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Impulsive Transfer between Elliptical Orbits

DEREK F. LAWDEN
University of Canterbury, Christchurch, New Zealand

11.1 Introduction

The solution to the general problem of programming the propellant expenditure and thrust direction for a rocket vehicle so that transfer between two specified points in a given gravitational field is effected most economically is given elsewhere.[1] In the same article, the results of the general theory were applied in the particular case of a simple inverse square law field to obtain conditions governing the optimal mode of transfer between two coplanar elliptical orbits. It was shown that, provided the magnitude of the motor thrust could be regarded as unlimited, optimal conditions were attained by the application of impulsive thrusts at certain *junction points*, the vehicle coasting freely between these points under the action of gravity alone. In this chapter it will be demonstrated that, provided the assumption is made that the motor thrusts are invariably impulsive, the conditions determining the positions of the junction points for optimal transfer between coplanar elliptical orbits can be obtained by employing an elementary method and without reference to the general theory.

The problem of optimal transfer between orbits in an inverse square law field was first studied by Hohmann[2] in the particular case when the orbits are coplanar and circular. Transfer between elliptical orbits has been investigated by Lawden,[1,3–7] Long,[8] Smith,[9] and Plimmer.[10] Vargo,[11] Munick *et al.*,[12] and Horner[13] have considered the related problem of optimal transfer between two *terminals* in an inverse square law field, a terminal being completely specified by its distance from the center of attraction and a velocity vector, i.e., the angle subtended at the center of attraction by two terminal points is assumed to be open to choice. The question of the number of impulsive thrusts which is optimal for a maneuver of this type has been taken up by Edelbaum,[14] Hoelker and Silber,[15] and Rider,[16] and it has been demonstrated that, in certain circumstances, as many as four impulses are necessary to achieve absolute minimal expenditure of fuel. This important question will not be pursued further in this chapter; for it will always be assumed that the number of impulses (usually two) has been decided prior to our calculations and that the problem is to discover the junction points at which these are to be applied and their magnitudes and directions at these points.

11.2 Impulsive Change in Orbital Elements

Let O be a center of inverse square law attraction, γ/r^2 being the attraction per unit mass at a distance r from the center. Let (r, θ) be polar coordinates with respect to O as pole in a given plane through this point. Then the polar equation for the orbit of a body which moves in this plane under the action of the attraction alone can be written

$$\frac{l}{r} = 1 + e \cos (\theta - \bar{\omega}) \tag{11.1}$$

where l is the semilatus rectum, e is the eccentricity, and $\bar{\omega}$ is the longitude of perihelion. It will be convenient to put

$$\frac{1}{r} = s, \quad \frac{1}{l} = p, \quad \frac{e}{l} = q \tag{11.2}$$

so that Eq. (11.1) becomes

$$q \cos (\theta - \bar{\omega}) = s - p \tag{11.3}$$

The parameters $(p, q, \bar{\omega})$ completely determine the orbit in the given plane.

If P is the position of the body at any instant and $(1/s, \theta)$ are its polar coordinates, let (v_r, v_θ) be the components of the body's velocity along and perpendicular (in the sense of the motion) to OP, respectively. Then it is a well-known consequence of the theory of orbits that

$$v_r = q \sqrt{\frac{\gamma}{p}} \sin (\theta - \bar{\omega}) \tag{11.4}$$

$$v_\theta = \sqrt{\frac{\gamma}{p}} \{p + q \cos (\theta - \bar{\omega})\} = s \sqrt{\frac{\gamma}{p}} \tag{11.5}$$

