite number of vaiables subject to certain constraints; we shall accordingly refer to $F$ as the *Lagrange expression* and to the functions $\lambda_i$, $\mu_k$ as the *Lagrange multipliers.* Then it follows that

$$\int_{t_0}^{t_1}\left[\lambda_i\left\{\dot{y}_i-\frac{\partial f_i}{\partial x_r}y_r-\frac{\partial f_i}{\partial \alpha_j}\beta_j\right\}+\mu_k\left\{\frac{\partial g_k}{\partial x_r}y_r+\frac{\partial g_k}{\partial \alpha_j}\beta_j\right\}\right]\mathrm{d}t$$
$$=\int_{t_0}^{t_1}\left[\lambda_i\dot{y}_i+\frac{\partial F}{\partial x_i}y_i+\frac{\partial F}{\partial \alpha_j}\beta_j\right]\mathrm{d}t \tag{1.21}$$

and, by eqs (1.11), (1.14), this is identically zero for all admissible variations along the minimal trajectory.

The Lagrange multipliers will be assumed to be continuous functions, apart from a finite number of finite discontinuities. Thus, integrating by parts, it follows that

$$\int_{t_0}^{t_1}\frac{\partial F}{\partial x_i}y_i\,\mathrm{d}t=\Big[y_iG_i\Big]_{t_0}^{t_1}-\int_{t_0}^{t_1}\dot{y}_iG_i\,\mathrm{d}t, \tag{1.22}$$

where

$$G_i=\int_{t_0}^{t}\frac{\partial F}{\partial x_i}\mathrm{d}\tau. \tag{1.23}$$

The expression (1.21) is accordingly identical with

$$y_{i1}\int_{t_0}^{t_1}\frac{\partial F}{\partial x_i}\,\mathrm{d}t+\int_{t_0}^{t_1}\left\{(\lambda_i-G_i)\dot{y}_i+\frac{\partial F}{\partial \alpha_j}\beta_j\right\}\mathrm{d}t. \tag{1.24}$$

We shall now choose the $n$ multipliers $\lambda_i$ to satisfy the $n$ equations

$$\lambda_i=\int_{t_0}^{t}\frac{\partial F}{\partial x_i}\,\mathrm{d}t+\lambda_{i0}, \tag{1.25}$$

where the $\lambda_{i0}$ are constants yet to be determined. It is proved in

Bliss (p. 200) that, when the $\lambda_{i0}$ are given, eqs (1.25) determined the $\lambda_i$ uniquely over the interval $[t_0, t_1]$. The $\lambda_i$ so determined are continuous, except at discontinuities of the $\alpha_j$ and do not vanish simultaneously for any $t$. The $p$ multipliers $\mu_k$ will be chosen to satisfy the $p$ equations

$$\frac{\partial F}{\partial \alpha_k}=0, \; k=1, 2, \ldots, p<m. \tag{1.26}$$

For this to be possible, it is necessary that the determinant of the coefficients of the $\mu_k$ in these equations should not vanish, i.e.

$$\left|\frac{\partial g_h}{\partial \alpha_k}\right| \neq 0, \tag{1.27}$$

where $h$ also ranges over the integers 1, 2, . . ., $p$. But, unless this condition is satisfied, the constraints (1.3) are not independent. We shall accordingly assume that the condition is satisfied. With these choices of the Lagrange multipliers, the expression (1.24) reduces to the form

$$\begin{aligned} y_{i1}(\lambda_{i1}-\lambda_{i0})+\int_{t_0}^{t_1}\lambda_{i0}\dot{y}_i\,\mathrm{d}t+\int_{t_0}^{t_1}\frac{\partial F}{\partial \alpha_t}\beta_t\,\mathrm{d}t \\ =y_{i1}\lambda_{i1}-y_{i0}\lambda_{i0}+\int_{t_0}^{t_1}\frac{\partial F}{\partial \alpha_t}\beta_t\,\mathrm{d}t, \end{aligned} \tag{1.28}$$

where the index $t$ ranges over the values $p+1, p+2, \ldots, m$ and $\lambda_{i1}=\lambda_i(t_1)$.

Let $\gamma_0$ be a constant to be determined later. Then, since the expression (1.28) is known to vanish for all admissible variations, it may be added to $\gamma_0 J_1$ as given by the expression (1.18) to yield

$$\gamma_0 J_1=\gamma_0\left\{\frac{\partial J}{\partial x_{s1}}(\dot{x}_{s1}u_1+y_{s1})+\frac{\partial J}{\partial t_1}u_1\right\}+y_{i1}\lambda_{i1}-y_{i0}\lambda_{i0}+\int_{t_0}^{t_1}\frac{\partial F}{\partial \alpha_t}\beta_t\,\mathrm{d}t. \tag{1.29}$$

This is the transformed expression required.

### 1.6. First Necessary Conditions for a Minimum of J

Let $y_i^{(\sigma)}(t)$, $\beta_j^{(\sigma)}(t)$, $u_1^{(\sigma)}$, where $\sigma=1, 2, \ldots, q+n+1=N$, be $N$ sets of admissible variations, each set of which satisfies the equations (1.11) and (1.14). We shall assume that an $N$-parameter



family of admissible sets of functions $x_i(t, \epsilon_1, \epsilon_2, \ldots, \epsilon_N)$, $\alpha_j(t, \epsilon_1, \epsilon_2, \ldots, \epsilon_N)$ can be found which includes the minimal set for the parameter values $\epsilon_1=\epsilon_2=\ldots=\epsilon_N=0$ and whose variations with respect to each parameter $\epsilon_\sigma$ are the functions $y_i^{(\sigma)}(t)$, $\beta_j^{(\sigma)}(t)$; i.e.

$$\frac{\partial x_i}{\partial \epsilon_\sigma}=y_i^{(\sigma)}, \; \frac{\partial \alpha_j}{\partial \epsilon_\sigma}=\beta_j^{(\sigma)}, \tag{1.30}$$

when $\epsilon_1=\epsilon_2=\ldots=\epsilon_N=0$. The reader should refer to Bliss (p. 198) for a rigorous proof. We shall also make $t_1$ dependent upon the parameters $\epsilon_\sigma$ in such a way that

$$\frac{\partial t_1}{\partial \epsilon_\sigma}=u_1^{(\sigma)}, \tag{1.31}$$

when all the $\epsilon_\sigma$ vanish. When the functions of this family are substituted, $J$ becomes a function of the $\epsilon_\sigma$, taking its minimum value $J_0$ when all the parameters vanish. We shall write

$$J(\epsilon_1, \epsilon_2, \ldots \epsilon_N)=J_0+U. \tag{1.32}$$

The conditions that a member of the family should satisfy the end conditions (1.2), (1.4) are as follows:

$$x_i[t_0, \epsilon_\sigma]=x_{i0}, \quad x_i[t_1(\epsilon_\sigma), \epsilon_\sigma]=x_{i1}, \tag{1.33}$$

where $\epsilon_\sigma$ denotes the whole set of parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$. For any set of parameters satisfying the conditions (1.33), $U$ must take a positive value, since $J_0$ is the minimum value of $J$ under these conditions. However, if $U=0$, the equations (1.32), (1.33) are known to possess a solution $\epsilon_1=\epsilon_2=\ldots=\epsilon_N=0$. It follows from well-known theorems relating to implicit functions that the eqs (1.32), (1.33) determined the $\epsilon_\sigma$ as continuous functions of $U$ in the neighbourhood of $U=0$ with $\epsilon_\sigma(0)=0$ $(\sigma=1, 2, \ldots, N)$, provided the determinant

$$\Delta=\begin{vmatrix} \dfrac{\partial J}{\partial \epsilon_1} & \dfrac{\partial J}{\partial \epsilon_2} & \cdots & \dfrac{\partial J}{\partial \epsilon_N} \\ \dfrac{\partial x_{i0}}{\partial \epsilon_1} & \dfrac{\partial x_{i0}}{\partial \epsilon_2} & \cdots & \dfrac{\partial x_{i0}}{\partial \epsilon_N} \\ \dfrac{\partial x_{i1}}{\partial \epsilon_1} & \dfrac{\partial x_{i1}}{\partial \epsilon_2} & \cdots & \dfrac{\partial x_{i1}}{\partial \epsilon_N} \end{vmatrix} \tag{1.34}$$

does not vanish for $\epsilon_1=\epsilon_2=\ldots=\epsilon_N=0$. This would imply that sets of values of the $\epsilon_\sigma$ enabling the end conditions (1.2), (1.4) to be satisfied and yet corresponding to negative values of $U$ existed. This cannot be the case and hence we conclude that $\Delta$ must vanish when $\epsilon_1=\epsilon_2=\ldots=\epsilon_N=0$.

From this point onwards, $\Delta$ will denote the determinant (1.34) after the parameters $\epsilon_\sigma$ have been put equal to zero. For any $N$ sets of admissible variations, let $R$ ($<N$) be the maximum rank which can be attained by $\Delta$. Suppose $\Delta$ has this rank for the variations $y_i^{(\sigma)}(t)$, $\beta_j^{(\sigma)}(t)$, $u_1^{(\sigma)}$. Then $R$ linearly independent columns can be selected from $\Delta$. It will be convenient to adopt a nomenclature so that these columns are the first $R$. The rows of $\Delta$ being linearly dependent, it follows that numbers $\gamma_0$, $\gamma_1, \ldots, \gamma_n$, $\nu_1, \nu_2, \ldots, \nu_q$, not all zero, can be found such that

$$\gamma_0\frac{\partial J}{\partial\epsilon_\sigma}+\gamma_i\frac{\partial x_{i0}}{\partial\epsilon_\sigma}+\nu_l\frac{\partial x_{l1}}{\partial\epsilon_\sigma}=0, \tag{1.35}$$

for $\sigma=1, 2, \ldots, R$. Now let $y_i(t)$, $\beta_j(t)$, $u_1$ represent a general set of admissible variations. We shall replace one of the sets of admissible variations corresponding to a column of $\Delta$ not included amongst the first $R$, by this new set. The replacement column will be denoted by

$$\left\{\frac{\partial J}{\partial\epsilon},\ \frac{\partial x_{i0}}{\partial\epsilon},\ \frac{\partial x_{l1}}{\partial\epsilon}\right\}, \tag{1.36}$$

$\epsilon$ representing the parameter corresponding to the new set of variations. After this modification, the rank of $\Delta$ continues to be $R$, since this is its maximum rank and a linearly independent set of $R$ columns can still be found. It follows that the column (1.36) must be linearly dependent upon the first $R$ columns of $\Delta$ and hence, in view of eqs (1.35), that

$$\gamma_0\frac{\partial J}{\partial\epsilon}+\gamma_i\frac{\partial x_{i0}}{\partial\epsilon}+\nu_l\frac{\partial x_{l1}}{\partial\epsilon}=0. \tag{1.37}$$

This equation must be valid for any set of admissible variations.

Differentiating the left-hand members of eqs (1.33) with respect to $\epsilon$ and then putting all parameters equal to zero, it is found that



$$\frac{\partial x_{i0}}{\partial\epsilon}=y_{i0}, \tag{1.38}$$

$$\frac{\partial x_{l1}}{\partial\epsilon}=\dot{x}_{l1}u_1+y_{l1}. \tag{1.39}$$

Substituting from these two equations into the condition (1.37) and for $\gamma_0 J_1=\gamma_0\partial J/\partial\epsilon$ from eq (1.29) into the same condition, this assumes the form:

$$(\gamma_i-\lambda_{i0})y_{i0}+(\nu_l+\lambda_{l1})y_{l1}+\left(\gamma_0\frac{\partial J}{\partial x_{s1}}+\lambda_{s1}\right)y_{s1}+$$

$$+\left(\gamma_0\frac{\partial J}{\partial x_{s1}}\dot{x}_{s1}+\gamma_0\frac{\partial J}{\partial t_1}+\nu_l\dot{x}_{l1}\right)u_1+\int_{t_0}^{t_1}\frac{\partial F}{\partial a_t}\beta_t\,dt=0. \tag{1.40}$$

We shall now choose the constants $\lambda_{i0}$ (which are still at our disposal) so that the coefficients of the $y_{i0}$ in the condition (1.40) all vanish. Thus, we take

$$\lambda_{i0}=\gamma_i. \tag{1.41}$$

Then, in the condition (1.40), the quantities $y_{i1}$ can be chosen arbitrarily, for the eqs (1.11) can always be satisfied by functions $y_i(t)$ taking any given values at $t=t_1$. Again, the functions $\beta_{p+1}, \beta_{p+2}, \ldots, \beta_m$ can be chosen arbitrarily in the $p$ eqs (1.14), leaving the remaining $p$ functions $\beta_1, \beta_2, \ldots, \beta_p$ completely determined. Finally, the value of $u_1$ can also be chosen without restriction. It follows that the condition (1.40) can only be satisfied for all admissible variations if the coefficients of these arbitrary quantities all vanish. That is, if

$$\nu_l=-\lambda_{l1}, \tag{1.42}$$

$$\gamma_0\frac{\partial J}{\partial x_{s1}}=-\lambda_{s1}, \tag{1.43}$$

$$\gamma_0\frac{\partial J}{\partial t_1}+\gamma_0\frac{\partial J}{\partial x_{s1}}\dot{x}_{s1}+\nu_l\dot{x}_{l1}=0, \tag{1.44}$$

$$\frac{\partial F}{\partial a_t}=0. \tag{1.45}$$

Eliminating the $\nu_l$ between eqs (1.42), (1.44) and substituting from eq (1.43) into eq (1.44), these conditions reduce to the form

$$\lambda_{s1} = -\gamma_0 \frac{\partial J}{\partial x_{s1}}, \tag{1.46}$$

$$\lambda_{i1}\dot{x}_{i1} = \gamma_0 \frac{\partial J}{\partial t_1}, \tag{1.47}$$

together with eq (1.45).

To summarize, therefore, the following conditions have necessarily to be satisfied for the minimal set of functions: The Lagrange multipliers $\lambda_i$, $\mu_k$ must satisfy the *characteristic equations*

$$\lambda_i = \int_{t_0}^{t} \frac{\partial F}{\partial x_i}\,\mathrm{d}t + \lambda_{i0}, \tag{1.48}$$

$$0 = \frac{\partial F}{\partial \alpha_j}, \tag{1.49}$$

and, in addition, the end conditions (1.46), (1.47). Together with the eqs (1.1), (1.3), eqs (1.48), (1.49) provide a set of $2n+m+p$ equations to be satisfied by the same number of functions $x_i(t)$, $\alpha_j(t)$, $\lambda_i(t)$, $\mu_k(t)$. Eqs (1.2), (1.4), (1.46) and (1.47) provide $2n+1$ end conditions which serve to determine the $n$ constants $\lambda_{i0}$, the $n$ constants of integration associated with the differential eqs (1.1) and the end point $t_1$.

At points where the $\alpha_j$ are continuous, the characteristic eqs (1.48) are equivalent to the differential equations

$$\dot{\lambda}_i = \frac{\partial F}{\partial x_i}. \tag{1.50}$$

If the minimization is not to be carried out with respect to the end point $t_1$, this being a given quantity, then $u_1 = 0$ in eq (1.40) and condition (1.47) is no longer applicable.

If the above necessary conditions can be satisfied with $\gamma_0 = 0$, the resulting solution is said to be *abnormal*. In the case of a *normal* solution $\gamma_0 \neq 0$ and we can, without loss of generality, take $\gamma_0 = 1$, for eq (1.35) can then be divided through by $\gamma_0$ and the ratios $\gamma_l/\gamma_0$, $\nu_l/\gamma_0$ replaced by new multipliers $\gamma_l$, $\nu_l$ respectively.



### 1.7. The Weierstrass-Erdmann Corner Conditions

In this section we shall study the conditions to be satisfied by the minimal set of functions at a point of discontinuity $t = a$ of some or all of the functions $\alpha_j(t)$.

As has been remarked previously, at such a point the functions $x_i(t)$ will remain continuous, but their derivatives $\dot{x}_i$ may be discontinuous. The Lagrange expression $F$ may be discontinuous and hence its partial derivatives $\partial F/\partial x_i$, $\partial F/\partial \alpha_j$ may also be discontinuous. However, it follows from eqs (1.25) that the multipliers $\lambda_i$ will, nonetheless, be continuous. Eqs (1.26) determine the remaining multipliers $\mu_k$ and it is evident that these functions may suffer a discontinuity at $t = a$.

If a discontinuity of this type occurs at $t = a$ for the minimal set, it is clear that $J$ must be a minimum with respect to small variations in the position of the discontinuity within the interval $[t_0, t_1]$. A further necessary condition to be satisfied at the discontinuity can be calculated by making use of this property of its position. The calculation is best performed by restating the problem in *parametric form* thus:

We make the variable $t$ depend upon a parameter $\theta$ in a manner not specifically stated, but such that as $\theta$ increases from $\theta = \theta_0$ to $\theta = \theta_1$, $t$ strictly increases from $t_0$ to $t_1$. The functions $x_i(t)$, $\alpha_j(t)$ can then be expressed as functions of $\theta$ and these we shall denote by $X_i(\theta)$, $A_j(\theta)$ respectively. The constraints (1.1), (1.3) are equivalent to the equations

$$X'_i = Tf_i(X_1, \ldots, X_n, A_1, \ldots, A_m, t), \tag{1.51}$$

$$t' = T, \tag{1.52}$$

$$g_k(X_1, \ldots, X_n, A_1, \ldots, A_m, t) = 0, \tag{1.53}$$

where $T = T(\theta)$ and primes denote differentiations with respect to $\theta$. The end conditions (1.2), (1.4) yield similar conditions

$$X_i = x_{i0} \tag{1.54}$$

to be satisfied at $\theta = \theta_0$, and

$$X_i = x_{i1} \tag{1.55}$$

to be satisfied at $\theta = \theta_1$. Our problem is now to choose control functions $A_1, A_2, \ldots, A_m, T$, determining by eqs (1.51) and

(1.52) further functions $X_1, X_2, \ldots, X_n$, $t$ and satisfying the constraints (1.53), such that

$$J(X_{q+1,\,1}, X_{q+2,\,1}, \ldots, X_{n1}, t_1) \tag{1.56}$$

is minimized. This is again a problem of the type we have been studying in the previous sections and results already obtained are applicable to it. However, since $t$ is now an arbitrary function of the parameter $\theta$, the position of any discontinuity of the control functions in the interval $[t_0, t_1]$ can be adjusted by variation of this function and will therefore automatically be selected to be optimal by application of the method of the earlier sections.

The new Lagrange expression is given by

$$\Phi = -\lambda_i T f_i - \lambda_{n+1} T + \mu_k T g_k, \tag{1.57}$$

where, for convenience, the constraints (1.53) have been multiplied by $T$ ($>0$, since $t$ is strictly increasing). It will be shown immediately that, as suggested by the notation, the new multipliers are identical with the old after these have been extended by the addition of $\lambda_{n+1}$ to their number. Thus, for the problem in this new form, the characteristic eqs (1.48) become

$$\begin{aligned} \lambda_i &= \int_{\theta_0}^{\theta} \frac{\partial \Phi}{\partial X_i}\, \mathrm{d}\theta + \lambda_{i0}, \\ &= \int_{\theta_0}^{\theta} \left( -\lambda_r \frac{\partial f_r}{\partial X_i} + \mu_k \frac{\partial g_k}{\partial X_i} \right) T\, \mathrm{d}\theta + \lambda_{i0}, \\ &= \int_{t_0}^{t} \frac{\partial F}{\partial x_i}\, \mathrm{d}t + \lambda_{i0}, \end{aligned} \tag{1.58}$$

where, in the final step, we have transformed to the original variables in the integral. These equations are identical with the eqs (1.48) for the problem as originally posed. The equation for the new multiplier $\lambda_{n+1}$ is similarly found to be

$$\lambda_{n+1} = \int_{t_0}^{t} \frac{\partial F}{\partial t}\, \mathrm{d}t + \lambda_{n+1,\,0}. \tag{1.59}$$



The new form for eqs (1.49) is

$$\frac{\partial \Phi}{\partial A_j} = 0, \quad \frac{\partial \Phi}{\partial T} = 0. \tag{1.60}$$

It is easily verified that these are equivalent to the equations

$$\frac{\partial F}{\partial \alpha_j} = 0, \quad F - \lambda_{n+1} = 0. \tag{1.61}$$

$\lambda_{n+1}$ can now be eliminated between the eqs (1.59), (1.61) to yield

$$F = \int_{t_0}^{t} \frac{\partial F}{\partial t}\, \mathrm{d}t + \text{constant}. \tag{1.62}$$

The eqs (1.48), (1.49) have accordingly been recovered and an additional eq (1.62) deduced. At a point which is not a discontinuity, the eq (1.62) is equivalent to the differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} \tag{1.63}$$

and it will be proved in the next section that this equation is already deducible from the eqs (1.48), (1.49) and does not represent an independent condition. However, at a discontinuity, eq (1.62) indicates that $F$ is continuous and this is a new condition. Since $g_k = 0$ and $f_i = \dot{x}_i$ for the minimal trajectory, the new condition requires that the quantity

$$\lambda_i \dot{x}_i \tag{1.64}$$

shall be continuous.

To summarize, therefore, at a discontinuity (or *corner* as it is called in the general theory of the calculus of variations) on the minimal trajectory, the multipliers $\lambda_i$ and the expression $\lambda_i \dot{x}_i$ must all be continuous. These *corner conditions* for other problems from the calculus of variations were discovered independently by Erdmann and Weierstrass.

## 1.8. A First Integral

Along the minimal trajectory, $F$ can be expressed as a function of $t$ alone. Differentiating eq (1.20) totally with respect to $t$, we obtain at all points which are not corners

$$\begin{aligned}\frac{dF}{dt} &= -\dot{\lambda}_i f_i - \lambda_i \frac{\partial f_i}{\partial x_r}\dot{x}_r - \lambda_i \frac{\partial f_i}{\partial \alpha_j}\dot{\alpha}_j - \lambda_i \frac{\partial f_i}{\partial t} + \\ &\quad + \dot{\mu}_k g_k + \mu_k \frac{\partial g_k}{\partial x_r}\dot{x}_r + \mu_k \frac{\partial g_k}{\partial \alpha_j}\dot{\alpha}_j + \mu_k \frac{\partial g_k}{\partial t}, \\ &= -\frac{\partial F}{\partial x_i} f_i + \frac{\partial F}{\partial x_r}\dot{x}_r + \frac{\partial F}{\partial \alpha_j}\dot{\alpha}_j + \frac{\partial F}{\partial t},\end{aligned} \tag{1.65}$$

since $g_k$ is identically zero and eq (1.50) is valid along a minimal trajectory. Referring to eqs (1.1), (1.49), it is now clear that eq (1.65) reduces to eq (1.63).

In particular, if the functions $f_i$, $g_k$ are not explicitly dependent upon $t$, $\partial F/\partial t = 0$ and eq (1.63) possesses a first integral

$$F = \text{constant}. \tag{1.66}$$

Since $g_k = 0$, $f_i = \dot{x}_i$, this first integral can be written

$$\lambda_i \dot{x}_i = c \text{ (constant)}. \tag{1.67}$$

$c$ must take the same value over the whole of the minimal trajectory for, as proved in the previous section, $\lambda_i \dot{x}_i$ is continuous at a corner.

### 1.9. Second Necessary Condition for a Minimum of J

In section 1.4, a certain class of admissible sets of functions embracing the minimal set was defined. In section 1.6, a subclass of admissible sets satisfying the end conditions (1.2) and (1.4) was considered and the conditions necessarily satisfied by that member which makes $J$ a minimum were established. The concept of the variations of the minimal set with respect to the class of admissible sets was introduced; in the general theory of the calculus of variations, variations of this type are termed *weak variations*. In this section, by widening the scope of the class of admissible sets to include members not included in the class defined earlier, a second necessary condition for a minimum of $J$ due to Weierstrass will be established. Variations with respect to this enlarged class are called *strong variations*.

Taking $N = n+q+1$, an $N$-parameter family of admissible sets of functions $x_i$, $\alpha_j$ with parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}, \delta$, is defined thus:



$$\left.\begin{aligned} &x_i = x_i(t, \epsilon_1, \ldots, \epsilon_{N-1}), \quad \alpha_j = a_j(t, \epsilon_1, \ldots, \epsilon_{N-1}), \quad t_0 \leqslant t \leqslant T, \\ &x_i = X_i(t, \epsilon_1, \ldots, \epsilon_{N-1}), \quad \alpha_j = A_j(t, \epsilon_1, \ldots, \epsilon_{N-1}), T \leqslant t \leqslant T+\delta, \\ &x_i = x_i(t, \epsilon_1, \ldots, \epsilon_{N-1}, \delta), \alpha_j = a_j(t, \epsilon_1, \ldots, \epsilon_{N-1}, \delta), T+\delta \leqslant t \leqslant t_1, \end{aligned}\right\} \tag{1.68}$$

where, for every set of values of the parameters, the $x_i$ are continuous in $t$ over the whole interval $[t_0, t_1]$ and the right-hand members of these equations satisfy the constraints (1.1) and (1.3). However, the $\dot{x}_i$, $\alpha_j$ may possess a finite number of discontinuities with respect to $t$. In each of the three intervals $[t_0, T]$, $[T, T+\delta]$, $[T+\delta, t_1]$, the functions $x_i$, $\dot{x}_i$, $\alpha_j$ possess first partial derivatives with respect to every parameter and these are continuous in all their arguments $t, \epsilon_1, \ldots, \epsilon_{N-1}, \delta$. The right-hand members of the eqs (1.68) are otherwise arbitrary. The minimal set is taken to correspond to the zero set of parameter values and we shall suppose that $t = T$ is not a corner of this set. However, since the $\alpha_j$ can be discontinuous at $t = T$ for members of the family for which $\delta \neq 0$, this point may be a corner for a general member of the family. In the case of the family of admissible sets defined previously, a point could be a corner for a general member of the family only if it was a corner for the minimal set. The new family is accordingly representative of a wider class of sets of admissible functions than was the old.

To permit variation of the end point $t_1$, it will be regarded as a function of the $n+q$ parameters $\epsilon_\sigma$ (not of $\delta$).

From this family of sets of functions, we shall now select a one-parameter family satisfying the end conditions (1.2) and (1.4). These $N-1$ conditions serve to determine the $N-1$ parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}$ as functions of the parameter $\delta$ thus:

$$\epsilon_\sigma = \epsilon_\sigma(\delta), \quad \sigma = 1, 2, \ldots, N-1, \tag{1.69}$$

where, clearly, $\epsilon_\sigma(0) = 0$ corresponding to the minimal set which certainly satisfies these end conditions. The condition that the end conditions should be solvable for the $\epsilon_\sigma$ in terms of $\delta$ is that the Jacobian

$$\frac{\partial(x_{10}, x_{20}, \ldots, x_{n0}, x_{11}, x_{21}, \ldots, x_{q1})}{\partial(\epsilon_1, \epsilon_2, \ldots \ldots \ldots \ldots \ldots \ldots, \epsilon_{n+q})} \tag{1.70}$$

should not vanish for the zero set of parameter values. It is proved in Bliss (p. 217) that, provided the minimal set is normal, an $N$-parameter family can always be found for which this condition is satisfied. Substituting from eqs (1.69) into (1.68), a one-parameter family of admissible sets of functions satisfying the end conditions is obtained. $\delta$ is the parameter of this family and the minimal set is the member of the family for which $\delta=0$.

If we now substitute this one-parameter family into the end conditions (1.2) and (1.4) and differentiate the resulting identities in $\delta$ with respect to $\delta$, it is found that

$$\frac{\partial x_{i0}}{\partial \epsilon_\sigma}\;\frac{d\epsilon_\sigma}{d\delta}=0, \tag{1.71}$$

$$\frac{\partial x_{i1}}{\partial \epsilon_\sigma}\;\frac{d\epsilon_\sigma}{d\delta}+\frac{\partial x_{i1}}{\partial \delta}=0. \tag{1.72}$$

Note that $x_{i0}$ is independent of $\delta$ so that $\partial x_{i0}/\partial\delta$ vanishes.

Substituting from eqs (1.68) into equation (1.6), $J$ can be expressed as a function of the $N$ parameters $\epsilon_\sigma$, $\delta$. A further substitution from eqs (1.69) permits $J$ to be expressed relative to the one-parameter family as a function of $\delta$ alone. Thus

$$J(\delta)=J[\epsilon_1(\delta), \epsilon_2(\delta), \ldots, \epsilon_{n+q}(\delta), \delta]. \tag{1.73}$$

Differentiating with respect to $\delta$, we can thus obtain

$$\frac{dJ}{d\delta}=\frac{\partial J}{\partial \epsilon_\sigma}\;\frac{d\epsilon_\sigma}{d\delta}+\frac{\partial J}{\partial \delta}. \tag{1.74}$$

With $\delta=0$, the family of admissible sets of functions defined by eqs (1.68) is of the type considered earlier in section 1.6. The arguments of this section are accordingly still relevant and, in particular, eq (1.35) is valid. Taking $\gamma_0=1$ (since the minimal trajectory is being assumed normal), it follows that

$$\frac{\partial J}{\partial \epsilon_\sigma}=-\gamma_i\frac{\partial x_{i0}}{\partial \epsilon_\sigma}-\nu_i\frac{\partial x_{i1}}{\partial \epsilon_\sigma}, \tag{1.75}$$

when all the parameters are put to zero.