Let V be the component of the body's velocity at P in a direction making

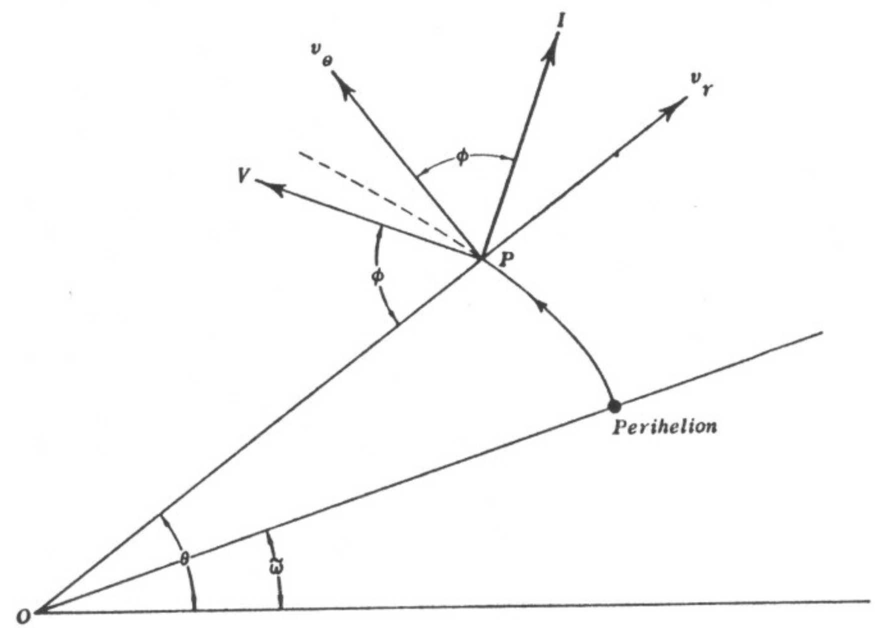

FIG. 1. Junction point.

an angle ϕ with PO (see Fig. 1). Then it follows from Eqs. (11.4) and (11.5) that

$$\begin{aligned} V &= v_\theta \sin \phi - v_r \cos \phi \\ &= s \sqrt{\frac{\gamma}{p}} \sin \phi - q \sqrt{\frac{\gamma}{p}} \sin (\theta - \bar{\omega}) \cos \phi \end{aligned} \tag{11.6}$$

By rearrangement, Eq. (11.6) can be expressed in the form

$$q \sin(\theta - \bar{\omega}) = (s - Zp^{1/2}) \tan \phi \tag{11.7}$$

where

$$Z = V/\gamma^{1/2} \sin \phi \tag{11.8}$$

Suppose, now, that an impulse I is applied to the orbiting body when it is at the point P and in a direction making an angle ϕ with the transverse velocity component v_θ. The component V of the body's velocity perpendicular to I will be unaffected by the impulse. It follows that Z, as given by Eq. (11.8), will be unaltered. Further, (s, θ) will undergo no appreciable change during the short period of thrust. Hence the values of the orbital parameters $(p, q, \bar{\omega})$ relating to the new orbit into which the body has been transferred by the impulse will still satisfy Eq. (11.7). By a similar argument it follows that these parameters will also satisfy Eq. (11.3).

To sum up, therefore, the parameters $(p, q, \bar{\omega})$ of any orbit into which the body may be transferred by application of an impulse at a point P $(1/s, \theta)$ in the direction making an angle ϕ with v_θ satisfy the equations

$$q \cos(\theta - \bar{\omega}) = s - p \tag{11.9}$$

$$q \sin(\theta - \bar{\omega}) = (s - Zp^{1/2}) \tan \phi \tag{11.10}$$

where Z is related to the body's velocity component V in the direction perpendicular to the impulse by Eq. (11.8). The requirement that the three parameters $(p, q, \bar{\omega})$ should satisfy Eqs. (11.9) and (11.10) leaves these quantities one degree of freedom. The parameters will be specified completely when the *magnitude* of the impulse is given.

A convenient graphical means of determining all those orbits $(p, q, \bar{\omega})$ into which an orbiting body may be transferred by application of a single impulse in a given direction at a given point is as follows: Construct the graph of the functional relationship $y = x^2$ (see Fig. 2). Fix the point A on the x-axis at a distance from the origin O given by the equation