Also

$$\frac{\partial J}{\partial \delta}=\frac{\partial J}{\partial x_{s1}}\frac{\partial x_{s1}}{\partial \delta}. \tag{1.76}$$



But, when all parameter values are zero (i.e. for the minimal trajectory), eq (1.46) is valid and hence

$$\begin{aligned}\frac{\partial J}{\partial \delta}&=-\lambda_{s1}\frac{\partial x_{s1}}{\partial \delta},\\ &=-\nu_i\frac{\partial x_{i1}}{\partial \delta}-\lambda_{i1}\frac{\partial x_{i1}}{\partial \delta},\end{aligned} \tag{1.77}$$

where we have employed eq (1.42) in the last step.

We can now calculate $dJ/d\delta$ for the case of zero parameter values from eq (1.74). Substituting from eqs (1.75) and (1.77), it will be found that

$$\begin{aligned}\frac{dJ}{d\delta}&=-\gamma_i\frac{\partial x_{i0}}{\partial \epsilon_\sigma}\;\frac{d\epsilon_\sigma}{d\delta}-\nu_i\left(\frac{\partial x_{i1}}{\partial \epsilon_\sigma}\;\frac{d\epsilon_\sigma}{d\delta}+\frac{\partial x_{i1}}{\partial \delta}\right)-\lambda_{i1}\frac{\partial x_{i1}}{\partial \delta},\\ &=-\lambda_{i1}\frac{\partial x_{i1}}{\partial \delta},\end{aligned} \tag{1.78}$$

after employing eqs (1.71) and (1.72).

In the one-parameter family, $J$ takes a minimum value when $\delta=0$. Negative values of $\delta$ are excluded and hence this only implies that $J$ is a non-decreasing function of $\delta$ at $\delta=0$. By eq (1.78), this leads to the condition

$$\lambda_{i1}\frac{\partial x_{i1}}{\partial \delta}\leqslant 0. \tag{1.79}$$

Over the interval $[T+\delta, t_1]$, the $N$-parameter family takes the form

$$x_i=x_i(t, \epsilon_\sigma, \delta),\; \alpha_j=\alpha_j(t, \epsilon_\sigma, \delta) \tag{1.80}$$

and, by the argument of section 1.4, the quantities

$$y_i=\left(\frac{\partial x_i}{\partial \delta}\right)_{\epsilon_\sigma=0,\;\delta=0},\; \beta_j=\left(\frac{\partial \alpha_j}{\partial \delta}\right)_{\epsilon_\sigma=0,\;\delta=0} \tag{1.81}$$

must satisfy eqs (1.11) and (1.14) along the minimal trajectory. It follows that

$$-\lambda_i\dot{y}_i=\frac{\partial F}{\partial x_r}y_r+\frac{\partial F}{\partial \alpha_j}\beta_j \tag{1.82}$$

is valid along the minimal trajectory, where $F$ is the Lagrange expression and hence, by eqs (1.49) and (1.50), at points where the $\alpha_j$ are continuous.

$$-\lambda_i \dot{y}_i = \dot{\lambda}_r y_r. \tag{1.83}$$

Thus

$$\frac{d}{dt}(\lambda_i y_i) = 0 \tag{1.84}$$

and hence

$$\lambda_i y_i = \text{constant} \tag{1.85}$$

between adjacent discontinuities of the control functions. At a point where the $\alpha_j$ are discontinuous, it has already been shown that the $\lambda_i$ are continuous; the $y_i$ are continuous at such a point on account of the continuity properties of the family as constructed. Hence, eq (1.85) is valid over the whole of the minimal trajectory from $t=T$ to $t=t_1$.

The condition (1.79) may be written

$$\lambda_{i1} y_{i1} \leqslant 0 \tag{1.86}$$

and hence, by eq (1.85), is equivalent to the inequality

$$\lambda_i y_i \leqslant 0 \text{ at } t=T+0. \tag{1.87}$$

Since $x_i$ is continuous at $t=T+\delta$ for every member of the $N$-parameter family, the equation

$$X_i(T+\delta, \epsilon_\sigma) = x_i(T+\delta, \epsilon_\sigma, \delta) \tag{1.88}$$

must be an identity in $\delta$ (N.B. the $\epsilon_\sigma$ are here to be regarded as independent of $\delta$). Differentiating with respect to $\delta$, we obtain

$$\frac{\partial X_i}{\partial t} = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial \delta} \tag{1.89}$$

at $t=T+\delta$. Making all parameters vanish, this equation can be written

$$\dot{X}_i - \dot{x}_i = y_i \tag{1.90}$$

at $t=T+0$. The condition (1.87) is accordingly equivalent to the requirement

$$\lambda_i \dot{x}_i \geqslant \lambda_i \dot{X}_i. \tag{1.91}$$

In this inequality, the quantities $\lambda_i$, $\dot{x}_i$ refer to the minimal trajectory at the point $t=T$. The quantities $\dot{X}_i$ are any set of values of the $\dot{x}_i$ obtainable from eqs (1.1) at $t=T$ by substitution of the minimal values of the $x_i$ and any set of values $A_j$ of the $\alpha_j$ consistent with the constraints (1.3). This is the *Weierstrass condition* for our problem and, since $t=T$ is any point on the minimal trajectory not a corner, it is necessarily satisfied at all such points. From considerations of continuity, it follows that the condition must also be satisfied at a corner.



## 1.10. Third Necessary Condition for a Minimum of J

This necessary condition for a minimum is easily derivable from the Weierstrass condition of the previous section. In the general theory of the calculus of variations, this new condition corresponds to the *Clebsch condition.*

The Weierstrass condition may be written

$$\lambda_i[f_i(x_r, \alpha_j, t) - f_i(x_r, A_j, t)] \geqslant 0. \tag{1.92}$$

Since both the $\alpha_j$ and the $A_j$ satisfy the constraints (1.3), this inequality is equivalent to

$$F(x_r, A_j, t, \lambda_i, \mu_k) - F(x_r, \alpha_j, t, \lambda_i, \mu_k) \geqslant 0. \tag{1.93}$$

Employing Taylor's theorem, the left-hand member of this inequality can be expanded to yield

$$\frac{\partial F}{\partial \alpha_j}(A_j - \alpha_j) + \frac{\partial^2 F}{\partial \alpha_h \, \partial \alpha_j}(A_h - \alpha_h)\,(A_j - \alpha_j) \geqslant 0, \tag{1.94}$$

where $h=1, 2, \ldots, m$ and $\partial^2 F/\partial\alpha_h \partial\alpha_j$ is to be calculated with the $\alpha_j$ replaced by $\alpha_j + \theta(A_j - \alpha_j)$, $(0<\theta<1)$. By condition (1.49), the first term of the inequality (1.94) vanishes and hence, putting

$$A_j = \alpha_j + \eta\pi_j, \tag{1.95}$$

the condition may be written

$$\frac{\partial^2 F}{\partial \alpha_h \, \partial \alpha_j}\pi_h \pi_j \geqslant 0. \tag{1.96}$$

Since the $A_j$ satisfy the constraints (1.3), we have

$$g_k(x_i, \alpha_j + \eta\pi_j, t) = 0. \tag{1.97}$$

Expanding the left-hand member by Taylor's theorem, we obtain

$$g_k(x_i, \alpha_j, t) + \frac{\partial g_k}{\partial \alpha_j}\eta\pi_j = 0, \tag{1.98}$$

where the $\alpha_j$ are to be replaced by $\alpha_j+\theta'\eta\pi_j$ in $\partial g_k/\partial\alpha_j$. Since the $\alpha_j$ also satisfy the constraints (1.3), eq (1.98) reduces to

$$\frac{\partial g_k}{\partial \alpha_j}\pi_j=0. \tag{1.99}$$

We now let $\eta\to 0$, so that $A_j\to\alpha_j$. In the limit the partial derivatives in conditions (1.96), (1.99) approach their values for the arguments $x_i$, $\alpha_j$, $t$, $\lambda_i$, $\mu_k$. The third necessary condition is, therefore, that the inequality (1.96) must be satisfied for all $\pi_j$ satisfying equation (1.99).

### 1.11. The Second Variation. Sufficiency Conditions

If the minimal set of functions is imbedded in a one-parameter family of sets of admissible functions which also satisfy the end conditions, $J$ can be computed for each member of the family and hence expressed as a function of the family parameter $\epsilon$. If $\epsilon=0$ for the minimal set, then for this value of $\epsilon$ we must have

$$\frac{dJ}{d\epsilon}=0, \frac{d^2J}{d\epsilon^2}\geqslant 0. \tag{1.100}$$

The consequences of the first of these conditions have already been explored. To examine the second inequality condition, it is necessary to calculate what is termed the *second variation* of $J$. It is shown in Bliss (p. 226) that this second variation can be expressed in terms of the variations of the minimal set with respect to the family of admissible sets and then the second of the conditions (1.100) requires that this expression shall be positive for all admissible variations. This requirement leads to an accessory minimum problem (Bliss, p. 228) and a fourth necessary condition. However, the application of this condition to the problems with which we shall be concerned is difficult to carry through and has, as yet (1962), received very little attention. The consequences of this additional necessary condition (first considered by Jacobi) will not therefore be explored in this book.

It is proved in Bliss (Chapter 9) that a set of sufficient conditions for a minimum of $J$ can be expressed in terms of the four necessary conditions mentioned in this chapter. Such a set of conditions only guarantees that the minimal set of functions makes the value of $J$ a *local* minimum, i.e. a minimum relative to sets of admissible functions lying in a sufficiently small neighbourhood of the minimal set. The problem of identifying the *absolute* minimal set still remains and can, at present, be solved only by the separate consideration of all possible local minimal sets and the selection from this class of the member yielding the absolute minimum value for $J$.

If it is a requirement that $J$ shall take a maximum, rather than a minimum value, the inequality signs occurring in the Weierstrass and Clebsch conditions and also in condition (1.100), must be reversed. The first necessary conditions remain unaltered.



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### *Exercises*

1. A continuous curve joins two fixed points and lies in the plane of rectangular axes $Oxy$. $s$ is distance measured along the curve from one of the fixed points $A$ to any other point $P(x, y)$. If $\theta$ is the angle made by the tangent to the curve at $P$ with $Ox$, deduce the equations

$$x'=\cos\theta,\ y'=\sin\theta,$$

where primes denote differentiations with respect to $s$. Regarding $s$ as independent variable, $x$, $y$ as state variables and $\theta$ as control variable for a Mayer problem, show that the length of the curve is a minimum when

$$\theta=\alpha \text{ (a constant)}$$

along the curve (i.e. the curve is a straight line). Verify that the corner and Weierstrass conditions are satisfied.

2. A uniform string having weight $w$ per unit length and length $l$ has its ends attached to two fixed points and hangs in a vertical plane under gravity. $Ox$, $Oy$ are horizontal and vertical axes respectively in this plane. $s$ is the distance measured along the string from any point $P(x, y)$ on it, to the end $A$ having the smallest $x$-co-ordinate. If $v$ is the potential energy of the arc $AP$ of the string (measured from $Ox$ as datum), obtain the equations

$$v'=wy,\ y'=\sin\theta,\ x'=\cos\theta,$$

where primes denote differentiations with respect to $s$ and $\theta$ is the angle made by the tangent to the string with $Ox$. Assuming that the shape of the string is such that its net potential energy is a minimum, show that the intrinsic equation for the string is of the form

$$s = a \tan \theta + b$$

$a$, $b$ being constants determined by the end conditions for the string. (I.e. the string hangs in an arc of a catenary.) Show, also, that the corner conditions require that $\theta$ shall be continuous and that the values of $a$ and $b$ shall not change along the string. Verify that the Weierstrass condition is satisfied.

3. A smooth bead is threaded on a wire which is fixed to lie in a vertical plane. $Ox$, $Oy$ are rectangular axes lying in this plane, the $x$-axis being horizontal and the $y$-axis pointing vertically downwards. The wire passes through the origin $O$ and the point $A$. The bead is released from $O$ with zero velocity and falls on the wire under gravity towards $A$. If $(x, y)$ are the coordinates of the bead at time $t$ after its motion commences, prove that

$$\dot{x} = v \sin \theta, \; \dot{y} = v \cos \theta, \; v^2 = 2gy,$$

where $v$ is the magnitude of the bead's velocity, $\theta$ is the angle it makes with $Oy$ and $g$ is the gravitational acceleration. Show that the time of passage of the bead from $O$ to $A$ is minimized by taking the parametric equations of the wire to be of the form

$$x = a(2\theta - \sin 2\theta) + b, \; y = a(1 - \cos 2\theta),$$

$a$ and $b$ being constants which are determined by the conditions that the wire should pass through $O$ and $A$ (i.e. the wire is in the shape of an arc of a cycloid). Verify that the corner and Weierstrass conditions are satisfied. (This is the *Brachistochrone Problem.*)

4. Solve the following Mayer problem: $x$ is the independent variable, $y$ and $z$ are state variables, $\alpha$ is the control variable and the constraining equations are

$$z' = f(x, y, \alpha), \; y' = \alpha,$$

where primes indicate differentiations with respect to $x$; $y$ takes prescribed values $y_0$, $y_1$ at $x = x_0$, $x_1$ respectively; $z = 0$ at $x = x_0$; the value of $z$ is not prescribed at $x = x_1$, but is to be minimized.

Deduce that, if $y = y(x)$ is an arbitrary function subject only to the end conditions $y = y_0$, $y_1$ at $x = x_0$, $x_1$, the integral

$$\int_{x_0}^{x_1} f(x, y, y')\, dx$$

is minimized for $y(x)$ satisfying the equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}.$$



Show, further, that at a corner of the function $y(x)$, the following quantities are continuous,

$$\frac{\partial f}{\partial y'}, f - y'\frac{\partial f}{\partial y'}.$$

Obtain the Weierstrass condition

$$f(x, y, y'^*) - f(x, y, y') - (y'^* - y')\frac{\partial f}{\partial y'} \geqslant 0,$$

where $y'^*$ is arbitrary.

# 2

# MISCELLANEOUS OPTIMAL TRAJECTORY PROBLEMS

## 2.1. The Equation of Rocket Motion

At time $t$, let $M$ be the net mass of the particles which lie inside the surface formed by the outer shell of a rocket vehicle and closed by imaginary planes drawn across the venturi exits of the motor. Thus, the exhaust gases which have not, at this instant, issued from the venturi contribute to $M$, whereas particles in the jet stream external to the rocket do not. The system of particles contributing to $M$ will be denoted by $\Sigma$. Relative to an inertial frame $S$, let $\mathbf{v}$ be the velocity of the mass centre of $\Sigma$ and let $\mathbf{F}$ be the vector sum of all external forces (e.g. gravity and atmospheric resistance) applied to $\Sigma$. During the time interval $(t, t+\delta t)$, if $M$ changes to $M+\delta M$ ($\delta M<0$), particles of mass $-\delta M$ enter the jet stream external to the vehicle. At points just beyond the venturi exits, let $\mathbf{c}$ be the velocity of the exhaust particles relative to the rocket (assumed the same for all such particles). Then the net linear momentum of those particles of $\Sigma$ which enter the external jet stream during the interval $\delta t$ is $-\delta M(\mathbf{c}+\mathbf{v})$ (relative to $S$) and the net linear momentum of the particles of $\Sigma$ at time $t+\delta t$ is accordingly

$$(M+\delta M)(\mathbf{v}+\delta\mathbf{v})-\delta M(\mathbf{c}+\mathbf{v}), \tag{2.1}$$

$\mathbf{v}+\delta\mathbf{v}$ being the velocity of the mass centre of the particles of mass $M+\delta M$ which comprise the rocket at this instant. At time $t$, the linear momentum of $\Sigma$ is $M\mathbf{v}$. Neglecting second order terms, it follows from eq (2.1) that the increase in the linear momentum of $\Sigma$ during the interval $\delta t$ is

$$M\delta\mathbf{v}-\delta M\mathbf{c}. \tag{2.2}$$

By a fundamental principle of dynamics, this can be equated to



the impulse applied to $\Sigma$ by the external forces operating over the interval $(t, t+\delta t)$. Thus, we have the equation

$$\mathbf{F}\delta t=M\delta\mathbf{v}-\delta M\mathbf{c}. \tag{2.3}$$

Dividing through by $\delta t$ and proceeding to the limit, we obtain the *rocket equation, viz.*

$$M\dot{\mathbf{v}}=\mathbf{F}+\mathbf{c}\dot{M}. \tag{2.4}$$

As propellent is consumed, the mass centre of the vehicle moves relative to its main structure. However, this motion takes place at a velocity which is quite negligible by comparison with the velocity of the vehicle as a whole, and $\mathbf{v}$ can accordingly be accepted, without any qualification, as the velocity of the rocket in its trajectory.

If the pressure $p_e$ in the exhaust jet at points just outside the venturi exits is atmospheric (or zero if the rocket is moving in a vacuum), the surface of $\Sigma$ will be subject everywhere to a constant normal pressure (in addition to the frictional and other forces caused by its motion through the atmosphere). By a well-known hydrostatic principle, the resultant of this system of forces vanishes and consequently only the forces generated by the motion through the atmosphere need be included in $\mathbf{F}$ (e.g. drag and lift forces). If, however, $p_e$ is not equal to the local atmospheric pressure $p_0$, an additional pressure $(p_e-p_0)$ must be taken to act over the venturi exits and hence, if $A$ is the net area of these exits, a force $(p_e-p_0)A$ acting along the axis of the exits and augmenting the motor thrust must be included in $\mathbf{F}$. We shall invariably neglect this component of $\mathbf{F}$; the net motor thrust is then $\mathbf{c}\dot{M}$.

In the case of a rocket moving under no forces but its motor thrust, we have

$$\dot{\mathbf{v}}=\frac{\mathbf{c}}{M}\frac{\mathrm{d}M}{\mathrm{d}t}. \tag{2.5}$$

If the exhaust velocity is maintained constant in magnitude and direction during the motion, this equation is integrable over the time interval $[t_0, t_1]$ to yield

$$\mathbf{v}_1-\mathbf{v}_0=-\mathbf{c}\log\frac{M_0}{M_1}. \tag{2.6}$$

Thus the magnitude of the velocity increment caused by an expenditure of propellent which reduces the rocket mass from $M_0$ to $M_1$ is equal to

$$V = c \log \frac{M_0}{M_1}. \tag{2.7}$$

$V$ is termed the *characteristic velocity* for the manœuvre.

For a general rocket manœuvre under non-vanishing external forces and during which the magnitude, but not necessarily the direction, of the exhaust velocity is maintained constant, the quantity $V$ defined by eq (2.7) is a convenient measure of the propellent expenditure and is still termed the characteristic velocity for the manœuvre. For many of the problems considered in this book, our object will be to find a manœuvre which satisfies certain requirements related to the rocket's mission and which also minimizes the characteristic velocity.

## 2.2. Maximization of the Range of a Rocket Missile

In this section we shall consider the case of a rocket whose propellent expenditure programme is predetermined, but whose thrust direction at any instant can be chosen freely subject only to the restriction that it shall lie in a given vertical plane through the launching point $O$. The rocket is launched from $O$ with zero initial velocity at $t=0$ and the motor is operative until the instant $t=T$; at this instant the motor is shut down and thereafter the vehicle moves under gravity alone along a ballistic trajectory. The gravitational field will be assumed uniform and atmospheric resistance will be neglected. Our problem is to calculate the thrust direction programme which yields a maximum range for the rocket on the horizontal plane through $O$.

Let $Ox$, $Oy$ be horizontal and vertical axes through $O$ lying in the plane of motion and let $(u, v)$ be the components of the rocket's velocity parallel to the axes at time $t$. If $(x, y)$ are the co-ordinates of the rocket at the same instant and $\theta$ is the angle made by the thrust direction with $Ox$, the equations of motion can be expressed in the form

$$\dot{u} = f \cos\theta, \tag{2.8}$$
$$\dot{v} = f \sin\theta - g, \tag{2.9}$$
$$\dot{x} = u, \tag{2.10}$$
$$\dot{y} = v, \tag{2.11}$$



where $f = -c\, \mathrm{d}M/M\, \mathrm{d}t$ is the acceleration caused by the motor thrust and is a given function of $t$.

End conditions to be satisfied at $t=0$ are:

$$x = y = u = v = 0. \tag{2.12}$$

There are no end conditions applicable at $t=T$.

If $(x_1, y_1)$ are the rocket co-ordinates and $(u_1, v_1)$ are its velocity components when its motor is closed down, it will move a further horizontal distance

$$\frac{u_1}{g}[v_1 + \sqrt{(v_1^2 + 2gy_1)}] \tag{2.13}$$

before striking the horizontal plane through $O$. Its total range on the horizontal plane through $O$ is accordingly given by

$$R = x_1 + \frac{u_1}{g}[v_1 + \sqrt{(v_1^2 + 2gy_1)}]. \tag{2.14}$$

We have now to choose the control function $\theta(t)$ determining the thrust direction programme in such a way that $R$ is maximized. When $\theta(t)$ is known, the first order differential eqs (2.8)–(2.11) together with the initial conditions (2.12) determine the state functions $u(t)$, $v(t)$, $x(t)$, $y(t)$ and hence also the value of $R$. The problem is clearly of the type considered in the first chapter and the results obtained there will now be applied.

The Lagrange expression for our problem is

$$F = -\lambda_u f \cos\theta - \lambda_v (f \sin\theta - g) - \lambda_x u - \lambda_y v \tag{2.15}$$

and eqs (1.50) take the form

$$\dot{\lambda}_u = -\lambda_x,\ \dot{\lambda}_v = -\lambda_y,\ \dot{\lambda}_x = \dot{\lambda}_y = 0. \tag{2.16}$$

From condition (1.49), it is found that

$$\lambda_u f \sin\theta - \lambda_v f \cos\theta = 0. \tag{2.17}$$

$t_0 = 0$, $t_1 = T$ are specified and thus condition (1.47) is not applicable. Conditions (1.46) require that, for a normal solution, at $t=T$

$$\lambda_{u_f} = -\frac{1}{g}(v_1 + r), \tag{2.18}$$

$$\lambda_{v_f} = -u_1 (v_1 + r)/gr, \tag{2.19}$$

$$\lambda_x = -1, \tag{2.20}$$
$$\lambda_y = -u_1/r, \tag{2.21}$$

where

$$r = \sqrt{(v_1^2 + 2gy_1)}. \tag{2.22}$$

Integrating eqs (2.16), we obtain

$$\lambda_u = at+b,\ \ \lambda_v = a't+b',\ \ \lambda_x = -a,\ \ \lambda_y = -a', \tag{2.23}$$

where $a$, $b$, $a'$, $b'$ are constants of integration. Hence, from eq (2.17)

$$\tan\theta = \frac{a't+b'}{at+b}. \tag{2.24}$$

The end conditions (2.18)–(2.21) fix the values of $a$, $b$, $a'$, $b'$ to be

$$\left.\begin{aligned} &a=1, \quad b=-T-\frac{1}{g}(v_1+r), \\ &a'=u_1/r, \quad b'=-\frac{u_1}{r}\Big[T+\frac{1}{g}(v_1+r)\Big]. \end{aligned}\right\} \tag{2.25}$$

Eq (2.24) therefore reduces to the simple form

$$\tan\theta = u_1/r, \tag{2.26}$$

indicating that maximum range will be achieved by directing the thrust at a constant angle to the horizontal.

Eq (2.26) determines the angle $\theta$ except for the possible addition of any multiple of $\pi$. The Weierstrass condition (1.91)† removes this remaining ambiguity. The condition requires that

$$\lambda_u\dot{u}+\lambda_v\dot{v}+\lambda_x\dot{x}+\lambda_y\dot{y} \leqslant \lambda_u\dot{U}+\lambda_v\dot{V}+\lambda_x\dot{X}+\lambda_y\dot{Y}, \tag{2.27}$$

where $\dot{U}$, $\dot{V}$, $\dot{X}$, $\dot{Y}$ are values for $\dot{u}$, $\dot{v}$, $\dot{x}$, $\dot{y}$ respectively, obtained from the constraining eqs (2.8)–(2.11) by replacing the optimal value of the control function $\theta$ by any other value $\theta^*$. Clearly $\dot{X}=\dot{x}$, $\dot{Y}=\dot{y}$ and hence the requirement is that

$$\lambda_u\dot{u}+\lambda_v\dot{v} \leqslant \lambda_u\dot{U}+\lambda_v\dot{V}. \tag{2.28}$$

Substituting from eqs (2.8), (2.9), (2.18), (2.19), this condition will be found to reduce to the form

† The inequality sign must be reversed since $R$ is to be maximized.



$$\cos\theta+\frac{u_1}{r}\sin\theta \geqslant \cos\theta^*+\frac{u_1}{r}\sin\theta^* \tag{2.29}$$

and is clearly satisfied for all $\theta^*$ if $\theta$ takes a value such that the left-hand member has its maximum value. Assuming $u_1>0$ (otherwise $R$ will be negative), this will be the case if $\theta$ is the positive acute angle solution of eq (2.26). Clearly, the form of the thrust acceleration function $f(t)$ is irrelevant and the condition is satisfied for all thrust magnitude programmes.

It remains to consider the possibility of the existence of corners. A corner is a point at which the control variable $\theta$ is discontinuous. However, $\theta$ cannot be discontinuous, since it is the unique acute angle solution of eq (2.26). We conclude that there are no corners on the optimal trajectory.

To complete the solution of our problem, the equations of motion (2.8)–(2.11) must be integrated for the case when $\theta$ takes the constant value given by eq (2.26). When this has been done, $u(T)$, $v(T)$, $x(T)$, $y(T)$ can be calculated and put equal to $u_1$, $v_1$ $x_1$, $y_1$, thus yielding four equations for these, as yet, unknown quantities. However, integration of the equations of motion cannot be performed until the form of the thrust acceleration function $f(t)$ is known. Here, we shall only consider the simplest case, *viz.*

$$f = \text{constant}. \tag{2.30}$$

The more realistic case of constant rate of propellent expenditure (i.e. $\dot{M}$=constant) will be found analysed in Lawden (1955).

If $f$ and $\theta$ are both constants, the integration of eqs (2.8)–(2.11) is elementary. Making use of the initial conditions (2.12), we find that

$$\left.\begin{aligned} &u_1 = fT\cos\theta, \quad v_1=(f\sin\theta-g)T, \\ &x_1=\tfrac{1}{2}fT^2\cos\theta, \quad y_1=\tfrac{1}{2}(f\sin\theta-g)T^2. \end{aligned}\right\} \tag{2.31}$$

Substituting from these equations into eq (2.26), it will be found that $\theta$ satisfies the equation

$$\sin^3\theta+\frac{f}{g}\cos 2\theta = 0. \tag{2.32}$$

If $f>g$, this equation possesses four roots in the range

$0<\theta<2\pi$, one in each quadrant. From what has already been said above, we know that only the positive acute solution is acceptable. If $f<g$, the only solutions of eq (2.32) lie in the third and fourth quadrants and are inadmissible; in this case the motor thrust is insufficient to lift the vehicle against gravity.

When $\theta$ has been calculated, the values of the quantities $u_1$, $v_1$, $x_1$, $y_1$ can be derived from eqs (2.31) and the optimal trajectory is determined by the parametric equations

$$x=\tfrac{1}{2}ft^2\cos\theta,\quad y=\tfrac{1}{2}(f\sin\theta-g)t^2. \tag{2.33}$$

It is clear that the optimal trajectory is a straight line making an acute angle $\phi$ with the horizontal given by

$$\tan\phi=\tan\theta-\frac{g}{f}\sec\theta. \tag{2.34}$$

This line is traversed by the rocket with constant acceleration $a$, where

$$a^2=f^2+g^2-2fg\sin\theta. \tag{2.35}$$

The maximum range is found from eq (2.14), after making use of eq (2.32), to be

$$R_{\max}=fT^2\left(\frac{f}{g}\cot\theta-\frac{1}{2}\cos\theta\right). \tag{2.36}$$

For an abnormal solution to this problem, the conditions (1.46) with $\gamma_0=0$ show that the $\lambda_i$ are all zero at $t=T$ and hence, by eqs (2.16), are identically zero. Eq (2.17) does not now lead to any restriction upon the thrust direction programme and any possible trajectory belongs to the class of abnormal trajectories. It is clear, therefore, that this class of arcs cannot contribute to the optimal trajectory.

## 2.3. Optimal Launching of a Satellite

The thrust magnitude programme will again be supposed predetermined, and will be assumed to divide into three distinct phases. The duration of the first phase will be taken to extend from the instant of launching $t=0$ to the instant $t=T$ when the first rocket stage ceases to function. During the second phase, the vehicle will coast under gravity alone to the vertex of its ballistic trajectory, at which point it will be moving horizontally. The motor of the second stage then commences to function and,



during the third phase, will accelerate the satellite payload to its orbital injection velocity.

To simplify the analysis, the motion will be assumed to take place in a vertical plane through the launching point $O$ and the rotation and curvature of the earth, atmospheric drag and non-uniformity of the gravitational field will all be neglected. In this simplified situation, it is possible to eliminate the third phase of the manœuvre entirely; for, if at the conclusion of the first phase the horizontal component of the vehicle's velocity is equal to the required orbital injection velocity, since the field is being assumed uniform, the vehicle's velocity at the vertex will then be exactly as required. This cancellation of the third phase is clearly advantageous, for it eliminates the necessity for transporting propellent to the orbital altitude and the overall propellent expenditure is thereby reduced. In the actual situation, of course, the horizontal component of the vehicle's velocity will fall slightly during the second or coasting phase and this loss will have to be remedied during a short final thrust phase. However, our idealization of the true situation can be expected to lead to a near-optimal thrust direction programme for the first phase and, when this has been found, an accurate trajectory making allowance for all the factors neglected can be computed numerically.

Taking axes $Ox$, $Oy$ through the launching platform as described in the last section, the eqs (2.8)–(2.11) remain valid for the first phase of the motion. However, it will here prove convenient to replace $y$ by a new state variable $z$ defined by the equation

$$z=y+v^2/2g. \tag{2.37}$$

At any instant during the motion, provided $v>0$, $z$ represents the maximum height to which the vehicle will rise if its motor ceases to operate. Differentiating eq (2.37), we obtain

$$\dot{z}=\dot{y}+v\dot{v}/g. \tag{2.38}$$

Eq (2.11) is therefore equivalent to

$$\dot{z}=v(1+\dot{v}/g)=\frac{f}{g}v\sin\theta, \tag{2.39}$$

where we have also employed eq (2.9).

Eqs (2.8), (2.9) and (2.39) constitute a set of constraints for the state variables $u$, $v$ and $z$. $\theta$ is again the only control variable. The appropriate Lagrange expression is given by

$$F=-\lambda_u f\cos\theta-\lambda_v(f\sin\theta-g)-\lambda_z\frac{f}{g}v\sin\theta. \tag{2.40}$$

At $t=0$, the following end conditions are to be satisfied

$$u=v=z=0. \tag{2.41}$$

At $t=T$, the horizontal velocity must equal the orbital injection velocity $u_c$ and the height to which the vehicle will rise as it coasts under gravity must equal the orbital height $h$. Thus, at this instant, it is necessary that

$$u=u_c,\quad z=h. \tag{2.42}$$

The propellent expenditure is to be minimized and, since this is an increasing function of the duration $T$ of the first phase, we shall attempt to minimize $T$. Thus we take

$$J=T. \tag{2.43}$$

Substituting the Lagrange expression (2.40) into the characteristic eqs (1.50), we obtain

$$\dot\lambda_u=\dot\lambda_z=0, \tag{2.44}$$

$$\dot\lambda_v=-\lambda_z\frac{f}{g}\sin\theta. \tag{2.45}$$

Also, from eq (1.49), it follows that

$$\lambda_u\sin\theta-\lambda_v\cos\theta-\lambda_z\frac{v}{g}\cos\theta=0. \tag{2.46}$$

From eqs (2.44), we deduce that

$$\lambda_u=c,\ \lambda_z=a, \tag{2.47}$$

where $a$ and $c$ are constants of integration. Thus, employing eq (2.9), eq (2.45) can be written

$$\dot\lambda_v=-\frac{a}{g}(\dot v+g) \tag{2.48}$$

and this integrates immediately to yield



$$\lambda_v=-\frac{a}{g}v-at-b, \tag{2.49}$$

where $b$ is another integration constant. Substituting from eqs (2.47), (2.49) into eq (2.46), we find that

$$\tan\theta=-(at+b)/c. \tag{2.50}$$

This equation determines the optimal thrust direction programme.

Now consider the end conditions (1.46), (1.47). From these it is concluded that, for a normal solution,

$$\lambda_{v_1}=0, \tag{2.51}$$

$$\lambda_u\dot u+\lambda_v\dot v+\lambda_z\dot z=1, \tag{2.52}$$

at $t=T$. Thus, from eqs (2.49), (2.51), it follows that

$$av_1=-g(aT+b)=gc\tan\theta_1, \tag{2.53}$$

the last equality being implied by eq (2.50). Also, substituting for $\dot u$, $\dot v$, $\dot z$ from eqs (2.8), (2.9), (2.39) and for $\lambda_u$, $\lambda_v$, $\lambda_z$ from eqs (2.47) and (2.49), in the end condition (2.52), and employing eq (2.50), it appears that

$$cf_1\sec\theta_1+av_1-gc\tan\theta_1=1. \tag{2.54}$$

Taken together with eq (2.53), this implies that

$$cf_1=\cos\theta_1. \tag{2.55}$$

If, now, the equations of motion (2.8), (2.9), (2.39) can be integrated with $\theta$ given by eq (2.50) and employing the initial conditions (2.41), expressions for $u_1$, $v_1$ and $z_1$ can be found. Eqs (2.42), (2.50) (with $\theta=\theta_1$ and $t=T$), (2.53) and (2.55) then provide five conditions fixing the unknowns $a$, $b$, $c$, $\theta_1$, $T$. The problem is then solved. The particular case when $f(t)$ is constant will be further considered later in this section.

Let us now apply the Weierstrass condition (1.91). This requires that

$$\lambda_u\dot u+\lambda_v\dot v+\lambda_z\dot z\geqslant\lambda_u\dot u^*+\lambda_v\dot v^*+\lambda_z\dot z^* \tag{2.56}$$

at all points on the trajectory, the 'starred' quantities being obtained from the constraining equations by replacing the optimal value of $\theta$ by any other value $\theta^*$. Substituting for

$\lambda_u$, $\lambda_v$, $\lambda_z$ from eqs (2.47), (2.49), and employing the equations of motion, this condition reduces to

$$c \cos \theta-(at+b) \sin \theta \geqslant c \cos \theta^*-(at+b) \sin \theta^*. \quad (2.57)$$

By eq (2.50), this is equivalent to

$$c \sec \theta \geqslant c \cos(\theta^*-\theta) \sec \theta. \quad (2.58)$$

But, this inequality is true if and only if

$$c \sec \theta \geqslant 0. \quad (2.59)$$

If $c$ is positive, this implies that $\theta$ must be acute and lie in the first or fourth quadrants. If $c$ is negative, $\theta$ must be obtuse and lie in the second or third quadrants. The ambiguity inherent in eq (2.50) is therefore eliminated. Since we shall assume that the satellite is to be projected in the positive sense of the $x$-axis, the possibility that $\theta$ is obtuse may be rejected. $c$ is accordingly positive.

At the instant $t=T$, the rocket has not yet arrived at the vertex of its trajectory and hence $v_1 \geqslant 0$. It follows from eq (2.53), therefore, that the signs of $a/c$ and $\theta_1$ are the same. Thus, if $\theta_1$ were a negative acute angle, eq (2.50) would imply that $\theta$ increased with $t$ and hence, that $\theta$ was a negative acute angle throughout the motion. In these circumstances, $v_1$ would clearly be negative and we have been led to a contradiction. Hence $\theta_1$ must be a positive acute angle, $a/c$ must also be positive (i.e. $a>0$) and must decrease monotonically as $t$ increases.

$\theta$ is now unambiguously specified at all points by eq (2.50) and cannot be discontinuous. There are accordingly no corners on the optimal trajectory.

If $f$ is constant, the equations of motion can be integrated in the following manner: from eq (2.50) by differentiating, we first obtain

$$\sec^2\theta \frac{d\theta}{dt}=-a/c. \quad (2.60)$$

The independent variable may now be transformed from $t$ to $\theta$ in eqs (2.8), (2.9) and (2.39), giving

$$-\frac{a}{c}\frac{du}{d\theta}=f \sec \theta, \quad (2.61)$$



$$-\frac{a}{c}\frac{dv}{d\theta}=f \tan \theta \sec \theta-g \sec^2\theta, \quad (2.62)$$

$$-\frac{a}{c}\frac{dz}{d\theta}=\frac{f}{g} v \tan \theta \sec \theta. \quad (2.63)$$

Integrating eqs (2.61), (2.62) with respect to $\theta$, over the interval $(\theta_0, \theta)$ and employing the initial conditions (2.41), we obtain

$$\frac{a}{gc}u=\frac{f}{g} \log \frac{\sec \theta_0+\tan \theta_0}{\sec \theta+\tan \theta}, \quad (2.64)$$

$$\frac{a}{gc}v=\frac{f}{g} \sec \theta_0-\tan \theta_0-\frac{f}{g} \sec \theta+\tan \theta. \quad (2.65)$$

Putting $\theta=\theta_1$, $v=v_1$ in eq (2.65) and employing eq (2.53), it follows that

$$\frac{f}{g} \sec \theta_0-\tan \theta_0=\frac{f}{g} \sec \theta_1. \quad (2.66)$$

Hence

$$\frac{a}{gc}v=\frac{f}{g}(\sec \theta_1-\sec \theta)+\tan \theta. \quad (2.67)$$

Substituting for $v$ in eq (2.63) from this last equation and again integrating, we now deduce that

$$\frac{a^2}{gc^2}z=\frac{f^2}{g^2} \sec \theta_1 (\sec \theta_0-\sec \theta)-\frac{f^2}{2g^2} (\tan^2 \theta_0-\tan^2 \theta)$$
$$+\frac{f}{2g}\left(\tan \theta_0 \sec \theta_0-\tan \theta \sec \theta+\log \frac{\sec \theta+\tan \theta}{\sec \theta_0+\tan \theta_0}\right). \quad (2.68)$$

Eliminating $c$ from eqs (2.64), (2.68) by means of eq (2.55) and then putting $\theta=\theta_1$, the following equations are obtained

$$au_e=\cos \theta_1 \log \frac{\sec \theta_0+\tan \theta_0}{\sec \theta_1+\tan \theta_1}, \quad (2.69)$$

$$a^2hg \sec^2 \theta_1=\sec \theta_1 (\sec \theta_0-\sec \theta_1)-\tfrac{1}{2} (\tan^2 \theta_0-\tan^2 \theta_1)$$
$$+\frac{g}{2f}\left(\tan \theta_0 \sec \theta_0-\tan \theta_1 \sec \theta_1-\log \frac{\sec \theta_0+\tan \theta_0}{\sec \theta_1+\tan \theta_1}\right). \quad (2.70)$$

This last pair of equations together with eq (2.66) determine the unknowns $\theta_0$, $\theta_1$ and $a$. The remaining unknown quantities are then easily calculated.

To solve this set of equations for given values of $f/g$, $h$ and $u_c$, it is best to proceed as follows: Assume a value for $\theta_0$ in eq (2.66) and calculate $\theta_1$. Now calculate $a$ from eq (2.69). A value for $h$ can then be derived from eq (2.70). $\theta_0$ is then adjusted until this value for $h$ agrees with that prescribed. As might be expected, as $\theta_0$ decreases, so does $h$. The minimum value which can be assumed by $\theta_0$ is that for which $\theta_1=0$; eq (2.66) indicates that $\theta_0$ is then given by

$$\theta_0 = 2\tan^{-1}(g/f). \tag{2.71}$$

With $\theta_0$, $\theta_1$ taking this pair of values, $h$ is a minimum and, by eq (2.53), $v_1=0$, i.e. the vehicle arrives at the vertex at $t=T$. The second phase of the manœuvre accordingly disappears. If the value required for $h$ is less than this critical value, it is necessary to modify our assumptions regarding the optimal manœuvre and to assume instead that the vehicle arrives at the vertex with the required orbital injection velocity at the instant $t=T$. The end conditions (2.42) must then be replaced by the set

$$u=u_c,\ v=0,\ y=h. \tag{2.72}$$

The results which are then obtained are given in Exercise 4 at the end of this chapter.

Although the details of the last paragraph will no longer be valid in the more practical case when it is the motor thrust, rather than $f$, which remains constant, the existence of a critical value for $h$ will not be affected and the choice of end conditions for the problem will depend, as before, upon whether $h$ exceeds or is less than this value.

### 2.4. Optimal Thrust Programme for a Sounding Rocket

In the previous sections of this chapter, we have been studying problems in which the thrust magnitude programme was assumed known and it was required to calculate an optimal thrust direction programme. In this section, we shall study a problem in which the thrust direction is known and it is the thrust magnitude programme which is to be optimized. The problem is to maximize the height attained by a sounding rocket fired vertically from the earth's surface. The thrust will be taken to be directed vertically upwards whenever the motor is operating and the motion will accordingly take place along a vertical straight line.



Let $y$ be the height of the rocket above the launching point at time $t$ after launching, and let $v$ be its velocity at the same instant. Then, if $m$ is the mass rate of propellent expenditure and $D$ is the atmospheric drag force, the equation of rocket motion can be written.

$$\dot{v} = \frac{1}{M}(cm - D) - g. \tag{2.73}$$

$D$ will be taken to depend upon the height and velocity of the vehicle and the gravitational acceleration $g$ to be a function of $y$ alone. Thus

$$D = D(y, v),\ g = g(y). \tag{2.74}$$

To complete the set of differential constraints, we have the equations

$$\dot{y} = v, \tag{2.75}$$

$$\dot{M} = -m. \tag{2.76}$$

When the control function $m(t)$ has been specified, the constraints (2.73), (2.75), (2.76), together with the initial conditions

$$v=0,\ y=0,\ M=M_0,\ t=0, \tag{2.77}$$

fix the functions $v(t)$, $y(t)$, $M(t)$ and hence completely determine the rocket's motion.

If $\bar{m}$ is the maximum rate of propellent expenditure, the control function $m$ is subject to the inequality constraints

$$0 \leqslant m \leqslant \bar{m}. \tag{2.78}$$

Since our general theory does not allow for the possibility of constraining inequalities, we transform the constraints (2.78) into the form of equation (1.3) by introducing a further control variable, $\alpha$, possessing no physical significance, and require that $m$ shall satisfy the equation

$$m(\bar{m} - m) - \alpha^2 = 0. \tag{2.79}$$

Since $\alpha^2 \geqslant 0$, eq (2.79) is equivalent to the inequalities (2.78). This device is due to Valentine (1937).

Let $M_1$ be the rocket mass when all propellent has been expended and let $t=T$ be the final instant when maximum height is attained. Thus we have the end conditions

$$M=M_1,\ t=T. \tag{2.80}$$

Clearly $v=0$ at this instant, but it is convenient to deduce this result rather than to impose it as an end condition. The quantity to be maximized is

$$J=y_1. \tag{2.81}$$

We now proceed to the solution of this problem by setting up the Lagrange expression

$$F=-\lambda_v\frac{1}{M}(cm-D-Mg)-\lambda_y v+\lambda_M m+\mu[m(\bar{m}-m)-\alpha^2] \tag{2.82}$$

and writing down the conditions (1.49), (1.50), *viz.*

$$\dot{\lambda}v=\frac{1}{M}\frac{\partial D}{\partial v}\lambda_v-\lambda_y, \tag{2.83}$$

$$\dot{\lambda}_y=\left(\frac{1}{M}\ \frac{\partial D}{\partial y}+\frac{\mathrm{d}g}{\mathrm{d}y}\right)\lambda_v, \tag{2.84}$$

$$\dot{\lambda}_M=\frac{1}{M^2}(cm-D)\lambda_v, \tag{2.85}$$

$$0=-\frac{c}{M}\lambda_v+\lambda_M+\mu(\bar{m}-2m), \tag{2.86}$$

$$0=-2\alpha\mu. \tag{2.87}$$

The end conditions (1.46), (1.47) take the form

$$\lambda_v=0,\ \lambda_y=-1,\ \lambda_v\dot{v}+\lambda_y\dot{y}+\lambda_M\dot{M}=0, \tag{2.88}$$

all to be satisfied at $t=T$. Since $t$ does not occur explicitly in $F$, a first integral in the form of eq (1.67) can also be written down. This implies that the third of the conditions (2.88) is valid over the whole trajectory and hence, substituting for $\dot{v}$, $\dot{y}$, $\dot{M}$ from the constraining equations into this equation, we obtain

$$\lambda_v\frac{1}{M}(cm-D-Mg)+\lambda_y v-\lambda_M m=0. \tag{2.89}$$



Eq (2.87) shows that there are two cases to consider, (i) $\mu=0$ and (ii) $\alpha=0$.

If $\mu=0$, by eq (2.86)

$$\lambda_M=\frac{c}{M}\lambda_v. \tag{2.90}$$

Substituting from this equation for $\lambda_M$ into eq (2.85), it is found that

$$\dot{\lambda}_v=-\frac{D}{cM}\lambda_v. \tag{2.91}$$

It now follows from eq (2.83) that

$$\lambda_y=\frac{1}{M}\left(\frac{\partial D}{\partial v}+\frac{D}{c}\right)\lambda_v. \tag{2.92}$$

Substitution for $\lambda_M$, $\lambda_y$ from eqs (2.90), (2.92) respectively into the first integral (2.89) yields the result

$$\left[-\frac{D}{M}-g+\frac{v}{M}\left(\frac{\partial D}{\partial v}+\frac{D}{c}\right)\right]\lambda_v=0. \tag{2.93}$$

Thus,

$$\text{either } \lambda_v=0, \tag{2.94}$$

$$\text{or } Mg+D-\left(\frac{\partial D}{\partial v}+\frac{D}{c}\right)v=0. \tag{2.95}$$

The possibility that $\lambda_v$ vanishes can be dismissed immediately for, by eq (2.92), this would imply the vanishing of $\lambda_y$ and then, since both $\lambda_v$ and $\lambda_y$ are continuous at a corner and eqs (2.83), (2.84) are valid over the whole trajectory, it would follow that $\lambda_v$ and $\lambda_y$ must vanish everywhere. This contradicts the end conditions (2.88). Eq (2.95) must therefore be accepted and, together with eq (2.73) expressed in the form

$$Mvv'+cvM'+D+Mg=0 \tag{2.96}$$

where primes denote differentiations with respect to $y$, determines $M$ and $v$ as functions of $y$. Only one constant of integration will appear in the solution and hence not both of the initial conditions $v=0$, $M=M_0$ (at $y=0$), can be satisfied. In fact, it will be clear upon substituting $v=0$ in eq (2.95), that $v$ cannot vanish on the type of arc we are considering. The complete optimal trajectory must therefore include arcs of other types.

Such arcs are obtained by satisfying eq (2.87) with $a=0$. It follows from eq (2.79) that the corresponding propellent expenditure programme is either $m=0$ or $m=\bar{m}$, i.e. the motor is either shut down and the vehicle falls freely, or it is at full throttle and the thrust is a maximum.

It has been shown, therefore, that the optimal trajectory may comprise arcs of three types, (i) intermediate thrust arcs; (ii) maximum thrust arcs; and (iii) null thrust arcs. Such arcs join one another at corner points. The order in which these arcs are arranged to form an optimal trajectory will now be investigated.

The Weierstrass condition (1.91) (with inequality reversed) requires that at all points

$$\lambda_v \frac{1}{M}(cm-D-g)+\lambda_y v-\lambda_M m \leqslant \lambda_v \frac{1}{M}(cm^*-D-Mg)+\lambda_y v - \lambda_M m^*, \quad (2.97)$$

where $m^*$ is any value for $m$ satisfying the constraints (2.78). This condition reduces immediately to

$$\kappa m \geqslant \kappa m^*, \quad (2.98)$$

where

$$\kappa=\lambda_M-\frac{c}{M}\lambda_v. \quad (2.99)$$

On a null thrust arc, $m=0$ and the condition is satisfied provided

$$\kappa \leqslant 0. \quad (2.100)$$

On an arc of maximum thrust, $m=\bar{m}$ and the condition is satisfied provided

$$\kappa \geqslant 0. \quad (2.101)$$

On an arc of intermediate thrust, $m^*$ can take values either greater than or less than $m$ and the condition can only be satisfied with

$$\kappa=0. \quad (2.102)$$

The conditions (2.88) show that

$$\dot{y}=-\lambda_M m. \quad (2.103)$$



at the terminal instant $t=T$. If, therefore, the final sub-arc were one of intermediate thrust on which $\kappa=0$, since $\lambda_v=0$ at $t=T$, $\lambda_M$ would also have to vanish at this instant and eq (2.103) would then indicate that $\dot{y}$ must vanish. However, we have seen that this cannot happen on such an arc. Again, if the final sub-arc were one of maximum thrust, then $\kappa \geqslant 0$ and hence at $t=T$

$$\lambda_M \geqslant \frac{c}{M}\lambda_v=0. \quad (2.104)$$

Then by eq (2.103), $y \leqslant 0$ at this instant. However, even if $\dot{y}=0$, provided the motor thrust is greater than the weight at this instant (as we shall assume), $\dot{y}$ would then be negative prior to $t=T$ and we shall exclude such a possibility (although this can arise if the motor thrust is initially of insufficient magnitude to lift the vehicle from its launching platform). The final sub-arc must therefore be one of null thrust. It follows that $m=0$ finally and hence, by eq (2.103), $\dot{y}$ vanishes at $t=T$ as was to be expected.

Since we are assuming that the rocket's velocity is directed in the upwards sense throughout its motion, the initial sub-arc cannot be one of null thrust. Neither can it be an arc of intermediate thrust, since $v$ cannot vanish on this type of arc. It must accordingly be an arc of maximum thrust.

At a corner separating two arcs of different types, the $\lambda_i$ and the left-hand member of the first integral (2.89) have to be continuous. This latter condition requires that the expression

$$\left(\frac{c}{M}\lambda_v-\lambda_M\right)m \quad (2.105)$$

shall be continuous and this will be the case at a corner where $m$ is discontinuous, only provided

$$\kappa=0 \quad (2.106)$$

at this point.

From eq (2.99)

$$\lambda_M=\frac{c}{M}\lambda_v+\kappa. \quad (2.107)$$

Substituting this expression for $\lambda_M$ into eq (2.85), it will be found that

$$\dot{\lambda}_v = -\frac{M}{c}\dot{\kappa} - \frac{D}{cM}\lambda_v. \tag{2.108}$$

Then, from eq (2.83) it follows that

$$\lambda_y = \frac{1}{M}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right)\lambda_v + \frac{M}{c}\dot{\kappa}. \tag{2.109}$$

Substitution from eqs (2.107), (2.109) for $\lambda_M$, $\lambda_y$ respectively into the first integral (2.89) now yields the equation

$$\dot{\kappa} - \frac{cm}{vM}\kappa = \frac{c}{vM^2}\lambda_v G, \tag{2.110}$$

where $G$ is the left-hand member of eq (2.95). This last equation can also be written

$$\dot{K} = \frac{c}{vM^2}\lambda_v G \exp\left(-\int \frac{cm}{vM}\mathrm{d}t\right), \tag{2.111}$$

where

$$K = \kappa \exp\left(-\int \frac{cm}{vM}\mathrm{d}t\right). \tag{2.112}$$

Consider the eqs (2.83), (2.84). These are equivalent to the equations

$$\dot{v} = -\lambda_y \exp\left(-\int \frac{1}{M}\frac{\partial D}{\partial v}\mathrm{d}t\right), \tag{2.113}$$

$$\dot{\lambda}_y = -H\omega, \tag{2.114}$$

where

$$\omega = \lambda_v \exp\left(-\int \frac{1}{M}\frac{\partial D}{\partial v}\mathrm{d}t\right), \tag{2.115}$$

$$H = -\left(\frac{1}{M}\frac{\partial D}{\partial y} + \frac{\mathrm{d}g}{\mathrm{d}y}\right)\exp\left(\int \frac{1}{M}\frac{\partial D}{\partial v}\mathrm{d}t\right). \tag{2.116}$$

Since $D$ and $g$ will be monotonically decreasing functions of $y$ for all values of $v$, $H$ will always be positive. At $t=T$, the end conditions (2.88) show that $\omega=0$, $\lambda_y=-1$. It now follows from eqs (2.113), (2.114) that both $\omega$ and $\lambda_y$ are strictly increasing functions in the interval $[0, T]$ and hence are negative throughout this interval ($\omega$ and $\lambda_y$ are continuous by the corner conditions). Eq (2.115) then shows that $\lambda_v<0$ over the whole trajectory. From eq (2.111), we can now conclude that $\dot{K}$ and $G$ always possess opposite signs or are zero together.



On an arc of intermediate thrust, $G$ vanishes identically. Provided we assume that the conditions of our problem are such that $G$ invariably decreases on an arc of maximum thrust and always increases on an arc of null thrust, we can now prove that the optimal trajectory comprises either (i) an arc of maximum thrust (MT-arc) followed by an intermediate thrust arc (IT-arc) and terminated by a null thrust arc (NT-arc) or (ii) an MT-arc followed by an NT-arc. That this assumption will generally be justified can be made plausible by relating the drag to the velocity and height by a law of the form

$$D = A\,e^{-\beta y}v^{\gamma}. \tag{2.117}$$

By proper choice of the parameters $A$, $\beta$, $\gamma$, this equation can be made to provide a good approximation for the actual law. Substituting this form of $D$ into the left-hand member of eq (2.95), it will be found that

$$G = Mg - \left(\frac{v}{c} + \gamma - 1\right)D. \tag{2.118}$$

On an NT-arc, the weight $Mg$ will remain constant, except for the negligible variation caused by the decrease of $g$ with increasing height; however, the drag $D$ can be expected to decrease as the velocity decreases. The net effect is for $G$ to increase. On an MT-arc, $Mg$ will decrease and $v$ will increase; however, if $v$ is large, there is the possibility that $D$ may be decreasing owing to the influence of the atmospheric density factor $\exp(-\beta y)$; nevertheless, in most circumstances, $D$ will be found to increase and then $G$ will certainly diminish.

Since $\lambda_v$, $\lambda_M$, $M$ are continuous over the whole optimal trajectory, it follows from eq (2.99) that $\kappa$ is also continuous. Further, by differentiating this equation and employing eqs (2.83), (2.85), we find that

$$\dot{\kappa} = -\frac{1}{M^2}\left(D + c\frac{\partial D}{\partial v}\right)\lambda_v + \frac{c}{M}\lambda_y \tag{2.119}$$

and this demonstrates that $\dot{\kappa}$ is continuous. It now follows from eq (2.112) that $K$ is continuous but, since $m$ is discontinuous, $\dot{K}$ is not continuous. Now suppose that $K$ is plotted against $t$ and that it possesses more than one zero. Let $t=t_2$, $t=t_3$ be successive zeros of $K$ and assume $K>0$ between these instants (Figure 2.1 (a)). Then $\kappa>0$ and the interval $(t_2, t_3)$ corresponds to a period of maximum thrust. $K$ cannot increase monotonically over the interval $(t_2, t_3)$. Suppose $K$ increases over the arc $AB$ and decreases over the arc $BC$, i.e. $\dot{K}>0$ over $AB$ and $\dot{K}<0$ over $BC$. Then $G<0$ over $AB$ and $G>0$ over $BC$. But, by definition, $G$ is continuous everywhere and hence must be increasing at

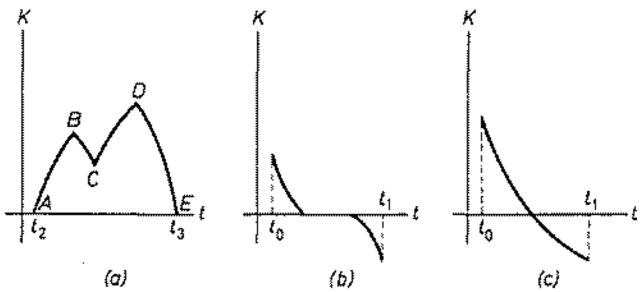


Figure 2.1.

some point between $A$ and $C$. By our assumptions, this implies that a period of null thrust occurs in the interval $(t_2, t_3)$ and this contradicts our earlier conclusion that the interval is a period of maximum thrust. Similarly, if $K<0$ over the interval $(t_2, t_3)$, we are again led to a contradiction. We conclude, therefore, that $K$ cannot possess two zeros separated by an interval over which it does not vanish.

Now, initially, the motor operates at maximum thrust and hence $K\geqslant 0$; finally, the motor is inoperative and hence $K\leqslant 0$. The graph of $K$ must accordingly take one of the forms indicated in Figure 2.1 (b) or (c) and $\dot{K}\leqslant 0$ everywhere. During any period when $K$ vanishes identically, $\dot{K}=0$ and hence $G=0$; such a period must therefore correspond to an IT-arc. It now follows that the initial MT-arc and the terminal NT-arc can be separated



by a single IT-arc at most. Thus, the initial phase of maximum thrust is followed until either the propellent is exhausted or $\nu$ satisfies eq (2.95), whichever occurs the sooner. If the propellent is expended first, the final NT-arc is then entered immediately and Figure 2.1 (c) is appropriate. If, however, $\nu$ satisfies $G=0$ before the propellent is completely expended, the MT-arc cannot be continued; for, if it were, by our hypothesis $G$ would then continue to decrease and would assume negative values; $\dot{K}$ would then become positive, whereas we have proved that $\dot{K}\leqslant 0$ everywhere. Hence, an IT-arc will be entered at this juncture and followed until the propellent is exhausted. Figure 2.1 (b) is appropriate in this case.

It will be noted that the sign of the quantity $\kappa$ or its vanishing along the trajectory, decides whether the appropriate thrust programme is one of maximum thrust, null thrust or intermediate thrust. $\kappa$ is accordingly termed the *thrust magnitude switching function*. Such switching functions feature in many solutions to optimal rocket trajectory problems.

For a detailed consideration of the optimal rocket trajectory corresponding to the atmospheric drag function for a practical vehicle, the reader should consult Leitmann (1957).

If it is possible to neglect the atmospheric drag (e.g. if the rocket is launched from a high altitude), eq (2.95) clearly possesses no solution and the optimal trajectory comprises two arcs only, *viz.* (i) an initial arc of maximum thrust during which all available propellent is consumed and (ii) a final arc of free fall under gravity to the vertex of the trajectory.

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### *Exercises*

1. Rework the problem discussed in section 2.2 with the modification that the rocket is launched from a point on an inclined plane and

its range on this plane is to be maximized. (Hint: Take the $x$-axis in the plane and the $y$-axis perpendicular to the plane.) If the plane of motion intersects the inclined plane in a line which makes an angle $a$ with the horizontal (and above it) and $\theta$ is the (constant) angle made by the thrust with the horizontal, show that the equation corresponding to eq (2.32) is

$$\sin(\theta-a)\sin^2\theta+\frac{f}{g}\cos(2\theta-a)=0.$$

2. Show that, in the case when $f$ is constant, the horizontal distance travelled by a rocket following the optimal trajectory discussed in section 2.3 is given by

$$x_1=\frac{\cos^2\theta_1}{a^2f}\left(\sec\theta_0-\sec\theta-\tan\theta\log\frac{\sec\theta_0+\tan\theta_0}{\sec\theta+\tan\theta}\right).$$

Hence show that the horizontal distance between the launching platform and the point where the satellite is injected into orbit is $g\tan\theta_0\cos^2\theta_1/a^2f^2$.