$$OA = \frac{s}{Z^2 \tan \phi} \tag{11.11}$$

and erect the perpendicular AB to OA such that

$$AB = \frac{s}{Z^2 \tan^2 \phi} \tag{11.12}$$

Let P be a point (x, y) on the curve $y = x^2$ and let PN be the perpendicular

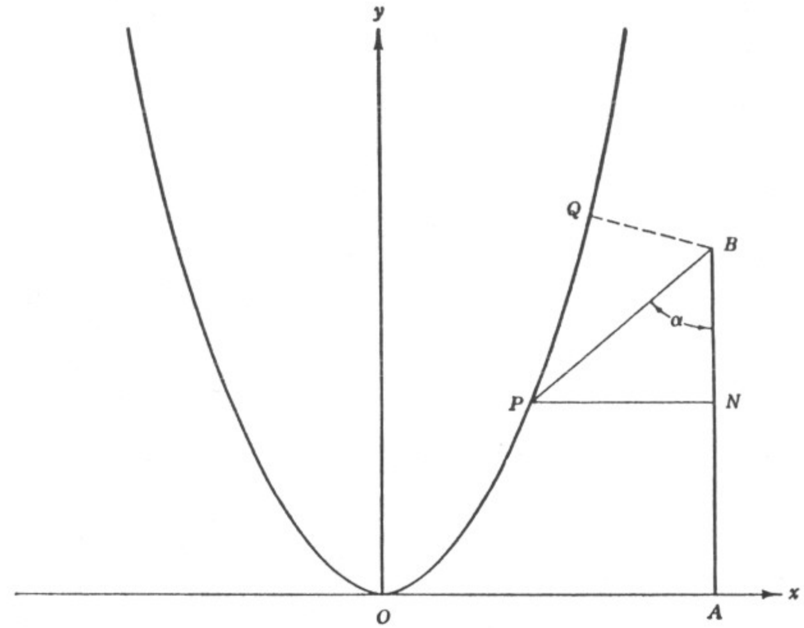

FIG. 2. Nomogram for orbital elements.

from P on to AB. Denoting $\angle PBA$ by α, we have by simple geometry

$$PB \cos \alpha = AB - y \tag{11.13}$$

$$PB \sin \alpha = OA - x = OA - y^{1/2} \tag{11.14}$$

If, therefore, we interpret PB, α, and y thus,

$$PB = \frac{q}{Z^2 \tan^2 \phi}, \quad \alpha = \theta - \bar{\omega}, \quad y = \frac{p}{Z^2 \tan^2 \phi} \tag{11.15}$$

Eqs. (11.13) and (11.14) are seen to be identical with Eqs. (11.9) and (11.10), respectively. It now follows that, as P describes the curve $y = x^2$, possible sets of values of the parameters $(p, q, \bar{\omega})$ are given for any position of P by AN, PB, and α in accordance with Eqs. (11.15). When an impulse of given magnitude is applied, the effect in the diagram will be to move P from one point to another along the curve. The possible ranges of values of $(p, q, \bar{\omega})$ for impulses applied at a given point on a body's orbit and in a given direction are now immediately obvious from inspection of the diagram. For example, the minimum value of q which can be attained corresponds to BQ.

Finally in this section, a formula will be obtained from which the magnitude of the impulse can be found. Referring to Fig. 1, it will be seen that if U is the body's component of velocity in the direction of the impulse, then

$$v_s = U \cos\phi + V \sin\phi \tag{11.16}$$

Employing Eq. (11.5), this implies that

$$U = s\sqrt{\frac{\gamma}{p}} \sec\phi - V \tan\phi \tag{11.17}$$

At the termination of the impulse, suppose (p, U) have new values (p', U'). (s, V) will be unaltered in value and the equation corresponding to (11.17), but applying to conditions immediately after the impulse, will be

$$U' = s\sqrt{\frac{\gamma}{p'}} \sec\phi - V \tan\phi \tag{11.18}$$

Subtracting Eq. (11.17) from (11.18) we now find that

$$\Delta U = U' - U = \gamma^{1/2} s(p'^{-1/2} - p^{-1/2}) \sec\phi \tag{11.19}$$

This last equation fixes the velocity increment caused by the impulse and hence its magnitude.

11.3 Dependence of Impulse on Orbital Elements

Suppose that a body moving in an orbit $(p, q, \bar{\omega})$ is transferred by a single impulse into a coplanar orbit $(p', q', \bar{\omega}')$. This impulse must be applied at a point of intersection (assumed real) of the two orbits. Let $(1/s, \theta)$ be the polar coordinates of this point and let ϕ be the angle made by the direction of the impulse with the perpendicular to the radius vector from the center of attraction. Equations (11.9) and (11.10) must be satisfied by the parameters of both orbits, thus yielding the equations

$$q \cos(\theta - \bar{\omega}) = s - p \tag{11.20}$$

$$q \sin(\theta - \bar{\omega}) = (s - Zp^{1/2}) \tan\phi \tag{11.21}$$

$$q' \cos(\theta - \bar{\omega}') = s - p' \tag{11.22}$$

$$q' \sin(\theta - \bar{\omega}') = (s - Zp'^{1/2}) \tan\phi \tag{11.23}$$

where Z is the quantity defined by Eq. (11.8) for this maneuver. Regarding the parameters determining the two orbits as given quantities, the four

equations (11.20)–(11.23) can be solved for the four unknowns (s, θ, Z, ϕ), thus expressing these quantities as functions of $p, q, \bar{\omega}, p', q', \bar{\omega}'$. The velocity increment $\Delta\bar{U} = \gamma^{1/2}X$ resulting from the impulse then follows from Eq. (11.19), viz.