3. In the problem of section 2.3 with $f$ constant, show that the critical value of $h$ is

$$\frac{u_c^2}{gL^2}\left(\frac{1}{n^2-1}-\frac{L}{2n}\right),$$

where $n=f/g$ and $L=\log\{(n+1)/(n-1)\}$. Show also that, for this value of $h$,

$$T=\frac{2u_c}{gL}\cdot\frac{1}{n^2-1}.$$

If $n$ is large and the satellite orbit is a circle of radius $R$, prove that the critical height is approximately $R/6n^2$.

4. If in the problem of section 2.3 $h$ is less than the critical height, optimize the trajectory with the end conditions (2.72). Show that the thrust direction programme is determined by an equation

$$\tan\theta=-(at+b).$$

In the particular case when $f$ is constant, show that $\theta_0$, $\theta_1$ and $a$ are determined by the equations

$$\tan\tfrac{1}{2}\theta_1=\frac{1-n\tan\frac{1}{2}\theta_0}{n-\tan\frac{1}{2}\theta_0},$$

$$au_c=ng\log\frac{\tan(\frac{1}{2}\theta_0+\frac{1}{4}\pi)}{\tan(\frac{1}{2}\theta_1+\frac{1}{4}\pi)},$$

$$2a^2h+au_c=ng(\sin\theta_0-\sin\theta_1)\sec\theta_0\sec\theta_1.$$

5. With $f=5g$, $u_c=5$ miles/sec, $g=32{\cdot}2$ ft/sec$^2$, compute values for $h$ and $T$ corresponding to values of $\theta_0$ lying in a range which includes the critical height value (use eqs (2.50), (2.55), (2.66), (2.69), (2.70)



and results from ex. 4). Plot $T$ against $h$ and observe that $T$ is a decreasing function of $h$.

6. A sounding rocket is fitted with a booster which gives it a rapid initial acceleration to a velocity satisfying eq (2.95). If $M_0$ is the mass of the rocket when the booster drops away and if the drag law is given by

$$D=Ae^{-\beta y}v^2,$$

show that the rocket velocity at the termination of the boost phase is $cV_0$, where

$$V_0^2+V_0^3=\frac{M_0g}{Ac^2}.$$

If the thrust is now programmed so that the rocket velocity continues to satisfy eq (2.95), assuming $g$ is constant, obtain the following equations for the height $y$ and time of flight $t$ in terms of the velocity $v=cV$:

$$\beta y=V-V_0+\tfrac{1}{2}(\gamma+\rho+3)\log\frac{2V-\gamma-\rho+1}{2V_0-\gamma-\rho+1}$$

$$+\tfrac{1}{2}(\gamma-\rho+3)\log\frac{2V-\gamma+\rho+1}{2V_0-\gamma+\rho+1},$$

$$gt/c=\log\frac{V_0}{V}+\tfrac{1}{2}(\gamma+\rho+1)\log\frac{2V-\gamma-\rho+1}{2V_0-\gamma-\rho+1}$$

$$+\tfrac{1}{2}(\gamma-\rho+1)\log\frac{2V-\gamma+\rho+1}{2V_0-\gamma+\rho+1}$$

where $\gamma=g/\beta c^3$ and $\rho^2=\gamma^2+6\gamma+1$.

# 3

# GENERAL THEORY OF OPTIMAL ROCKET TRAJECTORIES

### 3.1. The Extremal Arcs

In this chapter, the determination of the optimal trajectory for a rocket moving in a perfectly general gravitational field (which may vary with the time), when neither the thrust direction nor thrust magnitude programmes are predetermined, will be studied. The motion will be assumed to take place *in vacuo*, but will not, necessarily, be confined to a plane. Many of the conclusions we shall reach were first derived by Lawden (1953, 1954, 1959 and 1962) and Leitmann (1962).

Let $Ox_1x_2x_3$ be three rectangular Cartesian co-ordinate axes forming an inertial reference frame and let $x_i$ ($i=1, 2, 3$) be the rocket's co-ordinates relative to this frame at time $t$. Then, if $v_i$ are its velocity components at the same instant, its equations of motion may be written

$$\dot{v}_i=\frac{cm}{M}l_i+g_i, \tag{3.1}$$

$$\dot{x}_i=v_i, \tag{3.2}$$

$$\dot{M}=-m, \tag{3.3}$$

where $l_i$ are the direction cosines of the direction of motor thrust, $g_i$ are the components of gravitational acceleration, and $m$ is the rate of propellent expenditure. The $g_i$ will depend upon the position of the rocket in the field and, for a time-variable field, upon the instant $t$ it occupies this position. Thus

$$g_i=g_i(x_1, x_2, x_3, t). \tag{3.4}$$

The $l_i$ are subject to the constraint

$$l_1^2+l_2^2+l_3^2=1 \tag{3.5}$$



and it will be assumed, as in section 2.4, that the rate of propellent expenditure is limited so that $m$ is subject to a constraint

$$m(\bar{m}-m)-\alpha^2=0. \tag{3.6}$$

The control variables are $l_i$, $m$, $\alpha$ and when these are known functions of $t$; the quantities $v_i$, $x_i$, $M$ are determined by the differential constraints (3.1)–(3.3) and initial conditions

$$v_i=v_{i0},\ x_i=x_{i0},\ M=M_0 \tag{3.7}$$

to be satisfied at $t=t_0$.

We shall assume that some function of the final values of the $v_i$, $x_i$, $M$ at $t=t_1$, *viz.* $J$, is to be minimized. However, it will not be necessary to specify this function in detail until a later stage.

Constructing the Lagrange expression

$$F=-\lambda_i\left(\frac{cm}{M}l_i+g_i\right)-\lambda_{i+3}v_i+\lambda_7 m+\mu_1(l_1^2+l_2^2+l_3^2-1)$$
$$+\mu_2[m(\bar{m}-m)-\alpha^2], \tag{3.8}$$

where summation with respect to the index $i$ ($=1, 2, 3$) is understood, the necessary conditions (1.49), (1.50) yield the equations

$$\dot{\lambda}_i=-\lambda_{i+3}, \tag{3.9}$$

$$\dot{\lambda}_{i+3}=-\lambda_j\frac{\partial g_j}{\partial x_i}, \tag{3.10}$$

$$\dot{\lambda}_7=\frac{cm}{M^2}\lambda_i l_i, \tag{3.11}$$

$$0=-\frac{cm}{M}\lambda_i+2\mu_1 l_i, \tag{3.12}$$

$$0=-\frac{c}{M}\lambda_i l_i+\lambda_7+\mu_2(\bar{m}-2m), \tag{3.13}$$

$$0=-2\mu_2\alpha. \tag{3.14}$$

Eq (3.14) requires that either $\mu_2$ or $\alpha$ should vanish, or both. If $\alpha=0$, then $m=0$ or $\bar{m}$. If $\mu_2=0$, $m$ may assume intermediate values. Thus, the optimal trajectory may comprise arcs of three types, (i) null thrust arcs (NT-arcs) over which the vehicle falls freely under gravity; (ii) maximum thrust arcs (MT-arcs) and (iii) arcs of intermediate thrust (IT-arcs).

Elimination of $\lambda_{i+3}$ between eqs (3.9), (3.10), yields the equation

$$\dot{\lambda}_i = \lambda_j \frac{\partial g_j}{\partial x_i}. \tag{3.15}$$

Since the $g_j$ are components of a vector, $\partial g_j/\partial x_i$ behaves as a second rank tensor in respect to transformations between rectangular Cartesian frames such as we are employing and it follows, therefore, that the quantities $\lambda_i$ satisfying eq (3.15) behave like the components of a vector in such transformations. The vector whose components are $\lambda_i$ will be denoted by **p** and will be referred to as the *primer*.

Provided $m$, $\mu_1$ are both non-vanishing, eq (3.12) indicates that the $l_i$ are in the same ratio to one another as the $\lambda_i$ and hence that the thrust direction is always parallel to the primer. If $m=0$, the motor is not operating and the $l_i$ are indeterminate. If $\mu_1=0$, the primer vanishes.

Weierstrass's condition (1.91) requires that, at every point on the optimal trajectory,

$$\left(\frac{c}{M}\lambda_i l_i - \lambda_7\right) m \geq \left(\frac{c}{M}\lambda_i l_i^* - \lambda_7\right) m^* \tag{3.16}$$

for all possible values of the $l_i^*$, $m^*$ satisfying the constraints (3.5), (3.6).

Over an NT-arc, $m=0$ and this last condition requires that

$$\lambda_7 \geq \frac{c}{M}\lambda_i l_i^*. \tag{3.17}$$

The expression $\lambda_i l_i^*$ is the scalar product of the primer and the unit vector having components $l_i^*$. For variable $l_i^*$, its maximum value is obtained when the vector $l_i^*$ is aligned with the primer and this value is then the magnitude $p$ of the primer. Thus, it is necessary that

$$\lambda_7 \geq \frac{c}{M}p, \tag{3.18}$$

where

$$p = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}. \tag{3.19}$$



Over an MT-arc, $m=\bar{m}$ and hence, taking $m^*=\bar{m}$ in the condition (3.16), this is seen to require that

$$\lambda_i l_i \geq \lambda_i l_i^*. \tag{3.20}$$

Also, taking $l_i^*=l_i$ in the inequality (3.16), since $m^* \leq \bar{m}$ it is clear that this condition also requires that

$$\lambda_7 \leq \frac{c}{M}\lambda_i l_i. \tag{3.21}$$

Conversely, conditions (3.20), (3.21) are clearly sufficient to ensure the validity of condition (3.16). Condition (3.20) will be satisfied for all $l_i^*$ if, and only if, $\lambda_i l_i$ takes its maximum value with respect to variations of the $l_i$. But $\lambda_i l_i$ is the scalar product of the vectors $\lambda_i$, $l_i$ and these must, accordingly, be aligned. It follows, in agreement with an earlier result, that the directions of the primer and of the thrust must be identical. In this event,

$$l_i = \lambda_i/(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2} \tag{3.22}$$

and the condition (3.21) then reduces to the form

$$\lambda_7 \leq \frac{c}{M}p. \tag{3.23}$$

In the special case when the $\lambda_i$ all vanish, $p=0$ and the condition (3.23) follows immediately from (3.21).

Over an IT-arc, $m^*$ can take values either greater than or less than $m$. Taking $l_i^*=l_i$ in condition (3.16), it is clear that the inequality cannot be valid for all permissible $m^*$; however, equality is possible provided

$$\lambda_7 = \frac{c}{M}\lambda_i l_i. \tag{3.24}$$

With this value for $\lambda_7$, condition (3.16) then further requires that

$$\lambda_7 \geq \frac{c}{M}\lambda_i l_i^* \tag{3.25}$$

for all permissible $l_i^*$. The right-hand member of this inequality is a maximum when the vectors $\lambda_i$, $l_i^*$ are aligned and then it is necessary that

$$\lambda_7 \geqslant \frac{c}{M}p. \tag{3.26}$$

However, eq (3.24) implies that

$$\lambda_7 \leqslant \frac{c}{M}p, \tag{3.27}$$

with equality only when the vectors $\lambda_i$, $l_i$ are aligned. Hence

$$\lambda_7 = \frac{c}{M}p \tag{3.28}$$

and the thrust must be aligned with the primer, i.e. the $l_i$ are given by eqs (3.22). In the special case when the primer vanishes, eq (3.28) is still valid.

The results obtained from the Weierstrass condition can be summarized as follows: (i) Whenever the motor is operative and $p \neq 0$, the thrust must act in the direction and in the sense of the primer. (ii) Defining the function $\kappa$ by the equation

$$\kappa = \frac{c}{M}p - \lambda_7, \tag{3.29}$$

it is necessary that $\kappa \leqslant 0$ over an NT-arc, $\kappa \geqslant 0$ over an MT-arc and $\kappa = 0$ over an IT-arc.

At a corner separating two arcs of different types, the multipliers $\lambda_i$, $\lambda_{i+3}$, $\lambda_7$ must be continuous. By eq (3.9), this is equivalent to the statement that $\mathbf{p}$, $\dot{\mathbf{p}}$ and $\lambda_7$ are continuous. The locus of a point $P$, whose position vector relative to the origin of a set of rectangular axes is $\mathbf{p}$, will accordingly be a smooth curve without corners. This locus will be referred to as the *primer locus diagram*. Since the thrust is always in the direction of $\mathbf{p}$, it is clear that the $l_i$ cannot be discontinuous except at a point where the primer vanishes. At such a point, the primer locus passes through the origin and the direction of the primer is reversed. Thus, the $l_i$ will all change sign and the thrust direction will reverse.

At a corner, there is the further requirement that the expression (1.64) should be continuous. Substituting for the derivatives of the state variables from the constraints (3.1)–(3.3), since the $\lambda_i$, $\lambda_{i+3}$, $g_i$ and $v_i$ are known to be continuous, it follows that this requirement is satisfied if



$$\frac{cm}{M}\lambda_i l_i - \lambda_7 m = m\kappa \tag{3.30}$$

is continuous. Hence, if $m$ is discontinuous at a corner, then

$$\kappa = 0 \tag{3.31}$$

at this point.

It remains to take account of eqs (3.11) and (3.13). We shall eliminate $\lambda_7$ from the argument at this point, expressing it in terms of $\kappa$ by eq (3.29). Substituting for $\lambda_7$ in this manner into eqs (3.11) and (3.13) and employing eq (3.22), it will be found that

$$\dot{\kappa} = \frac{c}{M}\dot{p}, \tag{3.32}$$

$$\kappa = \mu_2(\bar{m} - 2m). \tag{3.33}$$

(N.B. Eq (3.22) will be employed to define the $l_i$ when $m = 0$.)

On NT- and MT-arcs, eq (3.33) serves only to determine $\mu_2$. On an NT-arc, $M$ is constant and eq (3.32) is immediately integrable to yield

$$\kappa = \frac{c}{M}p + \text{constant}. \tag{3.34}$$

On an IT-arc, $\mu_2 = 0$ and hence $\kappa = 0$. Eq (3.32) accordingly implies that

$$p = \text{constant}, \tag{3.35}$$

i.e. the primer is of constant magnitude.

The results of this section will now be collected together.

The primer $\mathbf{p}$ possesses components satisfying the eq (3.15) which, in vector notation, takes the form

$$\ddot{\mathbf{p}} = \nabla(\mathbf{p}\cdot\mathbf{g}), \tag{3.36}$$

where $\mathbf{g}$ is the gravitational acceleration. Whenever the rocket motor is operative, its thrust is aligned with $\mathbf{p}$ (provided $p \neq 0$). Over an NT-arc, $\kappa$ is negative and is determined by eq (3.34). Over an MT-arc, $\kappa$ is positive and is determined by eq (3.32). Over an IT-arc, $\kappa$ vanishes identically and the primer's magnitude is constant. $\kappa$ is the *thrust magnitude switching function*. At a corner between arcs of two different types where $m$ is discontinuous, $\kappa$ vanishes; in addition, the primer and its first derivative are continuous and, since $\lambda_7$ and $p$ are continuous, so is $\kappa$.

Also, it is evident from eq (3.32) that $\dot{\kappa}$ is continuous everywhere, except possibly at a point where **p** vanishes. At such a point, the thrust direction will reverse.

Finally, in this section, we shall take note of a first integral of the eqs (3.36) which exists in the case when the gravitational field is time-invariant. In these circumstances, the $g_i$ are independent of $t$ and a first integral in the form of eq (1.67) is valid over the whole optimal trajectory. Thus

$$\lambda_i \dot{v}_i + \lambda_{i+3}\dot{x}_i + \lambda_7 \dot{M} = C, \qquad (3.37)$$

where $C$ assumes the same constant value over the whole trajectory. Substituting for the time derivatives from eqs (3.1)–(3.3), for the $\lambda_{i+3}$ from eq (3.9), for $\lambda_7$ from eq (3.29) and for the $l_i$ from eq (3.22), the first integral is expressible in the form

$$\lambda_i g_i - \dot{\lambda}_i v_i + \kappa m = C. \qquad (3.38)$$

In vector notation, this reads

$$\mathbf{p}\cdot\mathbf{g} - \dot{\mathbf{p}}\cdot\mathbf{v} + \kappa m = C, \qquad (3.39)$$

where **v** is the rocket velocity.

On an arc of null thrust, $m = 0$ and on an arc of intermediate thrust, $\kappa = 0$. On such arcs, therefore, eq (3.39) reduces to

$$\mathbf{p}\cdot\mathbf{g} - \dot{\mathbf{p}}\cdot\mathbf{v} = C. \qquad (3.40)$$

### 3.2. Impulsive Thrusts

Letting $\bar{m} \rightarrow \infty$, the situation is approached where the thrust capability of the rocket motor is not bounded. A maximum thrust phase is then replaced by an impulsive thrust of negligible duration, serving only to change the vehicle velocity without in any way affecting its position. For the problem of interplanetary transfer employing a conventional rocket, capable of delivering a thrust comparable in magnitude with the vehicle's earth weight, the duration of a phase of maximum thrust is always small by comparison with the total time of transit between the planetary terminals. The motion of the rocket during these phases can therefore, for the purpose of calculating an optimal trajectory, often be neglected and then such a phase is replaceable by an impulsive thrust. In these circumstances, it is accordingly usual to regard $\bar{m}$ as being infinite and the results obtained in the previous section are then simplified as follows:



We shall, in future, refer to a point on the optimal trajectory at which an impulsive thrust is applied as a *junction point*. Across such a junction point, the rocket co-ordinates $x_i$ are continuous functions of $t$ and hence so are the coefficients $\partial g_j/\partial x_i$ in the eqs (3.15) governing the primer components. It follows from these equations, therefore, that the primer and its first and second order time derivatives are also continuous across a junction point. However, the third order derivative is given by the equation

$$\dddot{\lambda}_i = \dot{\lambda}_j \frac{\partial g_j}{\partial x_i} + \lambda_j \left( \frac{\partial^2 g_j}{\partial x_i \partial x_k}\dot{x}_k + \frac{\partial^2 g_j}{\partial x_i \partial t} \right) \qquad (3.41)$$

and, since the $\dot{x}_k$ are discontinuous, it accordingly is also discontinuous in general.

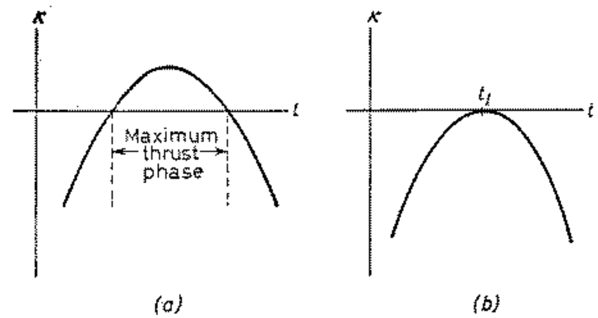


Figure 3.1.

Consider a phase of maximum thrust, preceded and succeeded by null thrust phases. During the phase of maximum thrust, we know that $\kappa \geqslant 0$ and hence the graph of this switching function takes the form indicated in Figure 3.1 (a). If the magnitude of the thrust be now increased and its duration decreased so that it approaches the impulsive condition, it appears that the graph of $\kappa$ must, in the limit, take the form shown in Figure 3.1 (b). This implies that, at the instant of application of an impulsive thrust $t = t_1$, it is necessary that

$$\kappa = \dot{\kappa} = 0 \qquad (3.42)$$

and that $\kappa$ is a maximum. If either or both the null thrust phases are replaced by an intermediate thrust phase over which $\kappa=0$ identically, this conclusion is not affected.

Since, however, the above argument does not entirely exclude the possibility that the $\kappa$-graph may have a cusp at $t=t_1$, we shall also present a more rigorous argument: For our present problem, eq (1.63) takes the form

$$\frac{dF}{dt}=-\lambda_i\frac{\partial g_i}{\partial t}. \tag{3.43}$$

Integrating over the infinitesimal duration of the impulsive thrust, since the right-hand member of this equation remains finite throughout this time interval, we obtain

$$F^+-F^-=0, \tag{3.44}$$

where $F^-$, $F^+$ are the values taken by $F$ immediately prior and immediately subsequent to the impulse. Thus $F$ is continuous across the impulse. But, on the optimal trajectory,

$$F=-\lambda_i g_i+\dot{\lambda}_i v_i-\kappa m. \tag{3.45}$$

Now $\kappa m=0$ on either an NT- or an IT-arc, and hence $\kappa m$ must vanish on both sides of the impulse. Also, $\lambda_i$ and $g_i$ are continuous across the junction. It follows that $\dot{\lambda}_i v_i$ is continuous across the impulse, i.e.

$$\dot{\lambda}_i v_i^- = \dot{\lambda}_i v_i^+,$$

or

$$\dot{\lambda}_i(v_i^+ - v_i^-)=0, \tag{3.46}$$

since the $\dot{\lambda}_i$ are also continuous across the impulse. The directions of the impulse and primer are known to be aligned and thus

$$v_i^+ - v_i^- = \beta\lambda_i, \tag{3.47}$$

where $\beta$ ($\neq 0$) is a proportionality factor. Hence, eq (3.46) is equivalent to

$$\lambda_i\dot{\lambda}_i=0,$$

or

$$\dot{p}=0. \tag{3.48}$$

The vanishing of $\dot{\kappa}$ now follows from eq (3.32).

If the impulse is applied at either of the terminals of the



trajectory, the above argument fails and condition (3.48) is invalid.

Consider an NT-arc joining two junction points. If $p=P$ at one junction point, eq (3.34) is valid on the arc and, since $\kappa=0$ at a junction,

$$\kappa=\frac{c}{M}(p-P) \tag{3.49}$$

over the arc. It follows that $p=P$ at the other junction also. Thus, if the optimal trajectory comprises NT-arcs separated by junction points alone, then $p=P$ at all junctions and eq (3.49) is valid over every NT-arc. Over an IT-arc, $\kappa$ vanishes and $p$ is constant; in respect to the values of $\kappa$ and $p$ therefore, an IT-arc behaves exactly like a junction point and our previous argument is accordingly in no way affected by the presence of these arcs in the optimal trajectory. We can now express all the conditions to be satisfied by the switching function $\kappa$ in terms of $p$. We shall require that $p=P$ at all junction points and over all IT-arcs. Also, over every NT-arc, we shall require that $p\leqslant P$. If these conditions are satisfied then, from eqs (3.32), (3.49) and the continuity of $\kappa$, it follows that $\kappa=0$ at all junctions and over every IT-arc and that $\kappa\leqslant 0$ over all NT-arcs. Finally, we shall require that $\dot{p}=0$ at all junctions; by eq (3.32), this implies that $\dot{\kappa}=0$ as required by condition (3.42) (this replaces the condition $\kappa\geqslant 0$ over an arc of maximum thrust of the general theory).

If impulsive thrusts are permitted, therefore, the following conditions must be satisfied over an optimal trajectory:

(*a*) The primer and its first time derivative must be continuous everywhere;

(*b*) Whenever the rocket motor is operative, the thrust must be aligned with the primer which must have a certain constant magnitude $P$;

(*c*) The magnitude $p$ of the primer must not exceed $P$ on any NT-arc;

(*d*) $\dot{p}=0$ at all junction points not coincident with the terminals.

These conditions were first obtained by Lawden (1953, 1954, 1959 and 1962).

Condition ($c$) implies that $p$ takes a maximum value $P$ at each junction and over each IT-arc. The condition ($d$) is equivalent to

$$\lambda_i \dot{\lambda}_i = 0, \tag{3.50}$$

or

$$\mathbf{p} \cdot \dot{\mathbf{p}} = 0; \tag{3.51}$$

i.e. the primer and its first derivative are orthogonal at a junction. It has been proved earlier in this section that condition ($d$) is equivalent to the requirement that the Lagrange expression $F$ should be continuous across a junction point. If, therefore, the gravitational field is time-invariant and the first integral (3.38) is employed to determine the optimal trajectory, the constancy of $C$ over the whole trajectory ensures the continuity of $F$ across all junctions and condition ($d$) is then superfluous.

The value of the maximum primer magnitude $P$ will often be determined by the end conditions. These will be considered in the next section.

### 3.3. Special Cases of the End Conditions

In this section we will examine the form taken by the end conditions (3.17), (3.18) and their implications for two problems of practical importance.

Consider first the problem of determining the manœuvre by which a rocket may escape most economically from a given gravitational field (supposed time-invariant). We shall assume that a body will escape from the field provided its total energy (kinetic and potential) exceeds the potential energy the body would possess at infinity, i.e. at a great distance from all the gravitating bodies responsible for the field. The problem will accordingly be posed in the following terms: The rocket is launched from a given point in the field with a given velocity and its final mass $M_1$, when all propellent has been consumed, is known. The values of the $x_i$ and $v_i$ at $t = t_1$ are not predetermined, but the manœuvre is to be such that the total energy of the vehicle, *viz.*

$$M[V(x_1, x_2, x_3) + \tfrac{1}{2}(v_1^2 + v_2^2 + v_3^2)], \tag{3.52}$$

calculated at $t = t_1$ is to be maximized. Here, $V$ is the potential function for the field. The time at the later terminal of the trajectory, *viz.* $t_1$, is open to choice.

Since $M_1$ is predetermined, we shall take



$$J = -V - \tfrac{1}{2}(v_1^2 + v_2^2 + v_3^2), \tag{3.53}$$

where it is understood that all quantities are to be evaluated at the instant $t = t_1$ and we shall then minimize this quantity. The conditions (1.46) require that

$$\left.\begin{aligned} &\lambda_1 = v_1, \quad \lambda_2 = v_2, \quad \lambda_3 = v_3, \\ &\lambda_4 = \frac{\partial V}{\partial x_1}, \quad \lambda_5 = \frac{\partial V}{\partial x_2}, \quad \lambda_6 = \frac{\partial V}{\partial x_3}. \end{aligned}\right\} \tag{3.54}$$

Since $\partial V/\partial x_i = -g_i$, these conditions are equivalent to the demand that

$$\mathbf{p} = \mathbf{v}, \ \dot{\mathbf{p}} = \mathbf{g} \tag{3.55}$$

at $t = t_1$. In addition, the condition (1.47) requires that

$$\lambda_i \dot{v}_i + \lambda_{i+3} \dot{x}_i + \lambda_7 \dot{M} = 0 \tag{3.56}$$

at $t = t_1$. Referring to eq (3.37), it is clear that eq (3.56) is a first integral for the optimal trajectory and is valid at all its points. It may also be expressed in the form

$$\mathbf{p} \cdot \mathbf{g} - \dot{\mathbf{p}} \cdot \mathbf{v} + \kappa m = 0 \tag{3.57}$$

as explained in connection with eq (3.37). Since the eqs (3.55) are valid at $t = t_1$, at this instant eq (3.57) reduces to

$$\kappa m = 0. \tag{3.58}$$

Along an NT-arc, the net energy of the vehicle remains constant. Any optimal escape manœuvre can accordingly always be extended by an arbitrary length of such an arc, and this suggests that if the conditions (3.55), (3.58) are satisfied at any one point on an NT-arc, they are then satisfied at all points on the arc. Since $m = 0$, condition (3.58) is trivial on such an arc. Suppose that on an NT-arc we choose the primer to be identical with the velocity vector, i.e.

$$\lambda_i = v_i. \tag{3.59}$$

Then, by the equations of motion (3.1), it follows that

$$\dot{\lambda}_i = g_i \tag{3.60}$$

on the arc. The eqs (3.15) for the primer are now satisfied provided

$$\dot{g}_i = v_j \frac{\partial g_j}{\partial x_i}. \tag{3.61}$$

But

$$v_j \frac{\partial g_j}{\partial x_i} = \dot{x}_j \frac{\partial^2 V}{\partial x_i \partial x_j} = \dot{x}_j \frac{\partial g_i}{\partial x_j} = \dot{g}_i. \tag{3.62}$$

Hence eq (3.59) provides a possible solution for the primer on an NT-arc. Since the equations governing the primer are of the second order, their solution is uniquely determined when the primer and its first derivative are given at a single point. But the conditions (3.55) are in conformity with the solution (3.59) and hence this must be the appropriate solution for the final NT-arc.

In view of these remarks, we shall dismiss the possibility that the final arc is one of null thrust and will suppose, therefore, that $m$ is non-vanishing at $t=t_1$. Then, by condition (3.58),

$$\kappa = 0 \tag{3.63}$$

at this terminal. Since the motor is operating at $t=t_1$, this must be the instant of 'all-burnt' and the first of the conditions (3.55) indicates that, at this moment, the thrust must operate in the direction of motion (Lawden, 1955).

If impulsive thrusts are permitted, the optimal trajectory must either terminate at a junction point or the final arc must be one of intermediate thrust. In either event, $\kappa=0$ at this terminal as required by condition (3.63) and this condition therefore adds nothing to the conditions (*a*)–(*d*) stated on p. 63. It may also be remarked at this juncture, that all these conditions can be satisfied in a very simple manner for the special case when the rocket's initial motion is in a direction which is perpendicular to the force field. For suppose that all propellent is expended instantaneously at the initial instant, the resulting impulsive thrust being directed along the line of motion. The trajectory will then comprise one junction point alone. At this point, the primer and its derivative must be given by eqs (3.55). Hence

$$\mathbf{p}\cdot\dot{\mathbf{p}} = \mathbf{v}\cdot\mathbf{g} = 0, \tag{3.64}$$

since $\mathbf{v}$ and $\mathbf{g}$ are perpendicular. This result implies that $\dot{p}=0$, as required by condition (*d*). Condition (*b*) is also clearly satisfied and conditions (*a*) and (*c*) are void. This simple manœuvre is accordingly a candidate for classification as the optimal manœuvre. In particular, consider a vehicle which is revolving initially in a circular orbit about a spherically symmetric attracting body. To effect escape from this attraction, an impulsive thrust could be applied at any point on the orbit in a direction which is tangential to the orbit. If the impulse is of sufficient magnitude, the rocket will then recede to an infinite distance along one branch of a hyperbola. Such an escape manœuvre satisfies all our conditions for an optimum; however, as will be shown later (p. 112), if the launching point is taken to be some point on the circular orbit preceding the junction point, so that this orbit is included in the optimal trajectory, then condition (*c*) is not satisfied on this orbit if the final velocity of recession exceeds a certain critical value. In these circumstances, an alternative manœuvre proves to be more economical.



For our second problem, we will try to minimize the characteristic velocity for a manœuvre which transfers the rocket from one given fixed terminal to another. The vehicle's velocity will be assumed specified at both terminals. Thus, by eq (2.7), we take

$$J = c \log \frac{M_0}{M_1} \tag{3.65}$$

and then condition (1.46) shows that

$$\lambda_7 = c/M_1 \tag{3.66}$$

at $t=t_1$. Also, if the time of arrival is not predetermined, condition (1.47) requires that

$$\lambda_i g_i - \dot{\lambda}_i v_i - \kappa m = 0 \tag{3.67}$$

at the terminal. In the case of a time-invariant field, eq (3.67) is valid over the whole trajectory.

By eq (3.29), the condition (3.66) requires that at $t=t_1$

$$\kappa = \frac{c}{M}(p-1). \tag{3.68}$$

Now suppose that impulsive thrusts are permitted. Then, if the final sub-arc is an NT-arc, eq (3.49) is valid over the arc and condition (3.68) will be satisfied provided

$$P = 1. \tag{3.69}$$

If the final sub-arc is an IT-arc, then $\kappa=0$ and $p=P$ over the arc and, again, condition (3.68) can only be satisfied by taking

$P=1$. We conclude, therefore, that an optimal transfer trajectory satisfies conditions (*a*)–(*d*) on p. 63 with $P=1$. Whenever the motor is operating, the primer is then the *unit* vector in the direction of the thrust.

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### *Exercises*

1. If at any point having position vector $\mathbf{r}$ with respect to an origin $O$, the gravitational acceleration is given by the equation

$$\mathbf{g} = -\omega^2\mathbf{r},$$

show that over the whole of an optimal trajectory in the field, the primer is given by an equation of the form

$$\mathbf{p} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t,$$

where $\mathbf{a}$, $\mathbf{b}$ are constant vectors. Deduce that the primer locus diagram is an ellipse, a circle or a straight line.

Show also that, if an IT-arc contributes to the trajectory, the primer locus is a circle, the magnitude of the motor thrust is indeterminate, and its direction rotates in a plane with constant angular velocity $\omega$.

If the primer locus is not a circle and impulsive thrusts are permitted, show that only two junction points can exist and that these are such that the line joining them is bisected at $O$. Prove, also, that the thrust directions at these junctions are parallel but in opposite senses.

If an NT-arc of the optimal trajectory has equation

$$\mathbf{r} = \mathbf{A}\cos\omega t + \mathbf{B}\sin\omega t$$

and the time of transit is not predetermined, show that

$$\mathbf{a}\cdot\mathbf{A} + \mathbf{b}\cdot\mathbf{B} = 0.$$

2. Prove that, at any junction between extremal arcs of different types at which no impulsive thrust is applied, the second and third order time derivatives of the primer are continuous, whereas that of the fourth order is not.

If $q = p^2$, deduce that at such a junction

$$\left|\frac{d^4 q}{dt^4}\right| = 2\mathbf{p}\cdot\left|\frac{d^4 \mathbf{p}}{dt^4}\right|,$$

where $|x|$ denotes the magnitude of the discontinuity in $x$ at the junction.

Hence show that at a point where the vehicle leaves an IT-arc and enters an NT-arc

$$\frac{d^4 q}{dt^4} = -2f\lambda_i\lambda_j\lambda_k\frac{\partial^2 g_j}{\partial x_i \partial x_k},$$

$f$ being the acceleration due to the motor in the IT-arc at the junction.

## 4

# OPTIMAL TRAJECTORIES IN A UNIFORM FIELD

### 4.1. General Theory

It has been shown by Leitmann (1959) that, if the gravitational field is uniform, a theory of rocket trajectories optimizing quantities belonging to a wide class can be presented in a fairly complete form.

In these circumstances, g is constant and the differential equation (3.36) for the primer reduces to the simple form

$$\ddot{\mathbf{p}}=0, \tag{4.1}$$

for which the general solution is evidently

$$\mathbf{p}=\mathbf{a}t+\mathbf{b}, \tag{4.2}$$

where **a**, **b** are constant vectors. Since **p**, $\dot{\mathbf{p}}$ are continuous over the whole optimal trajectory, a and **b** are the same on all sub-arcs and the primer locus diagram is accordingly a straight line which is described by the tracing point at a constant velocity **a**. In the special case when **a** vanishes, the locus degenerates to a single point, and the thrust direction will then never vary throughout the manœuvre. In general, the thrust direction will vary in the plane containing the locus and the origin of the locus diagram. If the axes $Ox_1$, $Ox_2$ are taken parallel to this plane, the components of the primer will be given by equations of the type

$$\lambda_1=a_1t+b_1,\ \lambda_2=a_2t+b_2,\ \lambda_3=0, \tag{4.3}$$

where the $a_i$ and $b_i$ are constants. Then, if $\theta$ is the angle made by the thrust direction with $Ox_1$ at time $t$,

$$\tan\theta=\lambda_2/\lambda_1=\frac{a_2t+b_2}{a_1t+b_1}. \tag{4.4}$$



This is the *bilinear tangent law* for the thrust direction in a uniform field already encountered in section 2.2 (eq (2.24)). Since $\lambda_3=0$, the thrust always has zero component in the direction of the $x_3$-axis, and the rocket's motion parallel to this axis must be one of free fall under gravity.

It is easy to prove that an optimal trajectory can include an IT-arc only in very special circumstances. For it has been shown in section 3.1 that, on such an arc, the primer is of constant magnitude (eq (3.35)). The primer locus cannot then be a straight line; it must therefore reduce to a point and the thrust direction must be invariable throughout the manœuvre. The $l_i$ now being constants, it follows that the equations of motion (3.1)–(3.3) are easily integrated over the period $(t_0, t_1)$ of the manœuvre and under the initial conditions (3.7) to yield

$$v_{i1}-v_{i0}=c\,l_i\log R+g_i(t_1-t_0), \tag{4.5}$$

$$x_{i1}-x_{i0}=c\,l_iS+v_{i0}(t_1-t_0)+\tfrac{1}{2}g_i(t_1-t_0)^2, \tag{4.6}$$

where

$$R=M_0/M_1,\ S=\int_{t_0}^{t_1}\log\frac{M_0}{M}\mathrm{d}t. \tag{4.7}$$

It is clear that the three quantities

$$v_{i1}-v_{i0}-g_i(t_1-t_0) \tag{4.8}$$

must be in the same ratios to one another as the quantities

$$x_{i1}-x_{i0}-v_{i0}(t_1-t_0)-\tfrac{1}{2}g_i(t_1-t_0)^2. \tag{4.9}$$

However, this requirement is often ruled out by the end conditions and, if this be the case, no IT-arc can be present in the optimal trajectory. If, on the other hand, eqs (4.5), (4.6) are not inconsistent with the end conditions, it is clear that any number of different thrust magnitude programmes can lead to the same values for $R$ and $S$ and hence to the same values for $x_{i1}$, $v_{i1}$ and $J$. It follows that the thrust magnitude programme will not then be unique.

Assuming that the end conditions rule out the possibility of an IT-arc, the optimal trajectory will comprise NT- and MT-arcs alone. It will be shown that there cannot be more than three of these sub-arcs. This result is due to Leitmann (1959).

First consider the case when the primer locus is a straight line which does not pass through the origin. Since $p$ is the distance of the tracing point from the origin, $p$ will first decrease to a minimum and then increase steadily. Hence, by eq (3.32), $\dot{\kappa}$ will first be negative, then zero and finally positive. $\kappa$ will accordingly first decrease steadily to a minimum and subsequently increase steadily. But, eq (3.32) indicates that both $\kappa$ and $\dot{\kappa}$ are continuous quantities and it follows, therefore, that the graph of $\kappa$ against $t$ can intersect the $t$-axis in at most two points. Since $\kappa$ vanishes at every corner between sub-arcs of different types, this implies that at most two corners can be present on an optimal trajectory and the maximum number of sub-arcs is therefore three. Also, if there are three sub-arcs, over the first $\kappa$ is positive, over the second it is negative, and over the third it is positive again. This implies that an NT-arc is preceded and followed by MT-arcs.

Now consider the special case when the primer locus passes through the origin. Let $W$ be the speed with which the tracing point traverses its straight path. Then

$$\left.\begin{aligned} \dot{p} &= -W,\ t<\tau, \\ &= +W,\ t>\tau, \end{aligned}\right\} \tag{4.10}$$

where $t=\tau$ is the instant the point passes through the origin. Thus, from eq (3.32),

$$\left.\begin{aligned} \dot{\kappa} &= -cW/M,\ t<\tau, \\ &= +cW/M,\ t>\tau, \end{aligned}\right\} \tag{4.11}$$

and hence $\dot{\kappa}$ never vanishes. In this case $\dot{\kappa}$ is discontinuous at $t=\tau$, being negative prior to this instant and positive afterwards. Thus $\kappa$ decreases monotonically for $t<\tau$ and increases monotonically for $t>\tau$ and its graph can intersect the $t$-axis in at most two points. The same conclusion now follows as before.

The end conditions (1.46), (1.47) alone depend upon the quantity $J$ being minimized and these have to be written down separately for each particular problem. This we shall do in the examples which follow in the next section.

## 4.2. Particular Problems

For our first three examples, the rocket motion will be restricted



to take place in a vertical plane. $Oxy$ will denote a set of rectangular axes lying in this plane, the $x$-axis being horizontal, and the $y$-axis vertical and in the upwards sense. Velocity components parallel to these axes will be denoted by $(u, v)$ and $\lambda_x$, etc., and will represent the Lagrange multipliers corresponding to the state variables $x, y, u, v, M$, respectively. It follows that the primer will have components $(\lambda_u, \lambda_v)$ and its derivative will have components $(-\lambda_x, -\lambda_y)$. $t=0$ will be taken to be the instant of launching and the manœuvre will be supposed to terminate at $t=T$. The angle made by the thrust direction with $Ox$ will be denoted by $\theta$.

Suppose first that the rocket is launched from the origin with zero initial velocity and that the thrust direction and propellent expenditure programmes are to be chosen so that the vehicle's final height is a maximum. We take

$$J = -y_1 \tag{4.12}$$

so that $J$ is to be minimized. At $t=T$, all propellent has been expended and

$$M = M_1 \tag{4.13}$$

is known. $u_1, v_1, x_1, y_1, T$ are all variable and the end conditions (1.46), (1.47) accordingly require that

$$\lambda_u=\lambda_v=\lambda_x=0,\ \lambda_y=1, \tag{4.14}$$

$$\lambda_u\dot{u}+\lambda_v\dot{v}+\lambda_x\dot{x}+\lambda_y\dot{y}+\lambda_M\dot{M}=0, \tag{4.15}$$

at $t=T$.

Eq (4.2), taken together with the conditions (4.14), implies that

$$\lambda_u=0,\ \lambda_v=T-t,\ \lambda_x=0,\ \lambda_y=1. \tag{4.16}$$

The primer is therefore directed along the positive $\lambda_v$-axis and decreases in magnitude from $T$ to 0 as the motion proceeds. The primer locus therefore comprises the segment of the $\lambda_v$-axis extending from the point $(0, T)$ to the origin. The thrust is always directed vertically upwards, therefore, and $\theta=\frac{1}{2}\pi$. The motion takes place along the $y$-axis.

From eq (3.32), it follows that

$$\dot{\kappa} = -c/M \tag{4.17}$$

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and thus $\kappa$ is always decreasing and the graph of $\kappa$ can intersect the $t$-axis at most once. Thus, either $\kappa > 0$ everywhere and the optimal trajectory comprises a single MT-arc, or $\kappa > 0$ during an initial phase and $\kappa < 0$ thereafter, when the trajectory comprises an MT-arc followed by an NT-arc.

At $t = T$, $p = 0$ and hence, by eq (3.29), $\kappa = -\lambda_M$. Condition (4.15) accordingly requires that

$$\dot{y} = \kappa \dot{M} \tag{4.18}$$

at $t = T$. If the trajectory is a single MT-arc then, at $t = T$, $\kappa \geqslant 0$ and $\dot{M} < 0$. Hence $\dot{y} \leqslant 0$ finally and this can only be the case if the motor thrust is less than the rocket weight during an initial phase of the motion. Clearly, this possibility is not realistic and therefore, in any practical case, the optimal thrust magnitude programme will comprise an initial phase of maximum thrust terminated at 'all-burnt', followed by a ballistic phase of free fall under gravity to the vertex.

For our second example, we shall suppose that the rocket is launched from the origin with velocity components $(u_0, v_0)$ and that, thereafter, it is required to programme the flight so that the vehicle's final velocity is as large as possible. Thus, we take

$$J = -(u_1^2 + v_1^2)^{1/2} \tag{4.19}$$

and minimize $J$. At $t = T$,

$$M = M_1 \tag{4.20}$$

is given, but $u_1$, $v_1$, $x_1$, $y_1$ are all open to choice. $T$ will be taken to be predetermined.

The end conditions (1.46) require that

$$\left.\begin{aligned} \lambda_u &= \frac{u_1}{(u_1^2 + v_1^2)^{1/2}} = \cos\psi, \\ \lambda_v &= \frac{v_1}{(u_1^2 + v_1^2)^{1/2}} = \sin\psi, \\ \lambda_x &= \lambda_y = 0, \end{aligned}\right\} \tag{4.21}$$

at $t = T$, $\psi$ being the angle made by the vehicle's final velocity with $Ox$. Thus the derivative of the primer vanishes and eq (4.2) shows that the eqs (4.21) are valid over the whole trajectory.



$\lambda_u$, $\lambda_v$ are constants and the primer is a unit vector making a constant angle $\psi$ with the $\lambda_u$-axis. Thus $\theta = \psi$ and the thrust direction never varies. The primer locus is a single point and hence this is the case when an IT-arc is a possibility and the thrust magnitude programme is arbitrary. The eqs (4.5) are applicable, and, in our present notation, take the form

$$u_1 = u_0 + c \cos\psi \log R, \tag{4.22}$$

$$v_1 = v_0 - gT + c \sin\psi \log R. \tag{4.23}$$

$\psi$ is now determined from the equation

$$\tan\psi = v_1/u_1. \tag{4.24}$$

Substituting for $u_1$, $v_1$ from eqs (4.22), (4.23), it will be found that $\psi$ must be such that

$$\tan\psi = (v_0 - gT)/u_0. \tag{4.25}$$

With this value for $\psi$, eqs (4.22), (4.23) yield the result

$$(u_1^2 + v_1^2)^{1/2} = \sqrt{[u_0^2 + (v_0 - gT)^2]} + c \log R, \tag{4.26}$$

indicating that $J$ is independent of the thrust magnitude programme selected.

For our third programme, suppose the rocket is launched from the origin with zero initial velocity and that its horizontal range $x_1$ is to be maximized. We shall assume that the vehicle is to arrive at its final position at a given instant and with a given horizontal velocity and mass, i.e. $T$, $u_1$ and $M_1$ are all predetermined. Taking

$$J = -x_1, \tag{4.27}$$

the end conditions (1.46) require that

$$\lambda_v = \lambda_y = 0, \ \lambda_x = 1, \tag{4.28}$$

at $t = T$. It follows from eq (4.2) that

$$\lambda_u = \tau - t, \ \lambda_v = 0, \ \lambda_x = 1, \ \lambda_y = 0, \tag{4.29}$$

where $\tau$ is the remaining unknown constant.

The primer is directed along the $\lambda_u$-axis, in the positive sense if $t < \tau$ and in the negative sense if $t > \tau$. Thus $\theta = 0$ or $\pi$ and the primer locus consists of the segment of the $\lambda_u$-axis lying between the points $(\tau, 0)$ and $(\tau-T, 0)$. Since $\lambda_u$, $\lambda_v$ both vanish at $t=\tau$, this is the case when $\dot{\kappa}$ can possess a discontinuity. From eqs (4.11), we calculate that

$$\left.\begin{aligned}\dot{\kappa}&=-c/M,\ t<\tau,\\ &=+c/M,\ t>\tau.\end{aligned}\right\} \tag{4.30}$$

Thus, $\kappa$ decreases to the instant $t=\tau$ and thereafter increases. Its graph must be as indicated in Figure 4.1 (a) or 4.1 (b). If Figure 4.1 (b) is correct, $\kappa>0$ everywhere and a phase of maximum horizontal thrust in the positive sense is immediately followed by a phase of maximum thrust in the opposite sense.

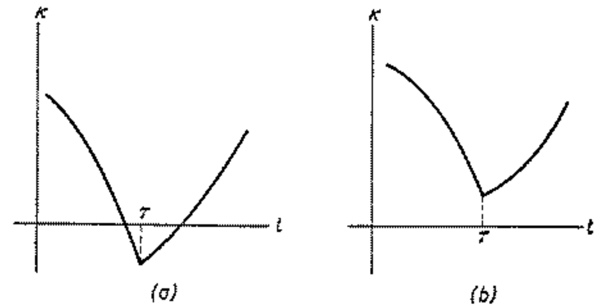


Figure 4.1.

The time to 'all-burnt' will then be completely fixed by the known maximum rate of propellent expenditure and this will, in general, be different from the required value $T$. Thus Figure 4.1 (a) must represent the normal case, i.e. a maximum thrust phase with $\theta=0$, followed by a null thrust phase, followed by a second maximum thrust phase with $\theta=\pi$.

Integration of the horizontal component of the equation of motion (with $\theta=0$ or $\pi$), shows that the final horizontal velocity generated by the two thrust phases is given by

$$u_1=c\log\frac{M_0}{M_i}-c\log\frac{M_i}{M_1}=c\log\frac{M_0M_1}{M_i^2}, \tag{4.31}$$



where $M_i$ is the rocket mass at the termination of the first phase. Provided the given value of $u_1$ does not exceed $c\log R$ in magnitude, this equation determines $M_i$ in the range $M_0>M_i>M_1$. The optimal trajectory is now completely fixed, the duration of the null thrust phase being the difference between the time required to expend all the propellent at the maximum rate and the overall time $T$.

As a final example, consider the problem of transferring a rocket between two given terminals in a uniform field with minimum propellent expenditure. The time of transit will be supposed specified and impulsive thrusts will be permitted. The vehicle's velocities at the terminals will be assumed known and such that the conditions (4.5), (4.6) cannot all be satisfied. In these circumstances, no IT-arc can contribute to the optimal trajectory and the rocket path must be formed from NT-arcs meeting at junction points; the primer locus diagram does not degenerate to a single point and hence is a straight line which is traversed at constant speed by the generating point. It follows that $p$ can have but one stationary value and that this is a minimum. However, we have proved that at a junction which is not coincident with a terminal, $p$ must take a maximum value of unity. We conclude that there can be no junctions which do not coincide with the terminals. If only one terminal were a junction, this would imply that transfer could be effected without variation of the thrust direction, and this is the excluded case when the conditions (4.5), (4.6) can all be satisfied. Hence, both terminals are junctions and it is easy to show that the impulses which must be applied at these points are uniquely determined by the terminal conditions and the time of transit. For an elementary treatment of this problem, the reader is referred to Lawden (1957). For an especially rigorous proof that the mode of transfer we have just described is optimal in the absolute sense, the reader should consult Ewing (1961).

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### *Exercises*

1. If $\mathbf{x}_0$, $\mathbf{x}_1$ are the position vectors of the departure and arrival terminals respectively in a uniform field $\mathbf{g}$ and if $\mathbf{v}_0$, $\mathbf{v}_1$ are the velocities at these terminals, show that the plane of the primer locus is parallel to the vectors

$$\mathbf{x}_1 - \mathbf{x}_0 - \mathbf{v}_0 T - \tfrac{1}{2}\mathbf{g}T^2,$$
$$\mathbf{v}_1 - \mathbf{v}_0 - \mathbf{g}T,$$

$T$ being the time of transit.

2. If $\mathbf{v}$ is the velocity of a rocket at any time, the locus of a point which moves so that $\mathbf{v}$ is its position vector at the same instant is called the *hodograph* of the vehicle. Show that, in the absence of a gravitational field, the mass ratio $R$ for a transfer manœuvre is related to the length $L$ of the hodograph for the manœuvre by the equation

$$c \log R = L.$$

3. Employing the notation of section 4.1 show that the problem of transfer between two terminals in a zero field with minimum expenditure of propellent is equivalent to the following: To join two points $A_0$, $A_1$ having respective position vectors $\mathbf{v}_0$, $\mathbf{v}_1$, by a hodograph locus having minimum length and such that for the manœuvre so specified

$$\int_{t_0}^{t_1} \mathbf{v}\, dt = \mathbf{x}_1 - \mathbf{x}_0.$$

Show that this latter condition is equivalent to the requirement that it should be possible to distribute a mass $(t_1 - t_0)$ along the hodograph so that its mass centre $G$ lies at the point having position vector $(\mathbf{x}_1 - \mathbf{x}_0)/(t_1 - t_0)$. Taking $t_0$, $t_1$ to be fixed, deduce that the hodograph for optimal transfer comprises the straight segments $A_0G$, $GA_1$ and show that this implies that optimal transfer is effected by the application of impulsive thrusts at the terminals. (Hint: For minimal hodograph length, the mass $t_1 - t_0$ must be concentrated at $G$.)

4. Employing the notation of section 4.2, consider the problem of achieving maximum range on a horizontal plane through the launching point $O$ for a rocket whose initial and final masses $M_0$, $M_1$ are specified, but for which the time to 'all-burnt' is variable. Show that the primer locus is a segment of a straight line through the origin of the diagram and that the primer magnitude steadily decreases. Deduce that, during the first phase, the propellent is consumed at maximum rate and the thrust is inclined at a constant angle to the horizontal and that the second phase is one of free fall under gravity.

# 5

# THE PRIMER IN AN INVERSE SQUARE LAW FIELD

## 5.1. The Arcs of Null Thrust

Suppose that the rocket moves under a gravitational attraction towards a fixed centre $O$, the magnitude of the attraction per unit mass being $\gamma/r^2$, where $r$ is the rocket's distance from $O$. Then it is well known that the trajectories of null thrust are Keplerian arcs in the shape of conics with their foci at $O$. Our problem in this section is to determine the form taken by the primer along such an arc.

Suppose that the Keplerian arc is a conic which lies in the $xy$-plane of a rectangular Cartesian inertial reference frame $Oxyz$. $O$ will be taken to be the centre of gravitational attraction and hence is a focus of the conic. Then, if $(r, \theta)$ are the polar co-ordinates of the rocket as it moves along the arc, relative to $O$ as pole and $Ox$ as initial line, the equations of motion of this body may be written

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\gamma}{r^2}, \tag{5.1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{5.2}$$

It is shown in textbooks of classical dynamics (e.g. Lawden, p. 89) that the polar equation of the conic orbit deducible from these equations takes the form

$$\frac{l}{r} = 1 + e \cos(\theta - \tilde{\omega}), \tag{5.3}$$

where $e$ is the eccentricity, $l$ is the semi-latus rectum and $\tilde{\omega}$ is the value of $\theta$ at the *pericentre* (i.e. point of nearest approach to the centre of attraction). $\tilde{\omega}$ will be referred to as the *longitude of the pericentre*. Also, eq (5.2) can be integrated immediately to yield an equation expressing the conservation of the rocket's angular momentum about the centre of attraction, *viz.*

$$r^2\dot{\theta} = \sqrt{(\gamma l)}. \tag{5.4}$$

It is here assumed that the sense of vehicle's motion is such that $\dot{\theta}$ is positive. Differentiating eq (5.3) with respect to the time and employing eq (5.4), a further equation of motion is obtained, *viz.*,

$$\dot{r} = e\sqrt{\left(\frac{\gamma}{l}\right)}\sin(\theta - \tilde{\omega}). \tag{5.5}$$

Consider now the eqs (3.15) governing the primer. It was shown at eq (3.62) that, since a scalar potential exists from which the field can be derived, then

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j}. \tag{5.6}$$

Eq (3.15) is accordingly equivalent to the equation

$$\ddot{\lambda}_i = \lambda_j \frac{\partial g_i}{\partial x_j} \tag{5.7}$$

which, in vector notation, reads as follows:

$$\ddot{\mathbf{p}} = \mathbf{p}\cdot\nabla\mathbf{g}. \tag{5.8}$$

If more convenient, this equation can be employed to replace eq (3.36). As a third alternative, it is shown in textbooks of vector analysis (e.g. Weatherburn, 1928) that the operator $\mathbf{p}.\nabla$ can be replaced by the operator $p\partial/\partial s$, where $\partial/\partial s$ denotes differentiation with respect to a displacement in the direction of $\mathbf{p}$ and hence eq (5.8) is also equivalent to

$$\ddot{\mathbf{p}} = p\frac{\partial \mathbf{g}}{\partial s}. \tag{5.9}$$

In the case we are considering

$$\mathbf{g} = -\frac{\gamma}{r^3}\mathbf{r}, \tag{5.10}$$

where $\mathbf{r}$ is the radius vector from $O$. Thus

$$\frac{\partial \mathbf{g}}{\partial s} = \frac{3\gamma}{r^4}\frac{\partial r}{\partial s}\mathbf{r} - \frac{\gamma}{r^3}\frac{\partial \mathbf{r}}{\partial s}. \tag{5.11}$$



If $\phi$ is the angle between $\mathbf{r}$ and the direction of differentiation $\mathbf{p}$, then

$$\frac{\partial r}{\partial s} = \cos\phi. \tag{5.12}$$

Also

$$\frac{\partial \mathbf{r}}{\partial s} = \hat{\mathbf{p}}, \tag{5.13}$$

where $\hat{\mathbf{p}}$ is the unit vector in the direction of $\mathbf{p}$. Hence

$$p\frac{\partial \mathbf{g}}{\partial s} = \frac{3\gamma}{r^5}\,\mathbf{p}\cdot\mathbf{r}\,\mathbf{r} - \frac{\gamma}{r^3}\mathbf{p}. \tag{5.14}$$

We shall now resolve eq (5.9) in the directions of rectangular Cartesian axes $OX$, $OY$, $Oz$, where $OX$ lies along the radius

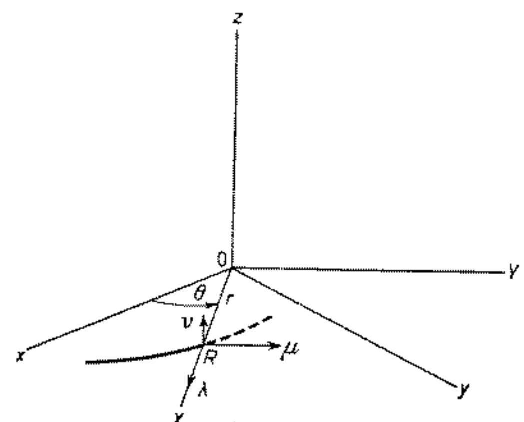


Figure 5.1.

vector from $O$ to the rocket $R$, $OY$ is the perpendicular to $OX$ lying in the plane of and in the same sense as the motion, and $Oz$ has been previously introduced (Figure 5.1). Let $(\lambda, \mu, \nu)$ be the components of $\mathbf{p}$ in the directions of these axes. Then it follows from eq (5.14) that the components of the right-hand member of eq (5.9) are

$$\frac{2\gamma}{r^3}\lambda,\ -\frac{\gamma}{r^3}\mu,\ -\frac{\gamma}{r^3}\nu. \tag{5.15}$$

$\dot{\mathbf{p}}$ denotes the time rate of change of $\mathbf{p}$ relative to a non-rotating frame such as $Oxyz$. Let $\mathring{\mathbf{p}}$ denote the rate of change of $\mathbf{p}$ relative to a rotating frame whose angular velocity relative to the inertial frame is $\Omega$ and whose axes momentarily coincide with those of the fixed frame. Then the relationship between these two rates is well known (see e.g. Lawden, p. 240) to be

$$\dot{\mathbf{p}} = \mathring{\mathbf{p}} + \Omega \times \mathbf{p}. \tag{5.16}$$

Replacing $\mathbf{p}$ in this formula by $\dot{\mathbf{p}}$, it will be found that

$$\ddot{\mathbf{p}} = \overset{\circ\circ}{\mathbf{p}} + \mathring{\Omega} \times \mathbf{p} + 2\Omega \times \mathring{\mathbf{p}} + \Omega \cdot \mathbf{p}\ \Omega - \Omega^2 \mathbf{p}. \tag{5.17}$$

$OXYz$ is a rotating frame whose angular velocity at any instant is given by the equation

$$\Omega = \dot{\theta}\ \mathbf{k}, \tag{5.18}$$

$\mathbf{k}$ being the unit vector along $Oz$. Substituting for $\Omega$ into eqs (5.16), (5.17), we obtain equations expressing $\dot{\mathbf{p}}$, $\ddot{\mathbf{p}}$ in terms of the corresponding rates measured relative to $OXYz$, *viz.*

$$\dot{\mathbf{p}} = \mathring{\mathbf{p}} + \dot{\theta}\ \mathbf{k} \times \mathbf{p}, \tag{5.19}$$

$$\ddot{\mathbf{p}} = \overset{\circ\circ}{\mathbf{p}} + \ddot{\theta}\ \mathbf{k} \times \mathbf{p} + 2\dot{\theta}\ \mathbf{k} \times \mathring{\mathbf{p}} + \dot{\theta}^2(\mathbf{k} \cdot \mathbf{p}\ \mathbf{k} - \mathbf{p}). \tag{5.20}$$

In the second of these equations, we now put

$$\mathbf{p} = r\mathbf{q} \tag{5.21}$$

to yield

$$\ddot{\mathbf{p}} = r\overset{\circ\circ}{\mathbf{q}} + 2\dot{r}\mathring{\mathbf{q}} + (\ddot{r} - r\dot{\theta}^2)\mathbf{q} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{k} \times \mathbf{q} + 2r\dot{\theta}\ \mathbf{k} \times \mathring{\mathbf{q}} + r\dot{\theta}^2 \mathbf{k} \cdot \mathbf{q}\ \mathbf{k}. \tag{5.22}$$

Making use of the equations of motion (5.1), (5.2), this reduces to

$$\ddot{\mathbf{p}} = r\overset{\circ\circ}{\mathbf{q}} + 2\dot{r}\mathring{\mathbf{q}} + 2r\dot{\theta}\ \mathbf{k} \times \mathring{\mathbf{q}} + r\dot{\theta}^2 \mathbf{k} \cdot \mathbf{q}\ \mathbf{k} - \frac{\gamma}{r^2}\mathbf{q}. \tag{5.23}$$

We shall now indicate differentiations with respect to $\theta$ within the frame $OXYz$ by primes. Then

$$\mathring{\mathbf{q}} = \dot{\theta}\ \mathbf{q}', \tag{5.24}$$

$$\overset{\circ\circ}{\mathbf{q}} = \dot{\theta}^2 \mathbf{q}'' + \ddot{\theta}\ \mathbf{q}'. \tag{5.25}$$

Substituting in eq (5.23), this further reduces with the assistance of equations (5.2) and (5.4) to the form



$$\ddot{\mathbf{p}} = \frac{\gamma l}{r^3}\left(\mathbf{q}'' + 2\mathbf{k} \times \mathbf{q}' + \mathbf{k} \cdot \mathbf{q}\ \mathbf{k} - \frac{r}{l}\mathbf{q}\right). \tag{5.26}$$

Thus, if $\mathbf{q}$ has components $(u, v, w)$ with respect to the frame $OXYz$, our last equation shows that the left-hand member of eq (5.9) has components

$$\frac{\gamma l}{r^3}\left(u'' - 2v' - \frac{r}{l}u,\ v'' + 2u' - \frac{r}{l}v,\ w'' + w - \frac{r}{l}w\right) \tag{5.27}$$

along these axes.