$$\Delta\bar{U} = \gamma^{1/2}X = \gamma^{1/2} s(p'^{-1/2} - p^{-1/2}) \sec\phi \tag{11.24}$$

and X can also be expressed as a function of the six orbital parameters. Thus

$$X = X(p, q, \bar{\omega}, p', q', \bar{\omega}'). \tag{11.25}$$

It is our purpose in this section to investigate this functional dependence of X upon the parameters by calculating expressions for the partial derivatives of X with respect to its six arguments.

Differentiating Eqs. (11.20)–(11.23) partially with respect to p, it will be seen that

$$-q \sin(\theta - \bar{\omega}) \frac{\partial\theta}{\partial p} = \frac{\partial s}{\partial p} - 1 \tag{11.26}$$

$$q \cos(\theta - \bar{\omega}) \frac{\partial\theta}{\partial p} = \left(\frac{\partial s}{\partial p} - \frac{\partial Z}{\partial p} p^{1/2} - \tfrac{1}{2}Zp^{-1/2}\right) \tan\phi + (s - Zp^{1/2}) \sec^2\phi \frac{\partial\phi}{\partial p} \tag{11.27}$$

$$-q' \sin(\theta - \bar{\omega}') \frac{\partial\theta}{\partial p} = \frac{\partial s}{\partial p} \tag{11.28}$$

$$q' \cos(\theta - \bar{\omega}') \frac{\partial\theta}{\partial p} = \left(\frac{\partial s}{\partial p} - \frac{\partial Z}{\partial p} p'^{1/2}\right) \tan\phi + (s - Zp'^{1/2}) \sec^2\phi \frac{\partial\phi}{\partial p} \tag{11.29}$$

By solving Eqs. (11.26) and (11.28) for $\partial s/\partial p$ and $\partial\theta/\partial p$ and employing Eqs. (11.21) and (11.23), it is easy to show that

$$\frac{\partial s}{\partial p} = \frac{s - Zp'^{1/2}}{Z(p^{1/2} - p'^{1/2})} \tag{11.30}$$

$$\frac{\partial\theta}{\partial p} = \frac{\cot\phi}{Z(p'^{1/2} - p^{1/2})} \tag{11.31}$$

By substituting for $\partial s/\partial p$ and $\partial\theta/\partial p$ in Eqs. (11.27) and (11.29), making use of Eqs. (11.20) and (11.22), and then eliminating $\partial Z/\partial p$ between these equations, it will be found that

$$sZ(p'^{1/2} - p^{1/2}) \tan \phi \sec^2 \phi \frac{\partial \phi}{\partial p}$$

$$= \left(s - Zp'^{1/2} + \frac{Z^2 p'^{1/2}}{2p^{1/2}}\right) \tan^2 \phi + s + (pp')^{1/2} \tag{11.32}$$

thus determining $\partial\phi/\partial p$.

Differentiating Eq. (11.24) with respect to p, we obtain

$$\frac{\partial X}{\partial p} = \frac{\partial s}{\partial p}(p'^{-1/2} - p^{-1/2}) \sec \phi + \tfrac{1}{2} s p^{-3/2} \sec \phi$$

$$+ s(p'^{-1/2} - p^{-1/2}) \sec \phi \tan \phi \frac{\partial \phi}{\partial p} \tag{11.33}$$

Upon substitution for $\partial s/\partial p$ and $\partial\phi/\partial p$ from Eqs. (11.30) and (11.32), respectively, this last result reduces to

$$\frac{\partial X}{\partial p} = \left\{\frac{s}{2p^{3/2}} \sec^2 \phi - \frac{Z}{2p} \tan^2 \phi - \frac{1}{p^{1/2}} - \frac{1}{Z}\right\} \cos \phi \tag{11.34}$$

By a very similar procedure, expressions for the partial derivatives of X with respect to q and $\bar\omega$ may be found in the form

$$\frac{\partial X}{\partial q} = -\frac{1}{p^{1/2} q} \left\{(s - p)\left(1 + \frac{p^{1/2}}{Z}\right) \cos \phi + (s - p^{1/2} Z) \tan \phi \sin \phi\right\} \tag{11.35}$$

$$\frac{\partial X}{\partial \bar\omega} = \left(Z - \frac{s}{Z}\right) \sin \phi \tag{11.36}$$