From eq (5.21), it follows that

$$\lambda = ru,\ \mu = rv,\ \nu = rw. \tag{5.28}$$

Thus the three components (5.15) of the right-hand member of eq (5.9) may be written

$$\frac{2\gamma}{r^2}u,\ -\frac{\gamma}{r^2}v,\ -\frac{\gamma}{r^2}w. \tag{5.29}$$

Equating these to the respective components (5.27) of the left-hand member, we obtain the equations

$$u'' - 2v' = \frac{3r}{l}u, \tag{5.30}$$

$$v'' + 2u' = 0, \tag{5.31}$$

$$w'' + w = 0, \tag{5.32}$$

determining the vector $\mathbf{q}$ and hence the primer $\mathbf{p}$ along the Keplerian arc.

Since the field is time invariant, a first integral of the equations (5.30)–(5.32) is known to exist. This is given by eq (3.40). Substituting from eq (5.21) into eq (5.19), it is found that

$$\dot{\mathbf{p}} = r\mathring{\mathbf{q}} + \dot{r}\mathbf{q} + r\dot{\theta}\ \mathbf{k} \times \mathbf{q}. \tag{5.33}$$

By eq (5.24), this is equivalent to

$$\dot{\mathbf{p}} = r\dot{\theta}(\mathbf{q}' + \mathbf{k} \times \mathbf{q}) + \dot{r}\mathbf{q}, \tag{5.34}$$

indicating that the components of $\dot{\mathbf{p}}$ in the directions of the axes of $OXYz$ are given by

$$\dot{\mathbf{p}} = [r\dot{\theta}(u' - v) + \dot{r}u,\ r\dot{\theta}(v' + u) + \dot{r}v,\ r\dot{\theta}w' + \dot{r}w]. \tag{5.35}$$

Also

$$\mathbf{p}=(ru, rv, rw), \tag{5.36}$$

$$\mathbf{g}=(-\gamma/r^2, 0, 0), \tag{5.37}$$

$$\mathbf{v}=(\dot{r}, r\dot{\theta}, 0). \tag{5.38}$$

Eq (3.40) now gives immediately

$$r\dot{r}\dot{\theta}u'+r^2\dot{\theta}^2v'+(\dot{r}^2+r^2\dot{\theta}^2+\frac{\gamma}{r})u=\text{constant}, \tag{5.39}$$

the constant right-hand member being zero when the transit time is not predetermined. Employing eqs (5.3)–(5.5), this first integral can be expressed in the form

$$u'e\sin f(1+e\cos f)+v'(1+e\cos f)^2+u(2+e^2+3e\cos f)=C \tag{5.40}$$

where $C$ is a constant and

$$f=\theta-\tilde{\omega} \tag{5.41}$$

is the *real anomaly*.

In the case $e=0$, the Keplerian arc is a circle of radius $l$ and the equations (5.30)–(5.32) then become

$$u''-2v'=3u, \tag{5.42}$$

$$v''+2u'=0, \tag{5.43}$$

$$w''+w\ =0. \tag{5.44}$$

These are easily solved and, when the solution is substituted in eqs (5.28), we obtain

$$\lambda=A\cos f+B\sin f+2C, \tag{5.45}$$

$$\mu=2B\cos f-2A\sin f-3Cf+D, \tag{5.46}$$

$$\nu=E\cos f+F\sin f, \tag{5.47}$$

where $A$, $B$, $C$, $D$, $E$, $F$ are constants of integration. The first integral (5.40) proves, in this case, to be identical with eq (5.43) after this has been integrated.

If the eccentricity is non-vanishing, eq (5.30) is discarded in favour of the first integral (5.40). Integrating eq (5.31), we obtain

$$v'=Ae-2u, \tag{5.48}$$



where $A$ is an arbitrary constant. $v'$ can now be eliminated from the first integral to yield

$$u'\sin f(1+e\cos f)+u(e-\cos f-2e\cos^2 f)=C-A(1+e\cos f)^2. \tag{5.49}$$

After multiplying through by an integrating factor $(1+e\cos f)^{-2}\sin^{-2}f$, this equation can be integrated immediately, leading to the result

$$u=(1+e\cos f)\,(A\cos f+Be\sin f+CI_1), \tag{5.50}$$

where

$$I_1=\sin f\int\frac{df}{\sin^2 f(1+e\cos f)^2}, \tag{5.51}$$

and $B$ is an arbitrary constant. This integral can be evaluated in terms of known functions and the result will be found stated in Lawden (1954). However, in this book, we shall invariably be concerned with problems for which the time of transit is not predetermined, but has to be chosen to yield an optimal manœuvre. For such problems, $C=0$ and $I_1$ does not enter into the calculation. Substituting for $u$ from eq (5.50) into eq (5.48) and integrating, it will be found that

$$v=(1+e\cos f)\left(-A\sin f+B(1+e\cos f)+\frac{D-A\sin f}{1+e\cos f}+CI_2\right), \tag{5.52}$$

where

$$I_2=\frac{\cot f}{e(1+e\cos f)}+\frac{1+e\cos f}{e\sin f}I_1 \tag{5.53}$$

and $D$ is an arbitrary constant.

Eq (5.44) can be integrated for $w$ without difficulty. The result is

$$w=E\cos f+F\sin f, \tag{5.54}$$

$E$ and $F$ being constants of integration.

It now follows that

$$\lambda=A\cos f+Be\sin f+CI_2, \tag{5.55}$$

$$\mu=-A\sin f+B(1+e\cos f)+\frac{D-A\sin f}{1+e\cos f}+CI_2, \tag{5.56}$$

$$\nu=(1+e\cos f)^{-1}(E\cos f+F\sin f), \tag{5.57}$$

a factor $l$ having been absorbed into the constants $A$, $B$, $C$, $D$, $E$ and $F$.

From eq (5.19), it follows that

$$\dot{\mathbf{p}}=(\mathbf{p}'+\mathbf{k}\times\mathbf{p})\dot{\theta}. \tag{5.58}$$

Hence, if $\dot{\mathbf{p}}$ possesses components $(\xi, \eta, \zeta)$ in the directions of the axes of the frame $OXYz$, then

$$\xi=(\lambda'-\mu)\dot{\theta},\ \eta=(\mu'+\lambda)\dot{\theta},\ \zeta=\nu'\dot{\theta}. \tag{5.59}$$

Substituting in these equations for $\dot{\theta}$ from eq (5.4) and for $(\lambda, \mu, \nu)$ from either eqs (5.45)–(5.47) or eqs (5.55)–(5.57), the form of the derivative of the primer can be established. If $e=0$, it will be found that

$$\xi=\frac{\gamma^{1/2}}{a^{3/2}}(A\sin f-B\cos f+3Cf-D), \tag{5.60}$$

$$\eta=-\frac{\gamma^{1/2}}{a^{3/2}}(A\cos f+B\sin f+C), \tag{5.61}$$

$$\zeta=\frac{\gamma^{1/2}}{a^{3/2}}(F\cos f-E\sin f), \tag{5.62}$$

where $a$ is the radius of the orbit. If $e\neq 0$, then

$$\xi=\frac{\sqrt{(\gamma l)}}{r^2}\left(\frac{A\sin f-D}{1+e\cos f}-B+CI_3\right), \tag{5.63}$$

$$\eta=\frac{\gamma^{1/2}}{l^{3/2}}\{-A(e+\cos f)+De\sin f+C\cos f\}, \tag{5.64}$$

$$\zeta=\frac{\gamma^{1/2}}{l^{3/2}}\{(e+\cos f)F-E\sin f\}, \tag{5.65}$$

where

$$I_3=\frac{e\sin f-\cos f}{e\sin f(1+e\cos f)^2}-\frac{I_1}{e}\operatorname{cosec} f. \tag{5.66}$$

## 5.2. The Arcs of Intermediate Thrust

The form taken by the primer on a general intermediate thrust arc lying in an inverse square law field of gravitational attraction to a single centre is not yet (1962) known. However, the problem



has been solved (Lawden, 1961, 1962) in the special case when the arc is restricted to lie in a plane through the centre of attraction and, in this section, we shall explain how this solution may be derived.

Suppose that, at time $t$, the rocket $R$ is at the point having polar co-ordinates $(r, \theta)$ on the intermediate thrust arc and that, at this instant, the direction of motor thrust makes an angle $\phi$ with the perpendicular to the radius vector drawn in the same

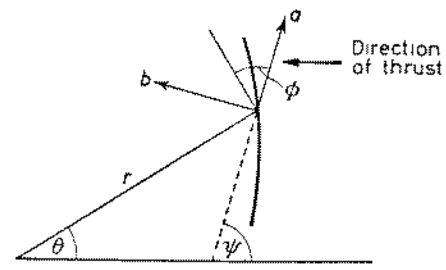


Figure 5.2.

sense as the motion (Figure 5.2). Let $\psi$ be the angle made by the thrust direction with the line $\theta=0$; then

$$\psi=\tfrac{1}{2}\pi-\phi+\theta. \tag{5.67}$$

We shall now resolve the equation of motion of the rocket and the equation governing the primer in directions along and perpendicular to the thrust.

Let $\mathbf{a}$, $\mathbf{b}$ be unit vectors in the direction of and perpendicular to the thrust respectively and in the senses indicated in Figure 5.2. At time $t$, these vectors are rotating with angular velocity $\dot{\psi}$ and it therefore follows that

$$\dot{\mathbf{a}}=\dot{\psi}\,\mathbf{b},\ \dot{\mathbf{b}}=-\dot{\psi}\,\mathbf{a}. \tag{5.68}$$

Let $(u, w)$ be the components of the rocket velocity in the direction of and perpendicular to the thrust respectively. Then

$$\mathbf{v}=u\mathbf{a}+w\mathbf{b} \tag{5.69}$$

and hence, differentiating and employing eqs (5.68)

$$\dot{\mathbf{v}}=(\dot{u}-w\dot{\psi})\mathbf{a}+(\dot{w}+u\dot{\psi})\mathbf{b}. \tag{5.70}$$

The equation of rocket motion (2.1) can now be resolved in the directions of the vectors **a** and **b** to yield the equations

$$\dot{u}-w\dot{\psi}=f-\frac{\gamma}{r^2}\sin\phi, \tag{5.71}$$

$$\dot{w}+u\dot{\psi}=\frac{\gamma}{r^2}\cos\phi, \tag{5.72}$$

where

$$f=cm/M \tag{5.73}$$

is the acceleration due to the motor.

We now transfer our attention to the eq (3.36) governing the primer. Along an intermediate thrust arc, the primer is known to be aligned with the thrust and to have a constant magnitude. Thus, we can write

$$\mathbf{p}=p\mathbf{a}, \tag{5.74}$$

where $p$ is constant. Differentiating this equation twice with respect to $t$ and making use of eqs (5.68), it follows that

$$\ddot{\mathbf{p}}=p(-\dot{\psi}^2\mathbf{a}+\ddot{\psi}\mathbf{b}). \tag{5.75}$$

The components of the left-hand member of eq (3.36) in the directions of **a** and **b** are accordingly $(-p\dot{\psi}, p\ddot{\psi})$ respectively. Referring to the result (5.15) of the last section, we see that the right-hand member of eq (3.36) has components

$$\frac{2\gamma}{r^3}p\sin\phi, -\frac{\gamma}{r^3}p\cos\phi \tag{5.76}$$

in the radial and transverse directions respectively. Since this member is a vector, we may further resolve these components in the directions of **a** and **b** to obtain

$$\frac{\gamma p}{r^3}(3\sin^2\phi-1), \ -\frac{3\gamma p}{r^3}\sin\phi\cos\phi, \tag{5.77}$$

respectively. Eq (3.36) can now be resolved in the directions of **a** and **b** to yield

$$\dot{\psi}^2=\frac{\gamma}{r^3}(1-3\sin^2\phi), \tag{5.78}$$

$$\ddot{\psi}=-\frac{3\gamma}{r^3}\sin\phi\cos\phi. \tag{5.79}$$



The radial and transverse components of **v** are $(\dot{r}, r\dot{\theta})$ respectively. Resolving these again in the directions of **a** and **b**, it follows that

$$u=\dot{r}\sin\phi+r\dot{\theta}\cos\phi, \tag{5.80}$$

$$w=-\dot{r}\cos\phi+r\dot{\theta}\sin\phi. \tag{5.81}$$

The seven eqs (5.67), (5.71), (5.72), (5.77), (5.78), (5.79) and (5.80) now serve to determine seven unknown functions of $t$, *viz.* $r, \theta, u, w, \phi, \psi, f$, and hence the intermediate thrust arc.

Since the field is time-invariant, a first integral of these equations can be written down immediately. Its form is given by eq (3.40). Resolving along **a** and **b**, we have

$$\left.\begin{aligned}&\mathbf{p}=(p, 0), \quad \mathbf{g}=\left(-\frac{\gamma}{r^2}\sin\phi, \frac{\gamma}{r^2}\cos\phi\right),\\ &\dot{\mathbf{p}}=(0, p\dot{\psi}), \ \mathbf{v}=(u, w).\end{aligned}\right\} \tag{5.82}$$

Substitution in eq (3.40) now yields

$$w\dot{\psi}=C-\frac{\gamma}{r^2}\sin\phi. \tag{5.83}$$

Another first integral can be arrived at as follows: From eqs (5.78) and (5.79), we first obtain the equation

$$\dot{\psi}^2\cos\phi-\ddot{\psi}\sin\phi=\frac{\gamma}{r^3}\cos\phi. \tag{5.84}$$

Eq (5.72) now implies that

$$\dot{w}+u\dot{\psi}-r\dot{\psi}^2\cos\phi+r\ddot{\psi}\sin\phi=0. \tag{5.85}$$

That this equation is equivalent to

$$\frac{d}{dt}(w+r\dot{\psi}\sin\phi)=0, \tag{5.86}$$

follows from eqs (5.67) and (5.80). An integration now provides another first integral, *viz.*

$$w+r\dot{\psi}\sin\phi=A, \tag{5.87}$$

$A$ being the integration constant.

Eliminating $\dot{\psi}$ between eqs (5.83), (5.87) and then again between eqs (5.78), (5.83), it follows that

$$w(A-w)=\left(Cr-\frac{\gamma}{r}\sin\phi\right)\sin\phi, \tag{5.88}$$

$$w^2(1-3\sin^2\phi)=\frac{r}{\gamma}\left(Cr-\frac{\gamma}{r}\sin\phi\right)^2. \tag{5.89}$$

$w$ and $r$ can now be evaluated in terms of $\phi$ without further integration.

Henceforward, in this section, it will prove convenient to treat $\phi$ as the independent variable; primes will denote differentiations with respect to this variable. From eqs (5.67) and (5.81), it follows that

$$w=-\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\phi)+r\dot{\psi}\sin\phi. \tag{5.90}$$

Whence, employing eq (5.87), we arrive at the result

$$A-2w=\frac{\mathrm{d}}{\mathrm{d}t}(r\cos\phi). \tag{5.91}$$

This may be rewritten as

$$(A-2w)t'=\frac{\mathrm{d}}{\mathrm{d}\phi}(r\cos\phi), \tag{5.92}$$

from which form, by an integration, $t$ may then be found in terms of $\phi$.

Also, from eq (5.87), we derive the equation

$$\psi'=t'(A-w)/(r\sin\phi). \tag{5.93}$$

An integration yields $\psi$ as a function of $\phi$ and eq (5.67) then leads immediately to an expression for $\theta$ in terms of this parameter.

Finally, $f$ and $u$ can be calculated in terms of $\phi$ from eqs (5.71) and (5.80) respectively.

Although the calculations which have just been outlined can be carried through in principle, the practical difficulties appear to be insurmountable in the general case, and we shall not pursue the matter further in this book. However, in the important case when the transit time for the manœuvre is not predetermined, $C$ vanishes and the calculation is then greatly simplified. Thus, from eqs (5.88), (5.89), we then find that



$$r=\frac{as^6}{1-3s^2}, \tag{5.94}$$

$$w=\pm\left(\frac{\gamma}{a}\right)^{1/2}\frac{1}{s^2}, \tag{5.95}$$

where $a=9\gamma/A^2$, $s=\sin\phi$, and the sign ambiguity will be resolved later. Clearly, $a$ is a positive quantity and possesses the physical dimensions of length.

From eq (5.92), it now follows that

$$t'=\pm\frac{a^{3/2}}{\gamma^{1/2}}\frac{s^7(5s^2-3)}{(1-3s^2)^2}, \tag{5.96}$$

the signs following those in eq (5.95). Integrating this equation, it will be found that

$$\pm\frac{\gamma^{1/2}}{a^{3/2}}t=-\frac{1}{9}c^5+\frac{31}{81}c^3-\frac{5}{9}c-\frac{2}{81}\sqrt{\frac{2}{3}}\log\frac{\sqrt{3}c-\sqrt{2}}{\sqrt{3}c+\sqrt{2}}-\frac{1}{81}\cdot\frac{c}{3c^2-2}+\text{constant}, \tag{5.97}$$

where $c=\cos\phi$.

Eq (5.93) gives

$$\psi'=3s^{-2}-5 \tag{5.98}$$

without ambiguity of sign, and hence

$$\psi=\text{constant}-5\phi-3\cot\phi. \tag{5.99}$$

From eq (5.67), we deduce that

$$\theta=\theta_0-4\phi-3\cot\phi, \tag{5.100}$$

where $\theta_0$ is a constant.

Eqs (5.94), (5.100) constitute polar parametric equations for the IT-arc. As $\phi$ increases from 0 to $\alpha=\sin^{-1}(1/\sqrt{3})$ ($\alpha$ positive acute), $\theta$ increases from $-\infty$ to $(\theta_0-4\alpha-3/2)$ and $r$ increases from 0 to $\infty$, all increases being monotonic. The corresponding part of the IT-arc is a spiral unwinding from the pole $O$ in an anti-clockwise sense and receding to an infinite distance after an infinite number of circuits of the pole have been made. If $\phi$ is replaced by $\phi-\pi$, the expression for $r$ is unchanged and the expression for $\theta$ is altered by the addition of a constant term only. It follows that when $\phi$ takes values lying in the third quadrant, the corresponding arc differs from the one already described in respect of orientation alone. If $\phi$ is replaced by $-\phi$ or $\pi-\phi$, the expression for $r$ is again unaltered, but that for $\theta$ now takes the form

$$\theta=\theta_0+4\phi+3\cot\phi, \tag{5.101}$$

where any constant term arising has been absorbed in $\theta_0$. This implies that when $\phi$ takes values lying in the second or fourth quadrants, the resulting arc is a mirror image of the spiral already described, i.e. a spiral which unwinds from the pole in a clockwise sense. Values of $\phi$ lying between the limits $(\alpha, \pi-\alpha)$ or $(\alpha-\pi, -\alpha)$ are not permitted, since such values lead to negative values for $r$, and $r$ has everywhere been treated as an essentially positive quantity. All values for $\phi$ have now received consideration. From this point onwards, we shall restrict $\phi$ to lie in one of the intervals $(0, \alpha)$ or $(\pi-\alpha, \pi)$. This restriction does not result in any loss of generality for, by changing the sign of the motor acceleration $f$, a $\phi$-value can always be transferred from the third and fourth quadrants into the first and second respectively; it will be shown later that, in fact, this restriction results in $f$ taking positive values only.

Since $\theta$ has been assumed to increase with $t$, the spiral obtained from $\phi$-values lying in the first quadrant represents the track of a rocket which is receding from the centre of attraction. The spiral corresponding to $\phi$-values lying in the second quadrant is traversed by a rocket which is approaching the centre of attraction. All intermediate thrust spirals are identical except for orientation (determined by $\theta_0$), scale (determined by $a$) and sense of unwinding from the pole (determined by the range of $\phi$-values).

The sign ambiguities noticed above can now be resolved. Differentiating eq (5.100), we obtain

$$\theta'=3s^{-2}-4\geqslant 5, \tag{5.102}$$

since $s^2\leqslant 1/3$. Hence

$$t'=\theta'/\dot\theta>0, \tag{5.103}$$

since we have assumed $\dot\theta$ to be positive everywhere. The inequality (5.103) is consistent with eq (5.96) only provided that the positive sign is taken when $s$ is negative (i.e. when $\phi$ lies in the second quadrant) and the negative sign is taken when $s$ is positive (i.e. when $\phi$ lies in the first quadrant). We conclude, therefore, that the negative sign is appropriate in eqs (5.95)–(5.97) in the



case when the rocket is receding from the pole and the positive sign when the rocket is approaching the pole.

$u$ can now be calculated from eq (5.80). The result is

$$u=\pm\left(\frac{\gamma}{a}\right)^{1/2}\frac{3-7s^2}{s^3(5s^2-3)}\cos\phi, \tag{5.104}$$

the sign being selected according to the rule just enunciated. $f$ now follows from eq (5.71) thus:

$$f=\frac{\gamma}{a^2}\left(\frac{1-3s^2}{3-5s^2}\right)^3 s^{-11}(27-75s^2+60s^4). \tag{5.105}$$

There is no sign ambiguity in this case and, as remarked above, $f$ is accordingly positive for all $\phi$ being considered. Eq (5.73) indicates that the characteristic velocity over any segment of the spiral can be calculated from

$$V=\int f\,dt=\pm\left|\left(\frac{\gamma}{a}\right)^{1/2}\frac{3(1-2s^2)(1-5s^2)}{s^3(3-5s^2)}\cos\phi\right|, \tag{5.106}$$

taken between the appropriate limits.

The net energy per unit mass $E$ of the vehicle at any instant is given by the equation

$$E=\tfrac{1}{2}(u^2+w^2)-\frac{\gamma}{r},$$

$$=-\frac{\gamma}{a}\cdot\frac{9-72s^2+169s^4-126s^6}{2s^6(3-5s^2)^2}. \tag{5.107}$$

In the case of a receding rocket, as $s$ increases from 0 to $1/\sqrt{3}$, $E$ increases through negative values to zero and ultimately becomes positive over the final segment of the spiral. This implies that the rocket ultimately escapes from the centre of attraction.

Finally, we will calculate the radial and transverse velocity components of the vehicle. These are given by

$$v_r=\dot r=\pm\left(\frac{\gamma}{a}\right)^{1/2}\frac{6(1-2s^2)}{s^2(5s^2-3)}\cos\phi, \tag{5.108}$$

$$v_\theta=r\dot\theta=\pm\left(\frac{\gamma}{a}\right)^{1/2}\frac{(3-4s^2)(1-3s^2)}{s^3(5s^2-3)}. \tag{5.109}$$

Hence, if $\chi$ is the angle between the direction of motion and the perpendicular to the radius vector,

$$\tan \chi = \frac{v_r}{v_\theta} = \frac{6s(1-2s^2)}{(3-4s^2)(1-3s^2)} \cos \phi. \tag{5.110}$$

Over those portions of the spiral where $\phi$ is small, we have approximately

$$\tan \chi = 2s = 2 \sin \phi,$$

or

$$\chi = 2\phi; \tag{5.111}$$

i.e. the direction of thrust bisects the angle between the perpendicular to the radius vector and the direction of motion. This is in conformity with a result relating to optimal escape from a centre of attraction employing a given small magnitude of thrust to be found in Lawden (1958).

## 5.3. The Arcs of Maximum Thrust

Bearing in mind the complexity of the MT-arcs in a uniform gravitational field (see section 4.1), it is to be expected that solving for the corresponding arcs in an inverse square law field will present great difficulty and, indeed, the solution may not be expressible in terms of known functions. To date (1962), no such solution has been found. However, if such arcs can be approximated by junction points at which impulsive thrusts are operative, the theory of section 3.2 is applicable and the lack of a solution for MT-arcs presents no obstacle to the calculation of optimal manœuvres in the neighbourhood of a centre of attraction. This is very often the case in practice, but even where such an approximation is not justified, an indication of the correct manœuvre can usually be found by making the approximation; then, since the precise details of an optimal manœuvre are generally not critical, a satisfactory thrust programme is easily devised. We shall make use of this approximation, therefore, throughout the following chapter.

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### *Exercises*

1. An optimal trajectory lies in a plane through a centre of inverse square law attraction. $P$ is a point on the trajectory where an IT-arc terminates and a Keplerian arc commences. No impulsive thrust is applied at $P$. If $q = p^2$, employing the result of Ex. 2, Chap. 3, show that

$$\frac{d^4 q}{dt^4} = -\frac{3\gamma f}{r^4} s(3-5s^2)$$

at $P$. Deduce that $p$ is a maximum at $P$. (Hint: In the result of Ex. 2, take $Ox_1$ along $OP$.)

2. Show that at any point on the IT-spiral, the circular orbital velocity $v_c$ is given by

$$v_c = \sqrt{\frac{\gamma}{a}} \cdot \frac{(1-3s^2)^{1/2}}{s^3}$$

and deduce that eqs (5.108), (5.109) may be written

$$v_r = \frac{6s(1-2s^2)\cos\phi}{(1-3s^2)^{1/2}(3-5s^2)} v_c,$$

$$v_\theta = \frac{(3-4s^2)(1-3s^2)^{1/2}}{3-5s^2} v_c.$$

Hence show that over the segment of the spiral where $\phi$ is small, the rocket velocity is approximately $v_c$.

3. Prove that the characteristic velocity for transfer along a segment of the IT-spiral from $s = s_1$ to $s = s_2$, where $s_1 < s_2$ and $s_2$ is small, is given approximately by

$$V = \sqrt{\frac{\gamma}{a}} \left( \frac{1}{s_1^3} - \frac{1}{s_2^3} \right).$$

Calculate the characteristic velocity for a Hohmann transfer (see Figure 6.4) between the two terminal orbits to the same order of approximation and verify that it is less than $V$.

# 6

# ORBITAL TRANSFER MANŒUVRES

### 6.1. Introduction

In this chapter, we shall make use of results established in the earlier chapters to solve a number of problems relating to the optimal transfer of a rocket between orbits in an inverse square law field. For simplicity, the motion will always be supposed confined to a plane through the centre of attraction, and the time of transit will be regarded as a variable whose value is also subject to optimization. The first assumption implies that we need take account of the primer components lying in the plane of motion alone. The second assumption requires that the constant $C$ be put equal to zero in all the formulae of the previous chapter. As an additional simplification, all phases of maximum thrust will be approximated by impulses.

### 6.2. Primer on a Circular Orbit

On an NT-arc which is a circular orbit, eqs (5.45), (5.46) show that

$$\lambda = A\cos f + B\sin f = R\sin(f+f_0), \tag{6.1}$$

$$\mu = 2B\cos f - 2A\sin f + D = 2R\cos(f+f_0) + D, \tag{6.2}$$

where $R=\sqrt{(A^2+B^2)}$ and $\tan f_0 = A/B$. By appropriate choice of the polar reference line $f=0$, these equations are reducible to the form

$$\lambda = R\sin f,\ \mu = 2R\cos f + D. \tag{6.3}$$

The locus of the point $P(\lambda, \mu)$ in the $\lambda\mu$-plane is accordingly an ellipse with its minor axis parallel to the $\lambda$-axis and of length $2R$, its major axis coincident with the $\mu$-axis and of length $4R$, and its centre at the point $(0, D)$. This is the primer locus diagram for a circular orbit and is shown in Figure 6.1. Examination of this figure reveals that, for each circuit of the locus, the primer



magnitude $p$ $(=OP)$ may pass through two maxima and two minima or only one maximum and one minimum, depending upon the value of $D$. Thus, if $D=0$, $O$ is at the centre of the ellipse and $p$ is a maximum when $P$ is at either extremity of the major axis and is a minimum when $P$ is at either extremity of the minor axis. However, if $O$ coincides with a focus, then $p$ is a maximum at one extremity of the major axis and a minimum at

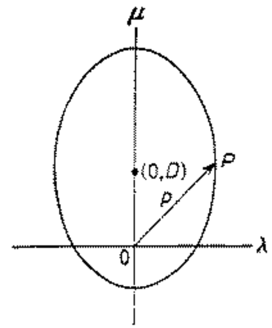


Figure 6.1.

the other extremity, and there are no other stationary points. Proceeding analytically, from eqs (6.3) it follows that

$$p^2 = \lambda^2 + \mu^2 = 3R^2\cos^2 f + 4RD\cos f + R^2 + D^2. \tag{6.4}$$

Thus $p^2$ is stationary where $f=0$ and $\pi$ and also where

$$3R\cos f + 2D = 0. \tag{6.5}$$

This latter equation possesses real solutions only when $|D| \leqslant 3R/2$. A little consideration will show that, if $|D| < 3R/2$, $p^2$ (and hence $p$) has maxima where $f=0$ or $\pi$ and minima where $f=\pi+\cos^{-1}(2D/3R)$. If $D \geqslant 3R/2$, $p^2$ has a maximum at $f=0$ and a minimum at $f=\pi$. If $D \leqslant -3R/2$, $p^2$ has a minimum at $f=0$ and a maximum at $f=\pi$. In the special case when $R=0$, the primer locus degenerates to a point and $p$ is constant over the circular orbit.

Taking these results in conjunction with the conditions (*a*)–(*d*) given on p. 63 it follows that, if a circular orbit forms part of an optimal trajectory, then the motor can only be operative at one or both of the points $f=0$ and $f=\pi$ (unless $R=0$, when any two points on the orbit can be junctions). However, in all cases, $\lambda=0$ at a junction and hence the thrust must be in the direction of the perpendicular to the radius vector. But, at no point on an intermediate thrust spiral is the thrust in such a direction and it follows, therefore, by condition (*a*), that a circular orbit can neither be entered from nor left into an IT-arc. Thus, only impulsive thrusts tangential to the circular orbit are permissible.

At any junction $p=P$ and is an absolute maximum. Hence, if the circular orbit has two junctions, then either $R=0$ and $D=\pm P$, or the junctions are at the opposite ends of a diameter where $f=0$, $\pi$ and $D=0$, $R=\pm\frac{1}{2}P$. Thus

$$\lambda=0,\ \mu=\pm P,\ \text{or}\ \lambda=\pm\tfrac{1}{2}P\sin f,\ \mu=\pm P\cos f. \qquad (6.6)$$

In the first case, the thrust supports the motion at both junctions or opposes it at both, depending upon the sign selected. In the second case, the thrust supports the motion at one junction and opposes it at the other. The first case will be proved inadmissible in section 6.4.

If, however, the circular orbit is a terminal arc of the optimal trajectory, it will contain but one junction point at which $p$ takes its absolute maximum value $P$ and this junction must be at $f=0$ or $f=\pi$. This we can ensure by taking

$$\lambda=\tfrac{1}{2}(P-D)\sin f,\ \mu=(P-D)\cos f\pm D, \qquad (6.7)$$

where $0\leqslant D\leqslant P$. Then, if the positive sign is accepted, the junction is at $f=0$, whereas if the negative sign is taken, the junction is at $f=\pi$; the thrust supports the motion in the first case and opposes it in the second. It will be observed that the two-junction possibilities (6.6) are included in the formulae (6.7) for the extreme values of $D$, *viz.* 0 and $P$.

If an impulse is applied to a body moving in a circular orbit and the direction of thrust is tangential to the orbit (but not necessarily in the sense of the motion), the body will be transferred into some conic orbit. Immediately after the application of the impulse, the body will be moving at right-angles to the radius from the centre of attraction and will accordingly be at an apse on the conic orbit. Similarly, it follows that a circular orbit can only be entered from a conic orbit by an impulsive thrust applied at an apse.



### 6.3. Primer on a Conic Orbit

We shall first consider the case where the conic orbit has been entered or will be left at an apse, by employing an impulsive thrust in a direction perpendicular to the radius vector. This will certainly be the case if the previous or subsequent arc is a circle. Taking the polar equation of the conic to be

$$\frac{l}{r}=1+e\cos f, \qquad (6.8)$$

eqs (5.55), (5.56) show that the primer is given on the orbit by the equations

$$\lambda=A\cos f+Be\sin f, \qquad (6.9)$$

$$\mu=-A\sin f+B(1+e\cos f)+\frac{D-A\sin f}{1+e\cos f}. \qquad (6.10)$$

One of the apses $f=0$ or $\pi$, must be a junction point at which the direction of thrust is perpendicular to the radius vector $r$. Suppose $f=0$ is a junction; then, at this point, $\lambda=0$ and $\mu=\pm P$. Hence, it is necessary that

$$A=0, \qquad (6.11)$$

$$B(1+e)^2+D=\pm P(1+e). \qquad (6.12)$$

Thus

$$\lambda=Be\sin f, \qquad (6.13)$$

$$\mu=B(1+e\cos f)+\frac{D}{1+e\cos f}, \qquad (6.14)$$

and it follows that

$$p^2=\lambda^2+\mu^2=B^2(1+e^2)+2BD+2B^2x+\frac{D^2}{(1+x)^2}, \qquad (6.15)$$

where $x=e\cos f$. Defining $\beta=p^2-B^2(1+e^2)-2BD$, we have

$$\frac{d\beta}{dx}=2B^2-\frac{2D^2}{(1+x)^3},\quad \frac{d^2\beta}{dx^2}=\frac{6D^2}{(1+x)^4}. \qquad (6.16)$$

Hence, if $D$ is non-vanishing, $\beta$ has a minimum where $x=x_0=(D^2/B^2)^{1/3}-1$ and its graph must take the form sketched in Figure 6.2. There are now two cases to consider: First, if the conic is an ellipse so that $e<1$, $x$ will vary over the range $(-e, e)$ on the orbit. Then, if $x_0$ lies inside this interval, it is evident from Figure 6.2 that $p^2$ (and hence also $p$) will assume its maximum values at the apses $f=0$ and $\pi$ and its minimum values at the two points where $e \cos f = x_0$. If $x_0$ lies outside the interval, $p^2$ will be a maximum at one apse and a minimum at the other and will possess no other stationary points on the orbit. Secondly, if the conic is a parabola or a hyperbola so that $e \geqslant 1$, $x$ will vary over the range $(-1, e)$ but will never attain the value $-1$. Then, if $x_0$ lies inside this interval, $p^2$ will be a maximum at $f=0$ and a

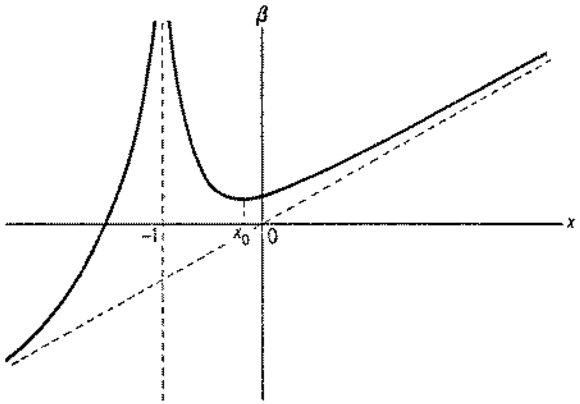


Figure 6.2.

minimum at the two points where $e \cos f = x_0$. If $x_0$ lies outside the interval, then $p^2$ will be a minimum at the apse $f=0$ and will possess no other stationary points. In either case, $p^2 \to \infty$ as $x \to -1$ and condition (c), p. 63, cannot be satisfied.

If $D=0$, eqs (6.13), (6.14) show immediately that the primer locus is a circle with centre at the point $(0, B)$ and radius $|B|e$. $p$ has a maximum at $f=0$ and a minimum at $f=\pi$. Thus, in all cases, there can be but one junction point and this at the apse $f=0$. This case $D=0$ is the only possible one, therefore, if the conic is a parabola or a hyperbola. Then, there can be but one junction point, the apse, and the arc must be a terminal one for the optimal trajectory. Putting $D=0$ in eqs (6.12)–(6.14), we obtain for the primer components the equations



$$\lambda = \pm \frac{Pe}{1+e} \sin f, \quad \mu = \pm \frac{P}{1+e}(1+e\cos f). \tag{6.17}$$

But, if $(v_r, v_\theta)$ are the radial and transverse components of the velocity $\mathbf{v}$ in the orbit, eqs (5.3)–(5.5) show that

$$v_r = e\sqrt{\frac{\gamma}{l}} \sin f, \quad v_\theta = r\dot{\theta} = \sqrt{\frac{\gamma}{l}}(1+e\cos f). \tag{6.18}$$

Comparing eqs (6.17), (6.18), it is clear that

$$\mathbf{p} = \pm \frac{P}{1+e}\sqrt{\frac{l}{\gamma}}\,\mathbf{v}. \tag{6.19}$$

Apart from a numerical factor, this is the particular solution for the primer obtained at eq (3.59).

Returning to the elliptical case with $D$ non-vanishing, unless $x_0$ lies in the interval $(-e, e)$, there can be but one junction point on the orbit and the ellipse must then be a terminal arc. If $D=0$, the same conclusion follows in all circumstances. If $x_0$ lies within this interval, both apses can be junction points and then it is also necessary that $p=P$ at $f=\pi$. This condition leads to the equation

$$B(1-e)^2 + D = \pm P(1-e). \tag{6.20}$$

If positive signs are chosen in both the eqs (6.12), (6.20), so that the impulse is assumed to support the motion at both apses, it will be found on solving for $B$ and $D$ that

$$B = \tfrac{1}{2}P, \; D = \tfrac{1}{2}P(1-e^2). \tag{6.21}$$

If negative signs are chosen in both cases, so that both impulses oppose the motion, the signs of $B$ and $D$ are reversed. In these two cases, it is found that $x_0 = (1-e^2)^{2/3} - 1$ and it is easily verified that $x_0$ does lie in the interval $(-e, e)$. Thus

$$\lambda = \pm \tfrac{1}{2}Pe \sin f, \tag{6.22}$$

$$\mu = \pm \tfrac{1}{2}P\left(1 + e\cos f + \frac{1-e^2}{1+e\cos f}\right). \tag{6.23}$$

If the positive sign is chosen in eq (6.12) and the negative sign in eq (6.20), so that the impulse supports the motion at the nearer apse and opposes it at the further apse, then

$$B=P/2e,\ D=-P(1-e^2)/2e. \tag{6.24}$$

$x_0=(1-e^2)^{2/3}-1$ as before. The signs of $B$ and $D$ are reversed if the negative sign is taken in eq (6.12) and the positive in eq (6.20). Thus

$$\lambda=\pm\tfrac{1}{2}P\sin f, \tag{6.25}$$

$$\mu=\pm\frac{P}{2e}\left(1+e\cos f-\frac{1-e^2}{1+e\cos f}\right). \tag{6.26}$$

Since $\lambda=0$ at $f=\pi$ in all cases, the impulse is again tangential to the orbit at the second apse.

It is clear, therefore, that neither the sub-arc preceding the conic orbit nor the sub-arc following it can be an arc of intermediate thrust, for the direction of thrust is never perpendicular to the radius vector on such an arc. Both these sub-arcs must, therefore, be conic orbits tangential at their apses to the conic orbit under consideration. The following conclusion can accordingly be drawn in respect of an optimal trajectory: If one of the constituent arcs of null thrust is a conic orbit which is entered or left by means of an impulsive thrust applied at an apse in a direction tangential to the orbit, then all the sub-arcs are arcs of null thrust and the junction points are always at apses. In particular, this will be the case if one sub-arc is a circle.

In the general case when the components of the primer are given on the conic orbit by eqs (5.55), (5.56), (with $C=0$), it is convenient to resolve the primer again in directions along and perpendicular to the major axis of the conic. Thus, if $(\lambda_0, \mu_0)$ are the respective components of $\mathbf{p}$ in these directions, then

$$\lambda_0=\lambda\cos f-\mu\sin f=A-Q\sin f, \tag{6.27}$$

$$\mu_0=\lambda\sin f+\mu\cos f=Be+Q\cos f, \tag{6.28}$$

where

$$Q=B+\frac{D-A\sin f}{1+e\cos f}. \tag{6.29}$$

Defining $y$ to be the perpendicular distance of a point on the conic orbit from its major axis, we have



$$y=r\sin f. \tag{6.30}$$

Taking this equation in conjunction with eq (6.8), it is evident that eq (6.29) is equivalent to the equation

$$Q=B+\frac{D}{l}r-\frac{A}{l}y. \tag{6.31}$$

Eqs (6.27), (6.28), show that the point $P(\lambda_0, \mu_0)$ is at a distance $Q$ from the point $(A, Be)$ in the $\lambda_0\mu_0$-plane as indicated in Figure 6.3. As $f$ varies, the locus of this point can be determined from eq (6.31) and this is the primer locus diagram. The precise form

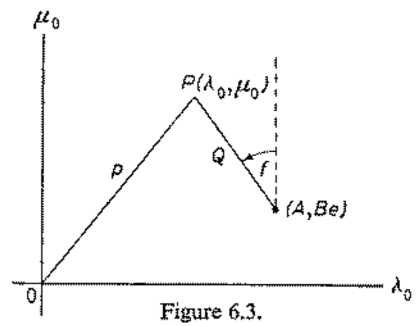


Figure 6.3.

of the locus will depend upon the values of the constants $B$, $D/l$, $A/l$ and $e$, but the plots in particular cases suggest that $p$ either possesses two maxima and two minima or only one maximum and one minimum, as was proved to be the case when $A=0$.

## 6.4. The Derivative of the Primer

At a junction $\dot{\mathbf{p}}$ is continuous. Before proceeding to apply the results we have obtained thus far, it will accordingly be convenient to calculate the components of this derivative at possible junctions for the cases considered in the last two sections.

Thus, corresponding to eqs (6.7) for the primer components on a circle of radius $a$, eqs (5.60), (5.61) show that

$$\xi=-\frac{\gamma^{1/2}}{a^{3/2}}[\tfrac{1}{2}(P-D)\cos f\pm D],\ \eta=-\frac{\gamma^{1/2}}{a^{3/2}}\tfrac{1}{2}(P-D)\sin f. \tag{6.32}$$

If there are two junctions, the thrust supporting the motion at one and opposing it at the other, then $D=0$ and the thrust supports the motion at $f=0$ and opposes it at $f=\pi$. At $f=0$, we find

$$\xi=-\gamma^{1/2}P/2a^{3/2},\ \eta=0, \tag{6.33}$$

whilst at $f=\pi$

$$\xi=\gamma^{1/2}P/2a^{3/2},\ \eta=0. \tag{6.34}$$

If the thrust supports or opposes the motion at both junctions, then $D=P$ and hence

$$\xi=\mp\gamma^{1/2}P/a^{3/2},\ \eta=0, \tag{6.35}$$

over the whole orbit, the negative sign being taken in the case when the thrusts support the motion and the positive sign otherwise.

If there is but one junction point on the circle and the thrust supports the motion at this point, then the positive sign is to be taken in eqs (6.32). The junction is then at $f=0$ and there

$$\xi=-\gamma^{1/2}(P+D)/2a^{3/2},\ \eta=0. \tag{6.36}$$

If the thrust opposes the motion at the junction, the negative sign is taken. The junction is then where $f=\pi$ and at this point

$$\xi=\gamma^{1/2}(P+D)/2a^{3/2},\ \eta=0. \tag{6.37}$$

Referring to eqs (5.63), (5.64), it is found that at the apse $f=0$ on a conic orbit, if a tangential thrust is applied there,

$$\xi=-\frac{\gamma^{1/2}}{l^{3/2}}(1+e)\,[D+B(1+e)],\quad \eta=0. \tag{6.38}$$

If the apse $f=\pi$ is a junction at which the thrust is tangential to the orbit, then at this point

$$\xi=-\frac{\gamma^{1/2}}{l^{3/2}}(1-e)\,[D+B(1-e)],\quad \eta=0. \tag{6.39}$$

If both these apses are junctions and the thrust supports the motion at both points, eqs (6.22), (6.23) are valid (with positive signs). It follows that at $f=0$

$$\xi=-\frac{P\gamma^{1/2}}{2l^{3/2}}(1+e)^2\,(2-e),\quad \eta=0, \tag{6.40}$$

and at $f=\pi$



$$\xi=-\frac{P\gamma^{1/2}}{2l^{3/2}}(1-e)^2\,(2+e),\quad \eta=0. \tag{6.41}$$

The sign of $\xi$ is reversed in both cases if the thrust opposes the motion at both junctions.

If the thrust supports the motion at $f=0$ and opposes it at $f=\pi$, eqs (6.25), (6.26) are valid (with positive signs). Thus, at $f=0$

$$\xi=-\frac{P\gamma^{1/2}}{2l^{3/2}}(1+e)^2,\quad \eta=0. \tag{6.42}$$

and at $f=\pi$

$$\xi=\frac{P\gamma^{1/2}}{2l^{3/2}}(1-e)^2,\quad \eta=0. \tag{6.43}$$

The sign of $\xi$ is reversed in both cases if the thrust opposes the motion at $f=0$ and supports it at $f=\pi$.

Finally, on a hyperbolic or parabolic orbit where an impulse is applied tangentially at the apse, eqs (6.17) are valid. Hence, from eqs (6.38), it follows that at this junction point

$$\xi=\mp\frac{\gamma^{1/2}}{l^{3/2}}(1+e)P,\quad \eta=0, \tag{6.44}$$

the negative sign being taken if the thrust supports the motion and the positive sign otherwise.

Employing some of the results obtained in this section, it can now be proved that a circular orbit having two junctions, at which the thrusts both support or both oppose the motion, cannot enter into an optimal trajectory.

For suppose the two thrusts support the motion and consider the junction at which the circular orbit is entered. The null thrust arc preceding the circular orbit must then be an ellipse, tangential at its further apse to the circle. This apse is a junction on the ellipse and eqs (6.39) give the components of $\dot{\mathbf{p}}$ at this point. The corresponding components at this junction on the circle are given by eqs (6.35) (taking the negative sign). For continuity, it is necessary that

$$P=\left(\frac{a}{l}\right)^{3/2}(1-e)\,[D+B(1-e)]. \tag{6.45}$$

The distance of the further apse on the ellipse from the centre of attraction is $l/(1-e)$ and hence

$$a=\frac{l}{1-e}. \tag{6.46}$$

Eq (6.45) can now be written

$$P(1-e)^{1/2}=D+B(1-e). \tag{6.47}$$

Also, at the apse, $\lambda=0$, $\mu=P$ and therefore, from eq (6.14), it follows that

$$P(1-e)=D+B(1-e)^2. \tag{6.48}$$

Solving eqs (6.47), (6.48) for $B$ and $D$, we find that

$$eB=[(1-e)^{-1/2}-1]P,\ eD=[1-e-(1-e)^{3/2}]P. \tag{6.49}$$

Thus $x_0$ (p. 99) is given by

$$x_0=\left(\frac{D^2}{B^2}\right)^{1/3}-1=-e \tag{6.50}$$

and, for this value, $f=\pi$ will be a minimum of $p$. Condition (*c*) p. 63 must accordingly be violated on the ellipse and the trajectory being considered cannot be optimal.

The case when both the thrusts applied on the circle oppose the motion, is similarly proved to be non-optimal.

We conclude, therefore, that if a null thrust arc belonging to an optimal trajectory is circular and is not a terminal arc, then the two junctions lying on it are at opposite ends of a diameter and the thrusts at these points are in opposite senses with respect to the motion. Thus, if a circular orbit forms part of an optimal trajectory, either as a terminal or an interior sub-arc, all other null thrust arcs will be conics with their axes aligned.

### 6.5. Transfer Between Two Circular Orbits

The manœuvre by which a rocket can be transferred with minimum propellent expenditure from one circular orbit about a centre of attraction into another coplanar and concentric with the first, was studied by Hohmann (1925). Since the terminal orbits are circles, it follows from the results of the previous sections of this chapter that the transfer orbits must be ellipses with their major axes aligned and tangential to one another and to the circles at their apses.



We will first consider the case where there is but one transfer orbit. This must be an ellipse tangential at its apses to the two circular orbits as indicated in Figure 6.4. The circular orbits will be assumed described by the rocket in the same sense and the transfer will be supposed to take place from the smaller circle to the larger. This is the mode of transfer discovered by Hohmann. There are two junction points, the apses $A$ and $B$ on the ellipse, and at both these points impulses are applied in the direction of motion.

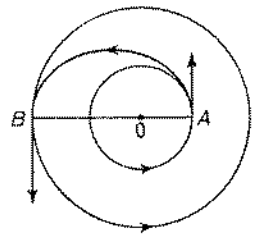


Figure 6.4.

Since the impulses at $A$ and $B$ support the motion, eq (6.36) give the components of $\dot{\mathbf{p}}$ at these points on the circles and eqs (6.40), (6.41) give the same components at these points on the transfer ellipse. Taking $P=1$ (eq (3.69)), for continuity of $\dot{\mathbf{p}}$ at $A$ it is necessary that

$$D+1=\left(\frac{a}{l}\right)^{3/2}(1+e)^2\,(2-e). \tag{6.51}$$

But, from the geometry of Figure 6.4, it is clear that

$$OA=a=l/(1+e). \tag{6.52}$$

Hence

$$D+1=(1+e)^{1/2}\,(2-e). \tag{6.53}$$

Similarly, if the outer circle is of radius $b$, eqs (6.36), (6.41) show that $\dot{\mathbf{p}}$ is continuous at $B$ provided that

$$D'+1=\left(\frac{b}{l}\right)^{3/2}(1-e)^2\,(2+e),$$
$$=(1-e)^{1/2}\,(2+e), \tag{6.54}$$

where $D'$ is the value of $D$ appropriate to the outer circle.

Eqs (6.53), (6.54) determine the constants $D$ and $D'$. But, as proved in section 6.2, each of these constants must take values lying in the interval [0, 1] if the transfer is to be optimal. It follows that $e$ must satisfy the inequalities

$$1 \leqslant (1+e)^{1/2}(2-e) \leqslant 2, \tag{6.55}$$

$$1 \leqslant (1-e)^{1/2}(2+e) \leqslant 2. \tag{6.56}$$

The graph of the function

$$y=(1+x)^{1/2}(2-x). \tag{6.57}$$

is shown in Figure 6.5. Consideration of this figure indicates that $y$ lies in the range [1, 2] provided $x$ lies in the interval $[-x_1, x_2]$.

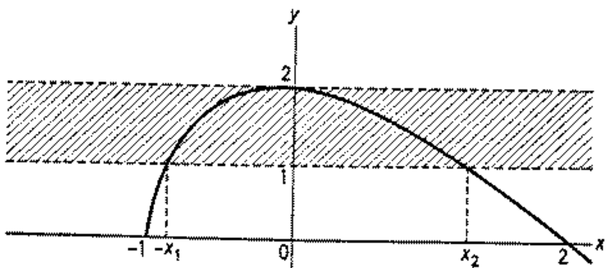


Figure 6.5.

It follows immediately that the inequalities (6.55), (6.56) are satisfied provided $e$ lies inside the interval $[0, e_0]$, where $e_0$ is the smaller of $x_1$ and $x_2$. Now $-x_1$, $x_2$ are roots of the equation

$$x^3-3x^2+3=0. \tag{6.58}$$

By numerical calculation, we find that $x_1=0{\cdot}8794\ \ldots$, $x_2=1{\cdot}347\ \ldots$ approximately. Thus

$$0 \leqslant e \leqslant 0{\cdot}8794. \tag{6.59}$$

Now, if $a$ is the semi-major axis of the transfer ellipse, it is evident from Figure 6.4 that



$$a=OA=\alpha(1-e),\ b=OB=\alpha(1+e). \tag{6.60}$$

Solving these equations for $\alpha$ and $e$, we obtain

$$\alpha=\tfrac{1}{2}(a+b), \quad e=\frac{\rho-1}{\rho+1}, \tag{6.61}$$

where $\rho=b/a$. The condition (6.59) can now be reinterpreted as a condition upon $\rho$, *viz.*

$$1 \leqslant \rho \leqslant 15{\cdot}6. \tag{6.62}$$

Thus, if the ratio of the radii of the circular orbits exceeds 15·6, then Hohmann transfer is certainly not optimal.

This result was first discovered by Hoelker and Silber (1959), who also showed that a three-impulse transfer can always be found which is more economical in propellent. The three-impulse transfer proceeds as follows (Figure 6.6). By the first impulse

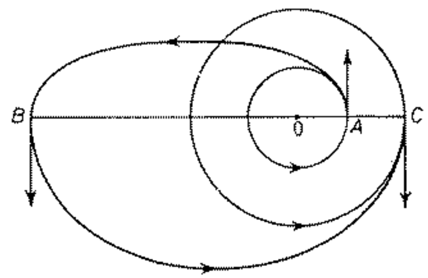


Figure 6.6.

applied at the junction $A$ in a tangential direction, the rocket is placed in an elliptical orbit whose major axis is greater than that of the Hohmann transfer ellipse. At the further apse $B$ on this orbit, the second tangential impulse is applied to place the vehicle in a second ellipse tangential at its nearer apse $C$ to the larger circle. The third impulse is applied at $C$ so as to *oppose* the motion and transfer the rocket into its final circular orbit. The greater the distance $OB$, the more economical is the manœuvre; in the limit, $B$ is at infinity, the transfer ellipses become parabolas and the impulse at $B$ is of zero magnitude. However, if $\rho > 15{\cdot}6$, any three-impulse transfer of this type for which $B$ lies outside the larger circle is superior to the Hohmann-type transfer. The Hohmann manœuvre can be regarded as a special case of the three-impulse manœuvre when $B$ lies on the larger circle and the impulse applied at $C$ vanishes. Thus, if $B$ lies just outside the larger circle, the three-impulse manœuvre can be thought of as being derived from the Hohmann manœuvre by a small variation of its thrust programme. This explains why the Hohmann transfer is found to be non-optimal in these circumstances.

If $\rho < 15{\cdot}6$, a small variation of the Hohmann programme of the type just considered, leads to an increase in propellent expenditure, as we expect. However, Hoelker and Silber show that if $11{\cdot}9 < \rho < 15{\cdot}6$, then provided $OB$ is sufficiently large, the three-impulse manœuvre can still be more economical than the Hohmann manœuvre. The variation in the Hohmann thrust programme necessary to convert it into the more economical three-impulse manœuvre is now a large one, however, and for this reason, the Hohmann manœuvre continues to be optimal with respect to small variations, the only variations covered by our theory.

If $\rho > 15{\cdot}6$ therefore, no optimal manœuvre exists. If the transfer is accomplished employing three impulses, by letting $OB$ increase indefinitely, the propellent expenditure can be made to decrease and approach a lower limit. Nevertheless, this lower limit is never actually attained for any finite value of $OB$ and hence does not correspond to a practicable manœuvre.

### 6.6. Optimal Escape Manœuvres

If a rocket has acquired sufficient velocity to escape from a gravitational field and is moving along its final NT-arc, it has been proved in section 3.3 (eq (3.59)) that, on this arc, the primer is given by the equation

$$\mathbf{p} = \mathbf{v}. \tag{6.63}$$

In particular, if the field is one obeying an inverse square law, then this equation is valid along the hyperbolic or parabolic escape arc. Now, by the conditions listed on p. 63, at any junction point on this arc or at a point where it meets an IT-arc, $\dot{p}$ must vanish. Hence, at such a point, $\dot{v}$ must also vanish. But,



$\dot{v}$ vanishes at the apse on the escape arc and at no other point. At this apse, $\mathbf{v}$ (and therefore $\mathbf{p}$) is perpendicular to the radius vector and it follows that this cannot be a point on an IT-arc. Thus the apse is a junction point at which an impulse is applied in a direction tangential to the trajectory. We can now conclude, as in section 6.3 (p. 102), that the optimal escape trajectory includes no IT-arcs and comprises only conic orbits with their axes aligned and tangential to one another at their apses.

Consider now the one-impulse mode of escape from a circular orbit already discussed in section 3.3. It was proved in this section that, if an impulsive thrust is applied to a rocket moving in a circular orbit in the direction of its motion so that the vehicle is transferred into a hyperbolic orbit and escapes from the centre of attraction, then all conditions for an optimal manœuvre are satisfied over the hyperbolic orbit. On this arc of null thrust, the primer and its derivative are given by eqs (3.55). It remains to decide whether the conditions for an optimal manœuvre are also satisfied over the initial circular orbit.