If, now, the two sets of variables $(p, q, \bar\omega)$ and $(p', q', \bar\omega')$ are interchanged in Eqs. (11.20)–(11.23), s, θ, Z, ϕ being unaltered, the same set of equations results. However, the sign of X as given by Eq. (11.24) is reversed. It follows that a repetition of the above argument will yield expressions for $\partial X/\partial p'$, $\partial X/\partial q'$, and $\partial X/\partial \bar\omega'$ which are identical with those for $\partial X/\partial p$, $\partial X/\partial q$, and $\partial X/\partial \bar\omega$, respectively, as given by Eqs. (11.34)–(11.36), with the exceptions that the quantities $(p, q, \bar\omega)$ will be replaced by the corresponding quantities $(p', q', \bar\omega')$ and the signs of the right-hand members will be reversed.

11.4 Optimal n-Impulse Transfer between Two Terminal Orbits

Suppose that a rocket is to be transferred between the coplanar orbits $(p_0, q_0, \bar\omega_0)$ and $(p_n, q_n, \bar\omega_n)$ by the application of a series of n impulsive thrusts. Let the first thrust effect transfer from the orbit $(p_0, q_0, \bar\omega_0)$ to the orbit $(p_1, q_1, \bar\omega_1)$ and be applied at the point having polar coordinates $(1/s_1, \theta_1)$, the second effect transfer from the orbit $(p_1, q_1, \bar\omega_1)$ to the orbit $(p_2, q_2, \bar\omega_2)$ and be applied at the point $(1/s_2, \theta_2)$, and so on, the nth impulse being applied at the point $(1/s_n, \theta_n)$ and finally transferring the rocket into the orbit $(p_n, q_n, \bar\omega_n)$. Let $\gamma^{1/2} X_i$ be the velocity increment due to the ith impulse so that, by Eq. (11.19),

$$X_i = s_i(p_i^{-1/2} - p_{i-1}^{-1/2}) \sec \phi_i \tag{11.37}$$

where ϕ_i is the angle determining the direction of the thrust. The characteristic velocity for the over-all maneuver is then given by the equation

$$W = \gamma^{1/2} \sum_{i=1}^{n} X_i \tag{11.38}$$

As explained in the last section, X_i is a function of the six orbital parameters $(p_{i-1}, q_{i-1}, \bar\omega_{i-1}, p_i, q_i, \bar\omega_i)$ and hence W is a function of the $3n - 3$ variables $(p_1, q_1, \bar\omega_1, \cdots, p_{n-1}, q_{n-1}, \bar\omega_{n-1})$, the quantities $(p_0, q_0, \bar\omega_0, p_n, q_n, \bar\omega_n)$ being given constants. W is stationary with respect to its variable arguments provided

$$\frac{\partial W}{\partial p_i} = \frac{\partial W}{\partial q_i} = \frac{\partial W}{\partial \bar\omega_i} = 0 \tag{11.39}$$

for $i = 1, 2, \cdots, (n - 1)$. The values of the parameters $(p_i, q_i, \bar\omega_i)$ $(i = 1, 2, \cdots, (n - 1))$ determining the optimal n-impulse mode of transfer must necessarily satisfy the Eqs. (11.39). Hence, by solving these equations and computing the value of W to which each solution leads, this optimal mode can, in principle, be determined.

Now

$$\frac{\partial W}{\partial p_i} = \gamma^{1/2}\left(\frac{\partial X_i}{\partial p_i} + \frac{\partial X_{i+1}}{\partial p_i}\right) \tag{11.40}$$

with similar expressions for $\partial W/\partial q_i$ and $\partial W/\bar\omega_i$. It follows from the results obtained in the last section that

$$\frac{\partial X_i}{\partial p_i} = -\left\{\frac{s_i}{2p_i^{3/2}} \sec^2 \phi_i - \frac{Z_i}{2p_i} \tan^2 \phi_i - \frac{1}{p_i^{1/2}} - \frac{1}{Z_i}\right\} \cos \phi_i$$

$$\frac{\partial X_{i+1}}{\partial p_i} = \left\{\frac{s_{i+1}}{2p_i^{3/2}} \sec^2 \phi_{i+1} - \frac{Z_{i+1}}{2p_i} \tan^2 \phi_{i+1} - \frac{1}{p_i^{1/2}} - \frac{1}{Z_{i+1}}\right\} \cos \phi_{i+1} \tag{11.41}$$

Z_i and Z_{i+1} being the Z-values appropriate to the ith and $(i+1)$th impulses, respectively. It follows that $\partial W/\partial p_i$ vanishes if

$$\left\{\frac{s_i}{2p_i^{3/2}}\sec^2\phi_i - \frac{Z_i}{2p_i}\tan^2\phi_i - \frac{1}{p_i^{1/2}} - \frac{1}{Z_i}\right\}\cos\phi_i$$

$$= \left\{\frac{s_{i+1}}{2p_i^{3/2}}\sec^2\phi_{i+1} - \frac{Z_{i+1}}{2p_i}\tan^2\phi_{i+1} - \frac{1}{p_i^{1/2}} - \frac{1}{Z_{i+1}}\right\}\cos\phi_{i+1} \qquad (11.42)$$

Similarly, it may be proved that $\partial W/\partial q_i$ and $\partial W/\partial\bar{\omega}_i$ vanish if

$$(s_i - p_i)\left(1 + \frac{p_i^{1/2}}{Z_i}\right)\cos\phi_i + (s_i - p_i^{1/2}Z_i)\tan\phi_i\sin\phi_i$$

$$(11.43)$$

$$= (s_{i+1} - p_i)\left(1 + \frac{p_i^{1/2}}{Z_{i+1}}\right)\cos\phi_{i+1} + (s_{i+1} - p_i^{1/2}Z_{i+1})\tan\phi_{i+1}\sin\phi_{i+1}$$

$$\left(Z_i - \frac{s_i}{Z_i}\right)\sin\phi_i = \left(Z_{i+1} - \frac{s_{i+1}}{Z_{i+1}}\right)\sin\phi_{i+1} \qquad (11.44)$$

Equations (11.20)–(11.23) determine the values of the quantities $(s_i, \theta_i, Z_i, \phi_i)$ appropriate to the ith impulse and take the form

$$q_{i-1}\cos(\theta_i - \bar{\omega}_{i-1}) = s_i - p_{i-1} \qquad (11.45)$$

$$q_{i-1}\sin(\theta_i - \bar{\omega}_{i-1}) = (s_i - Z_i p_{i-1}^{1/2})\tan\phi_i \qquad (11.46)$$

$$q_i\cos(\theta_i - \bar{\omega}_i) = s_i - p_i \qquad (11.47)$$

$$q_i\sin(\theta_i - \bar{\omega}_i) = (s_i - Z_i p_i^{1/2})\tan\phi_i \qquad (11.48)$$

With $i = 1, 2, \cdots, (n-1)$, Eqs. (11.42)–(11.48) are $7n - 7$ in number and determine the $7n - 7$ unknowns $p_i, q_i, \bar{\omega}_i, s_i, \theta_i, Z_i, \phi_i$ $(i = 1, 2, \cdots, (n-1))$ for an optimal n-impulse mode transfer between the given orbits.

Equation (11.42) can be replaced by a simpler, but equivalent, condition as follows: Multiply Eq. (11.42) through by $2p_i^{3/2}$ and subtract from Eq. (11.43); the result will be found to be

$$p_i\left(1 + \frac{s_i + p_i}{Z_i p_i^{1/2}}\right)\cos\phi_i = p_i\left(1 + \frac{s_{i+1} + p_i}{Z_{i+1}p_i^{1/2}}\right)\cos\phi_{i+1}$$

or

$$\left(1 + \frac{s_i + p_i}{Z_i p_i^{1/2}}\right)\cos\phi_i = \left(1 + \frac{s_{i+1} + p_i}{Z_{i+1}p_i^{1/2}}\right)\cos\phi_{i+1} \qquad (11.49)$$

11.5 Optimal Two-Impulse Transfer

Transfer between two coplanar orbits by means of a single impulsive thrust is only possible in the case when the orbits intersect and in this case there is no optimization problem since the impulse necessary at a point of intersection is completely fixed by the two given terminal orbits.