On the circular orbit, the primer components will be given by eqs (6.7) and, since the thrust supports the motion, the positive sign before $D$ must be taken and $f=0$ is the junction. The components of $\dot{\mathbf{p}}$ at the junction are then found from eqs (6.36). If $V$ is the velocity of the vehicle immediately after the impulse, then eqs (3.55) show that $\mathbf{p}$, $\dot{\mathbf{p}}$ are continuous across the junction provided

$$P = V, \quad \frac{\gamma^{1/2}}{2a^{3/2}}(P+D) = \frac{\gamma}{a^2}, \tag{6.64}$$

where $a$ is the orbital radius. Thus, if $V_c = \gamma^{1/2}/a^{1/2}$ is the vehicle velocity in the circular orbit, then

$$D = 2V_c - V. \tag{6.65}$$

But, for optimal conditions, it is necessary that $0 \leqslant D \leqslant P(=V)$ and hence

$$V_c \leqslant V \leqslant 2V_c. \tag{6.66}$$

The first of these inequalities is obviously satisfied. The significance of the second inequality is best understood as follows.

Let $V_\infty$ be the velocity at infinity on the hyperbolic orbit, i.e. the ultimate velocity of escape from the centre of attraction.

Since the total energy of the rocket remains constant over the hyperbolic orbit, we can write down the equation

$$\tfrac{1}{2}V^2 - \frac{\gamma}{a} = \tfrac{1}{2}V_\infty^2 \tag{6.67}$$

or

$$V_\infty^2 = V^2 - 2V_c^2. \tag{6.68}$$

It is now clear that the rocket can only escape from the field, provided $V \geqslant \sqrt{2}V_c = V_e$. $V_e$ is termed the *escape velocity* at the circular orbit. If $V \leqslant 2V_c$, then eq (6.68) shows that

$$V_\infty \leqslant V_e. \tag{6.69}$$

Thus, the escape manœuvre we have been considering cannot be optimal if the ultimate velocity with which the vehicle recedes from the centre of attraction exceeds the escape velocity at the circular orbit. This result is due to Lawden (1953).

If $V_\infty > V_e$, it is shown in Lawden (1953) that escape can be effected more economically by means of a two-impulse manœuvre. A thrust opposing the motion is first applied to the vehicle, transferring it into an elliptical orbit tangential to the circular orbit at its further apse. When the rocket arrives at the apse on this ellipse nearer to the centre of attraction, the second impulsive thrust operates in the direction of motion to increase the velocity to the value necessary to achieve escape with ultimate velocity $V_\infty$. The closer the second junction point is to the centre of attraction, the more economical is the manœuvre; however, since this junction can never coincide with the centre of attraction, optimal conditions are not attainable and no optimal manœuvre exists.

## 6.7. Optimal Transfer Manœuvre in the General Case

If the axes of two coplanar orbits are not aligned, optimal transfer between them cannot take place via a series of ellipses tangential to one another at their apses, and the theory of the foregoing sections is not applicable. It appears that IT-arcs can no longer be excluded from the optimal trajectory and, indeed, it seems possible that more than one such arc could be present. Also, it should be remembered, the conditions we have found for an optimal trajectory serve only to identify manœuvres which are



optimal relative to *small* variations of the thrust programme. This implies that a number of such optimal trajectories may be available in any particular case and we have no criterion, other than that of a direct comparison of the characteristic velocities corresponding to each, for deciding which represents the absolute optimal mode of transfer. Thus, if two terminal orbits intersect, transfer between them can be achieved by the application of a single impulse at a point of intersection. This impulse is completely calculable when the orbital elements are known, and hence the primer will also be known at the point of intersection. By substituting this information into eqs (5.55), (5.56) at the point of intersection on the two orbits, four conditions are obtained. Also, by writing down equations stating that $\dot{\mathbf{p}}$ is continuous at the junction, two more conditions become available. These six conditions serve to determine the values of the constants $A$, $B$, $D$ on the two orbits and then the primer is specified everywhere. Provided the magnitude of the primer is now found never to exceed unity, this mode of transfer constitutes a relative optimum. However, modes employing two or more impulses and satisfying all the conditions may also be found and these, also, will represent relative optima. It is clear that the theory, in its present state, requires that we should first choose the order in which the various thrust phases shall succeed one another, and our conditions will then select the relative optima from the class of all programmes for which the thrust sequence follows the chosen pattern. No criteria are known by which the absolute optimal pattern can be ascertained beforehand.

Analysis of the special case when the gravitational field is uniform which has been carried out in section 4.1, encourages us to conjecture that any phase of intermediate thrust can always be replaced, without increase of propellent expenditure, by two phases of impulsive thrust separated by a null thrust phase. However, numerical computations suggest that there exist pairs of orbits between which transfer can be effected via a single IT-arc and for which the best two-impulse transfer involves a greater expenditure of propellent. It appears probable, therefore, that an IT-arc or arcs will, in general, be present in an optimal trajectory. In this connection, the reader is referred to section X of Lawden (1962), Chap. 3, where it is proved that the magnitude of the primer is always a maximum, as required, at any junction between an IT-arc and an NT-arc. However, it is to be expected that the Jacobi-type condition (section 1.11) which we have not studied in this book, will require that any IT-arc present in an optimal trajectory shall not be too long. This would explain why the transfer between two nearly circular orbits via an IT-arc, discussed in section XI, Lawden (1962), Chap. 3, is less economical than a two-impulse mode of transfer, for the former mode of transfer involves a large number of turns of the intermediate thrust spiral.

Since we have, as yet, no means of deciding the pattern of the thrust phases for an absolute optimal transfer between two coplanar orbits, we shall, in the final sections of this chapter, study relative optimal transfers conforming to the two simplest possible patterns, *viz.* transfers via one or two impulses, there being no phase of intermediate thrust.

## 6.8. One-Impulse Transfers

Any elliptical orbit in a given plane through the centre of attraction can be completely specified in terms of three elements, *viz.* $l$, the semi-latus rectum, $e$, the eccentricity and $\tilde{\omega}$, the longitude of the pericentre measured from the reference line $\theta=0$. Consider the two orbits having elements $(l', e', \tilde{\omega}')$, $(l, e, \tilde{\omega})$, which we shall assume to intersect at a point $P$. A suitable impulse applied at $P$ will effect transfer from the first orbit to the second and, as explained in the previous section, this impulse will also determine the primer at all points on both orbits. In this section, equations will be found from which the primer can be conveniently calculated along the two orbits. If the magnitude of the primer is then found to be nowhere in excess of unity, the transfer will be known to be relatively optimal.

Let $(r, \theta)$ be the polar co-ordinates of $P$ and let $f', f$ be the real anomalies of the point on the two orbits. Then we have the equations

$$e' \cos f' = \frac{l'}{r} - 1, \tag{6.70}$$

$$e \cos f = \frac{l}{r} - 1, \tag{6.71}$$

$$f' = \theta - \tilde{\omega}', \tag{6.72}$$

$$f = \theta - \tilde{\omega} \tag{6.73}$$



determining $r$, $\theta$, $f$ and $f'$. Let $\phi$ be the angle made by the impulsive thrust $T$ with the perpendicular to the radius vector $OP$, drawn in the sense of the orbital motions (Figure 6.7). The component $W$ of the rocket's velocity at $P$ in a direction perpendicular to the thrust is unaffected by the impulse and is therefore the same for both orbits. If $(v_r, v_\theta)$ are the radial and transverse components of the rocket's velocity at $P$ in the orbit $(l, e, \tilde{\omega})$, then

$$W = v_\theta \sin\phi - v_r \cos\phi. \tag{6.74}$$

Employing eqs (6.18), this equation can be written in the form

$$e \sin f = \frac{l}{r}\tan\phi - W\sqrt{\frac{l}{\gamma}}\sec\phi. \tag{6.75}$$

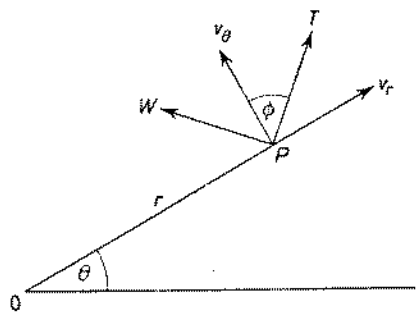


Figure 6.7.

Similarly, by consideration of the other orbit, we obtain the equation

$$e' \sin f' = \frac{l'}{r}\tan\phi - W\sqrt{\frac{l'}{\gamma}}\sec\phi. \tag{6.76}$$

Eqs (6.75), (6.76) serve to determine $W$ and $\phi$.

Equations can now be written down expressing that $\mathbf{p}$ is a unit vector in the direction of the impulse at $P$ and that $\dot{\mathbf{p}}$ is continuous at this point. Referring to eqs (5.55), (5.56), (5.63) and (5.64), it will be seen that the required conditions are as follows:

$$\left.\begin{aligned}
&A'\cos f' + B'e'\sin f' = A\cos f + Be\sin f = \sin\phi,\\
&-A'\sin f' + B'(1+e'\cos f') + \frac{D'-A'\sin f'}{1+e'\cos f'}\\
&\quad = -A\sin f + B(1+e\cos f) + \frac{D-A\sin f}{1+e\cos f} = \cos\phi,\\
&l'^{1/2}\left(\frac{A'\sin f' - D'}{1+e'\cos f'} - B'\right) = l^{1/2}\left(\frac{A\sin f - D}{1+e\cos f} - B\right),\\
&l'^{-3/2}\{-A'(e'+\cos f') + D'e'\sin f'\}\\
&\qquad = l^{-3/2}\{-A(e+\cos f) + De\sin f\}.
\end{aligned}\right\} \quad (6.77)$$

These provide sufficient equations for calculating the constants $A, B, D, A', B', D'$.

Solving the eqs (6.77) for $A$ and employing eqs (6.70), (6.71), (6.75) and (6.76) in the reduction, it will be found that

$$e\sqrt{\frac{\gamma}{l}}\,A = W - \frac{\gamma}{Wr}\sin^2\phi. \quad (6.78)$$

By the second of eqs (6.77), we have

$$Be\sin f = \sin\phi - A\cos f. \quad (6.79)$$

Substituting for $A$ from eq (6.78) and making use of eqs (6.71), (6.75), it can now be shown that

$$e^2B = \left(\frac{l}{r}-1\right)\left(1+\frac{\gamma^{1/2}}{Wl^{1/2}}\sin\phi\right)\cos\phi + \left(\frac{l}{r}\sin\phi - \frac{Wl^{1/2}}{\gamma^{1/2}}\right)\tan\phi. \quad (6.80)$$

Writing the fourth of eqs (6.77) in the form

$$B + D = (1+e\cos f)\cos\phi + (2+e\cos f)(A\sin f - Be\cos f) \quad (6.81)$$

and substituting in the right-hand member of this equation for $B$ from eq (6.79), by employing eqs (6.71), (6.75) and (6.78), it follows that

$$B + D = -\left\{1 + \frac{\gamma^{1/2}}{Wl^{1/2}}\left(\frac{l}{r}+1\right)\sin\phi\right\}\cos\phi. \quad (6.82)$$



The constants $A$, $B$, $D$ may now be conveniently calculated from eqs (6.78), (6.80) and (6.82). When this has been done, the primer is fully determined on the orbit $(l, e, \tilde{\omega})$. Replacing $A, B, D, l, e$ in eqs (6.78), (6.80), (6.82) by the corresponding quantities with primes, equations from which $A', B', D'$ can be calculated are immediately obtained and the primer is then known over the second orbit also.

To determine the characteristic velocity for the manœuvre, it is only necessary to calculate the components of the velocities at $P$ of the vehicle in the two orbits; the difference is then the velocity increment caused by the impulse. Let $U$ be the component of the orbital velocity in the direction of $T$ for the case of the orbit $(l, e, \tilde{\omega})$. Then, referring to Figure 6.7, it is clear that

$$v_\theta = U\cos\phi + W\sin\phi. \quad (6.83)$$

Hence

$$U = \frac{\sqrt{(\gamma l)}}{r}\sec\phi - W\tan\phi. \quad (6.84)$$

Similarly, for the same component $U'$ in the orbit $(l', e'\,\tilde{\omega}')$, we find that

$$U' = \frac{\sqrt{(\gamma l')}}{r}\sec\phi - W\tan\phi. \quad (6.85)$$

Subtracting the last two equations, we obtain for the characteristic velocity

$$V = U - U' = \frac{\gamma^{1/2}}{r}(l^{1/2} - l'^{1/2})\sec\phi. \quad (6.86)$$

### 6.9. Two-Impulse Transfers

Let $(l_1, e_1, \tilde{\omega}_1)$, $(l_2, e_2, \tilde{\omega}_2)$ be the orbital elements of two coplanar orbits to be described by a rocket in the same sense. Assuming that transfer between these orbits is effected by the application of two impulses, let $(l, e, \tilde{\omega})$ be the elements of the orbit along which the transfer takes place. The transfer orbit intersects the terminal orbits at two junctions $P_i$, where $i=1$ or 2, depending upon whether the junction lies on the first or the second terminal orbit. The polar co-ordinates of $P_i$ will be denoted by $(r_i, \theta_i)$, $\phi_i$ will denote the angle made by the impulse at $P_i$ with the perpendicular to the radius vector through the point and $W_i$ the component of velocity in a direction perpendicular to the impulse.

Then, the values of the constants $A$, $B$, $D$ on the transfer orbit must be the same whether calculated from data relating to $P_1$ or from data relating to $P_2$. Thus, from eqs (6.78), (6.80), (6.81), three conditions to be satisfied by the transfer orbit may be derived. These three conditions will be sufficient to determine the three elements of this orbit. Before writing down these conditions, it is convenient to introduce new quantities $p$, $q$, $s_i$, $Z_i$ defined by the following equations

$$p = 1/l,\ q = e/l, \tag{6.87}$$

$$s_i = 1/r_i,\ Z_i = W_i/(\gamma^{1/2}\sin\phi_i). \tag{6.88}$$

The three conditions may then be expressed thus:

$$(Z_1 - s_1/Z_1)\sin\phi_1 = (Z_2 - s_2/Z_2)\sin\phi_2, \tag{6.89}$$

$$\begin{aligned}(s_1-p)(1+p^{1/2}/Z_1)\cos\phi_1 + (s_1 - Z_1p^{1/2})\sin\phi_1\tan\phi_1 \\ = (s_2-p)(1+p^{1/2}/Z_2)\cos\phi_2 + (s_2 - Z_2p^{1/2})\sin\phi_2\tan\phi_2,\end{aligned} \tag{6.90}$$

$$\left(1+\frac{s_1+p}{Z_1p^{1/2}}\right)\cos\phi_1 = \left(1+\frac{s_2+p}{Z_2p^{1/2}}\right)\cos\phi_2. \tag{6.91}$$

An alternative derivation of these conditions can be made (Lawden, 1962) by expressing the characteristic velocity $V$ of the transfer manœuvre as a function of the elements ($l$, $e$, $\bar{\omega}$) and writing down the conditions that $V$ should be stationary with respect to all small variations in these elements.

To complete the set of equations determining the optimal transfer orbit, it is necessary to write down equations relating the quantities $s_i$, $\theta_i$, $Z_i$, $\phi_i$ to the orbital elements. Referring to the polar eq (6.8) for an orbit, four conditions that the junctions ($r_i$, $\theta_i$) should lie on their respective orbits can first be written down in the form

$$q_1\cos(\theta_1 - \bar{\omega}_1) = s_1 - p_1, \tag{6.92}$$

$$q\cos(\theta_1 - \bar{\omega}) = s_1 - p, \tag{6.93}$$

$$q\cos(\theta_2 - \bar{\omega}) = s_2 - p, \tag{6.94}$$

$$q_2\cos(\theta_2 - \bar{\omega}_2) = s_2 - p_2. \tag{6.95}$$

Then, at each junction, two equations of the type (6.75) are available. These can be written



$$q_1\sin(\theta_1 - \bar{\omega}_1) = (s_1 - p_1^{1/2}Z_1)\tan\phi_1, \tag{6.96}$$

$$q\sin(\theta_1 - \bar{\omega}) = (s_1 - p^{1/2}Z_1)\tan\phi_1, \tag{6.97}$$

$$q\sin(\theta_2 - \bar{\omega}) = (s_2 - p^{1/2}Z_2)\tan\phi_2, \tag{6.98}$$

$$q_2\sin(\theta_2 - \bar{\omega}_2) = (s_2 - p_2^{1/2}Z_2)\tan\phi_2. \tag{6.99}$$

There are now eleven equations, *viz.* (6.89)–(6.99), from which the eleven unknown quantities $s_1$, $s_2$, $\theta_1$, $\theta_2$, $Z_1$, $Z_2$, $\phi_1$, $\phi_2$, $p$, $q$, $\bar{\omega}$ may be evaluated.

Solution of this set of equations has been performed (Lawden, 1962, Plimmer, 1960 and Smith, 1959) for the cases when (i) the axes of the terminal orbits are aligned and (ii) the terminal orbits are identical apart from their orientations. In our final section, we shall obtain an approximate solution for the case when the terminal orbits are both of small eccentricity.

## 6.10. Transfer between Terminal Orbits of Small Eccentricity

Let ($p_1$, $\epsilon q_1$, $\bar{\omega}_1$), ($p_2$, $\epsilon q_2$, $\bar{\omega}_2$), $\epsilon$ being a small quantity, be elements of two almost circular coplanar orbits about the same centre of force. We shall assume that each of the eleven quantities involved in the optimal two-impulse transfer and which are governed by eqs (6.89)–(6.99), are regular functions of $\epsilon$ for sufficiently small values of this quantity and can accordingly be expanded as power series in $\epsilon$. It will here be convenient to denote these eleven unknowns by the symbols $\bar{s}_1$, $\bar{s}_2$, . . ., $\bar{\bar{\omega}}$, so that the power series expansions can be written

$$\left.\begin{aligned}\bar{s} &= s_1 + \epsilon s_1' + \epsilon^2 s_1'' + \dots,\\ \bar{\bar{\omega}} &= \bar{\omega} + \epsilon\bar{\omega}' + \epsilon^2\bar{\omega}'' + \dots\end{aligned}\right\} \tag{6.100}$$

We shall proceed to calculate these expansions as far as terms of the first order in $\epsilon$. The results to be found were first obtained by Smith (1959). (N.B. Primes in eq (6.100) indicate the order, with respect to $\epsilon$, of the various terms and are not intended to signify differentiations as formerly.)

Replacing $q_1$, $q_2$ in eqs (6.89)–(6.99) by $\epsilon q_1$, $\epsilon q_2$ respectively and substituting the power series expansions (6.100), by equating terms of zero order from the two members of each equation, the following equations are derived:

$$(Z_1 - s_1/Z_1)\sin\phi_1 = (Z_2 - s_2/Z_2)\sin\phi_2, \tag{6.101}$$

$$(s_1-p)(1+p^{1/2}/Z_1)\cos\phi_1+(s_1-Z_1p^{1/2})\sin\phi_1\tan\phi_1$$
$$=(s_2-p)(1+p^{1/2}/Z_2)\cos\phi_2+(s_2-Z_2p^{1/2})\sin\phi_2\tan\phi_2, \quad (6.102)$$

$$\left(1+\frac{s_1+p}{Z_1p^{1/2}}\right)\cos\phi_1=\left(1+\frac{s_2+p}{Z_2p^{1/2}}\right)\cos\phi_2, \quad (6.103)$$

$$0=s_1-p_1, \quad (6.104)$$

$$q\cos(\theta_1-\tilde{\omega})=s_1-p, \quad (6.105)$$

$$q\cos(\theta_2-\tilde{\omega})=s_2-p, \quad (6.106)$$

$$0=s_2-p_2, \quad (6.107)$$

$$0=(s_1-Z_1p_1^{1/2})\tan\phi_1, \quad (6.108)$$

$$q\sin(\theta_1-\tilde{\omega})=(s_1-Z_1p^{1/2})\tan\phi_1, \quad (6.109)$$

$$q\sin(\theta_2-\tilde{\omega})=(s_2-Z_2p^{1/2})\tan\phi_2, \quad (6.110)$$

$$0=(s_2-Z_2p_2^{1/2})\tan\phi_2, \quad (6.111)$$

It is clear that these equations are identical with eqs (6.89)–(6.99) after $q_1$ and $q_2$ have been put equal to zero. Their solution accordingly corresponds to the optimal two-impulse transfer between two circular orbits, *viz.* the Hohmann mode of transfer. If we assume that transfer is to take place from the smaller orbit to the larger, then we know that for a Hohmann manœuvre

$$\left.\begin{aligned}\phi_1=\phi_2=0,\ \theta_1-\tilde{\omega}=0,\ \theta_2-\tilde{\omega}=\pi,\\ s_1=p_1,\ s_2=p_2.\end{aligned}\right\} \quad (6.112)$$

Eqs (6.104)–(6.111) are now all satisfied provided we take

$$p=\tfrac{1}{2}(p_1+p_2),\ q=\tfrac{1}{2}(p_1-p_2). \quad (6.113)$$

If $a_1$, $a_2$ are the radii of the circular orbits, then $p_1=1/a_1$, $p_2=1/a_2$ and, by eqs (6.113), the semi-latus rectum and eccentricity of the transfer orbit are given, to zero order in $\epsilon$, by

$$l=\frac{1}{p}=\frac{2a_1a_2}{a_1+a_2}, \quad (6.114)$$

$$e=\frac{q}{p}=\frac{a_2-a_1}{a_2+a_1}. \quad (6.115)$$

These results also follow from the geometry of Figure 6.4. $\tilde{\omega}$ is, as yet, indeterminate, the orientation of the Hohmann transfer ellipse being arbitrary.



Eq (6.101) is satisfied, since $\phi_1=\phi_2=0$. Eqs (6.102), (6.103) reduce to

$$2\sqrt{2}+(p_1+p_2)^{1/2}\left(\frac{1}{Z_1}+\frac{1}{Z_2}\right)=0, \quad (6.116)$$

$$\frac{3p_1+p_2}{Z_1}=\frac{p_1+3p_2}{Z_2}. \quad (6.117)$$

It follows that

$$Z_1=-\frac{\sqrt{2}(p_1+p_2)}{p_1+3p_2},\quad Z_2=-\frac{\sqrt{2}(p_1+p_2)}{3p_1+p_2}. \quad (6.118)$$

We now turn to a consideration of the terms of the first order in $\epsilon$. It is found from eqs (6.89)–(6.99) that, after some substitution of the values just obtained for the zero order terms, the first order equations take the form:

$$(Z_1-p_1/Z_1)\phi_1'=(Z_2-p_2/Z_2)\phi_2', \quad (6.119)$$

$$(s_1'-p')(1+p^{1/2}/Z_1)+(p_1-p)\left(\frac{p'}{2p^{1/2}Z_1}-\frac{p^{1/2}Z_1'}{Z_1^2}\right)$$
$$=(s_2'-p')(1+p^{1/2}/Z_2)+(p_2-p)\left(\frac{p'}{2p^{1/2}Z_2}-\frac{p^{1/2}Z_2'}{Z_2^2}\right), \quad (6.120)$$

$$\frac{s_1'+p'}{Z_1p^{1/2}}-(p_1+p)\left(\frac{Z_1'}{Z_1^2p^{1/2}}+\frac{p'}{2Z_1p^{3/2}}\right)$$
$$=\frac{s_2'+p'}{Z_2p^{1/2}}-(p_2+p)\left(\frac{Z_2'}{Z_2^2p^{1/2}}+\frac{p'}{2Z_2p^{3/2}}\right), \quad (6.121)$$

$$q_1\cos(\tilde{\omega}-\tilde{\omega}_1)=s_1', \quad (6.122)$$

$$q'=s_1'-p', \quad (6.123)$$

$$-q'=s_2'-p', \quad (6.124)$$

$$-q_2\cos(\tilde{\omega}-\tilde{\omega}_2)=s_2', \quad (6.125)$$

$$q_1\sin(\tilde{\omega}-\tilde{\omega}_1)=(p_1-Z_1p_1^{1/2})\phi_1', \tag{6.126}$$

$$q(\theta_1'-\tilde{\omega}')=(p_1-Z_1p^{1/2})\phi_1', \tag{6.127}$$

$$-q(\theta_2'-\tilde{\omega}')=(p_2-Z_2p^{1/2})\phi_2', \tag{6.128}$$

$$-q_2\sin(\tilde{\omega}-\tilde{\omega}_2)=(p_2-Z_2p_2^{1/2})\phi_2'. \tag{6.129}$$

Dividing eq (6.129) by eq (6.126), we obtain

$$-\frac{q_2\sin(\tilde{\omega}-\tilde{\omega}_2)}{q_1\sin(\tilde{\omega}-\tilde{\omega}_1)}=\frac{(p_2-Z_2p_2^{1/2})\phi_2'}{(p_1-Z_1p_1^{1/2})\phi_1'}. \tag{6.130}$$

Eliminating the ratio $\phi'_2/\phi'_1$ by the use of eq (6.119) and substituting for $Z_1$, $Z_2$ from eq (6.118), this equation can be written in the form

$$-\frac{\sin(\tilde{\omega}-\tilde{\omega}_2)}{\sin(\tilde{\omega}-\tilde{\omega}_1)}=\frac{q_1}{q_2}P\left(\frac{p_2}{p_1}\right), \tag{6.131}$$

where, in its simplest form,

$$P(x)=\frac{1+3x-\sqrt{2}(1+x)^{3/2}}{3+x-\sqrt{(2/x)}(1+x)^{3/2}}. \tag{6.132}$$

Eq (6.131) now determines $\tilde{\omega}$ and, solving for $\tan\tilde{\omega}$, it is found that

$$\tan\tilde{\omega}=\frac{Q\sin\tilde{\omega}_1+\sin\tilde{\omega}_2}{Q\cos\tilde{\omega}_1+\cos\tilde{\omega}_2}, \tag{6.133}$$

where

$$Q=\frac{q_1}{q_2}P\left(\frac{p_2}{p_1}\right). \tag{6.134}$$

If $P(x)$ is graphed against $x$, it will appear that, very approximately

$$P(x)=x. \tag{6.135}$$

It follows, therefore, to the same degree of approximation that

$$Q=\frac{q_1p_2}{q_2p_1}=\frac{e_1}{e_2}, \tag{6.136}$$



$e_1$, $e_2$ being the eccentricities of the terminal orbits. Eq (6.133) can therefore be written approximately thus:

$$\tan\tilde{\omega}=\frac{e_1\sin\tilde{\omega}_1+e_2\sin\tilde{\omega}_2}{e_1\cos\tilde{\omega}_1+e_2\cos\tilde{\omega}_2}. \tag{6.137}$$

It is now clear that, if $e_1\gg e_2$, then $\tilde{\omega}=\tilde{\omega}_1$ approximately, whereas, if $e_2\gg e_1$, then $\tilde{\omega}=\tilde{\omega}_2$. This means that *the axis of the transfer ellipse tends to align itself with the axis of the terminal orbit having the greater eccentricity*. This is the *Rule of Attraction of Eccentricities*. In particular, if one terminal is a circle, then the axis of the transfer ellipse is aligned with that of the elliptical terminal; this we have already shown to be the case in the earlier sections of this chapter.

Having calculated $\tilde{\omega}$, the first order terms in the expansions are easily calculated from eqs (6.119)–(6.129). However, $\tilde{\omega}'$ remains indeterminate at this stage and, if required, can only be found by a consideration of second order terms.

It may be shown (Smith, 1959) that the first order solution for the transfer orbit which has been determined by this method is identical, to the same order of approximation, with the optimal ellipse tangential to both terminals. This cotangential ellipse was the subject of an investigation in Lawden (1952). However, in general, such an ellipse does not constitute the optimal transfer orbit, although it is usually near-optimal (Bender, 1962).

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### *Exercises*

1. $\rho = b/a (>1)$ is the ratio of the radii of two coplanar circular orbits and transfer between them is to be effected according to the Hoelker-Silber three-impulse mode. If $c\ (>b)$ is the distance from the centre of attraction of the junction which does not lie on either circle, show that the characteristic velocity for the manœuvre is given by

$$V = \sqrt{\frac{\gamma}{b}}\left[\frac{\sqrt{2}(\rho x - 1)}{x^{1/2}(\rho x + 1)^{1/2}} + \sqrt{2}x^{-1/2}(1+x)^{1/2} - \rho^{1/2} - 1\right],$$

where $x = c/b$. Prove that the signs of $dV/dx$ and the expression

$$3\rho + 1 + 6\rho(\rho+1)x - \rho^2(\rho - 9)x^2$$

are identical for $x \geqslant 1$ and deduce that $V$ is monotonically decreasing for all $x$ if $\rho > 15{\cdot}6$. Deduce that, for such values of $\rho$, the three-impulse mode is more economical than the Hohmann mode for all $x \geqslant 1$. Sketch the graph of $V$ against $x$ when $\rho < 15{\cdot}6$ and show that, if $\rho > 11{\cdot}94$, the three-impulse mode is the more economical for sufficiently large values of $x$.

2. A rocket is moving in a circular orbit of radius $a$ with velocity $v_c$ and is to escape to infinity with remaining velocity $v$. Show that the characteristic velocity for the two-impulse manœuvre described at the end of section 6.6. is given by

$$V = v_c\{\sqrt{2}(x + \rho^2)^{1/2} - \sqrt{2}(x+1)^{1/2} + 1\},$$

where $\rho = v/(\sqrt{2}v_c)$ and $x = a/a'$, $a'\ (<a)$ being the distance of the second junction point from the centre of attraction. Deduce that, if $\rho > 1$, $V$ decreases monotonically as $x$ increases, and hence that the two-impulse manœuvre is more economical than escape via a single tangential impulse, for all $x > 1$.

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