The next simplest mode is that of two-impulse transfer and this is always possible between any two terminal orbits. It will be convenient to denote the orbital parameters for the terminals by $(p_1, q_1, \bar{\omega}_1)$ and $(p_2, q_2, \bar{\omega}_2)$ and the parameters for the single variable transfer orbit by $(p, q, \bar{\omega})$. Equations determining the optimal mode can then be written down as explained in the previous section. These take the form

$$q_1\cos(\theta_1 - \bar{\omega}_1) = s_1 - p_1 \qquad (11.50)$$

$$q_1\sin(\theta_1 - \bar{\omega}_1) = (s_1 - Z_1 p_1^{1/2})\tan\phi_1 \qquad (11.51)$$

$$q\cos(\theta_1 - \bar{\omega}) = s_1 - p \qquad (11.52)$$

$$q\sin(\theta_1 - \bar{\omega}) = (s_1 - Z_1 p^{1/2})\tan\phi_1 \qquad (11.53)$$

$$q\cos(\theta_2 - \bar{\omega}) = s_2 - p \qquad (11.54)$$

$$q\sin(\theta_2 - \bar{\omega}) = (s_2 - Z_2 p^{1/2})\tan\phi_2 \qquad (11.55)$$

$$q_2\cos(\theta_2 - \bar{\omega}_2) = s_2 - p_2 \qquad (11.56)$$

$$q_2\sin(\theta_2 - \bar{\omega}_2) = (s_2 - Z_2 p_2^{1/2})\tan\phi_2 \qquad (11.57)$$

$$(s_1 - p)\left(1 + \frac{p^{1/2}}{Z_1}\right)\cos\phi_1 + (s_1 - Z_1 p^{1/2})\sin\phi_1\tan\phi_1$$

$$= (s_2 - p)\left(1 + \frac{p^{1/2}}{Z_2}\right)\cos\phi_2 + (s_2 - Z_2 p^{1/2})\sin\phi_2\tan\phi_2, \qquad (11.58)$$

$$\left(Z_1 - \frac{s_1}{Z_1}\right)\sin\phi_1 = \left(Z_2 - \frac{s_2}{Z_2}\right)\sin\phi_2 \qquad (11.59)$$

$$\left(1 + \frac{s_1 + p}{Z_1 p^{1/2}}\right)\cos\phi_1 = \left(1 + \frac{s_2 + p}{Z_2 p^{1/2}}\right)\cos\phi_2 \qquad (11.60)$$

where $s_1, \theta_1, Z_1, \phi_1$ refer to the first impulse and $s_2, \theta_2, Z_2, \phi_2$ to the second. These are 11 equations for the 11 unknowns $p, q, \bar{\omega}, s_1, \theta_1, Z_1, \phi_1, s_2, \theta_2, Z_2, \phi_2$.

The solution to Eqs. (11.50)–(11.60) is not known for the general case, but it has been obtained in a number of special cases and these we proceed to consider in the following sections.

11.6 Optimal Slewing of the Orbital Axis

Suppose that it is desired to rotate the major axis of a body's orbit through an angle 2α in the orbital plane, the shape and size of the orbit to remain unaffected. This is the problem of transferring a body from an orbit $(P, Q, -\alpha)$ to an orbit (P, Q, α). These orbits intersect, so that it is possible for the transfer to be effected by the application of a single impulse. However, this mode of transfer proves invariably to be uneconomical by comparison with the optimal two-impulse mode and we shall therefore concentrate our attention upon the latter.

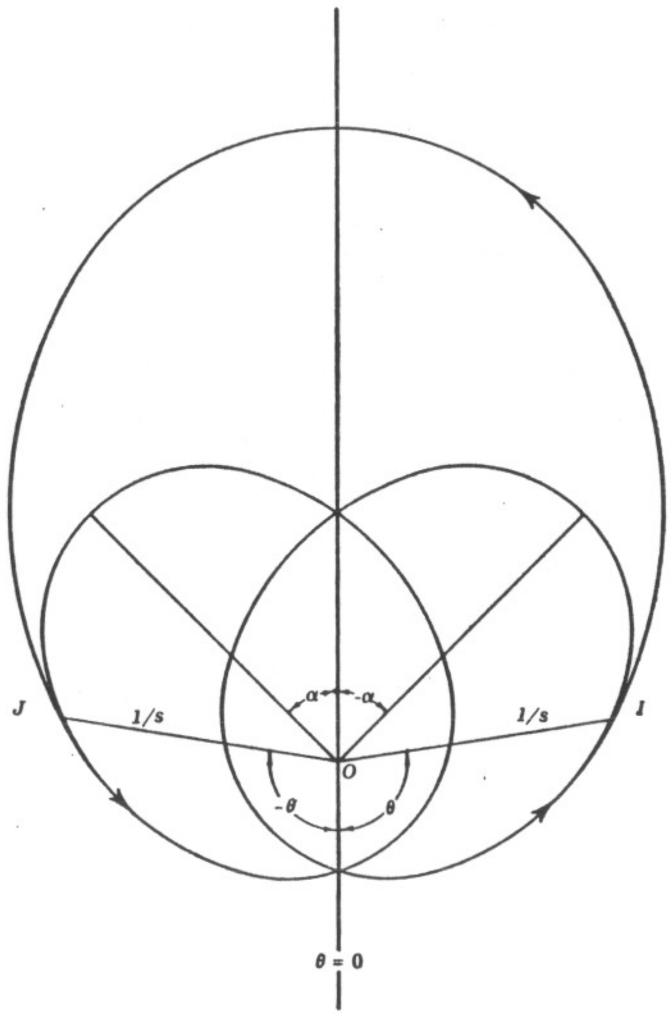

FIG. 3. Transfer between equal ellipses.

In view of the symmetry which exists about the line $\theta = 0$, it will be assumed that the intermediate ellipse of transfer has parameters $(p, q, 0)$ or (p, q, π), i.e., that its major axis lies along the axis of symmetry. The transfer ellipse will then intersect the terminal ellipses in two junction points I, J (Fig. 3) having polar coordinates $(1/s, \theta)$, $(1/s, -\theta)$, respectively. By further consideration of the symmetry, it will be evident that the values of Z at the junction points as given by Eq. (11.8) are identical and that, if ϕ determines the direction of the thrust at the first junction point, $\pi - \phi$ determines the direction at the second junction point. Equations (11.50)–(11.53) now become identical with Eqs. (11.54)–(11.57), condition (11.59) is automatically satisfied, and hence there remain the equations

$$Q \cos(\theta + \alpha) = s - P \tag{11.61}$$

$$Q \sin(\theta + \alpha) = (s - ZP^{1/2}) \tan \phi \tag{11.62}$$

$$q \cos \theta = s - p \tag{11.63}$$

$$q \sin \theta = (s - Zp^{1/2}) \tan \phi \tag{11.64}$$

$$(s - p)\left(1 + \frac{p^{1/2}}{Z}\right) \cos \phi + (s - Zp^{1/2}) \sin \phi \tan \phi = 0 \tag{11.65}$$

$$\left(1 + \frac{s + p}{Zp^{1/2}}\right) \cos \phi = 0 \tag{11.66}$$

When deriving these equations it has been assumed that the transfer ellipse has parameters $(p, q, 0)$. Since the transfer ellipse (p, q, π) is identical with the ellipse $(p, -q, 0)$, both possibilities are allowed for by Eqs. (11.61)–(11.66), provided negative values of q are permitted. These six equations then determine the six unknowns p, q, s, θ, Z, ϕ.

Rejecting the possiblity $\cos \phi = 0$, since this would require $\tan \phi$ to be infinite, Eq. (11.66) requires that

$$\frac{Z}{p^{1/2}} = -\frac{(x + 1)}{x} \tag{11.67}$$

where

$$x = \frac{p}{s} = \frac{r}{l} \quad \text{(by Eq. (11.2))} \tag{11.68}$$

Substituting for $Z/p^{1/2}$ in Eq. (11.65), it will be found that

$$\tan^2 \phi = \frac{x-1}{(x+1)(x+2)} \tag{11.69}$$

and, clearly therefore, $x \geq 1$. It will be found that $\tan^2 \phi$ is a maximum for $x = 1 + \sqrt{6} = 3{\cdot}449 \cdots$. Thus, when ϕ is positive acute, $\phi_{\max} = 17°38'$. Equations (11.63) and (11.64) can be written in the form

$$e \cos \theta = \frac{1}{x} - 1 \tag{11.70}$$

$$e \sin \theta = \frac{x+2}{x} \tan \phi \tag{11.71}$$

where $e = q/p$ is the eccentricity of the transfer ellipse. Dividing Eq. (11.71) by Eq. (11.70), we also obtain

$$\tan \theta = -\frac{x+2}{x-1} \tan \phi \tag{11.72}$$

Similarly, Eqs. (11.61) and (11.62) are equivalent to the following equations:

$$E \cos (\theta + \alpha) = \frac{y^2}{x} - 1 \tag{11.73}$$

$$E \sin (\theta + \alpha) = \frac{y(x+y+1)}{x} \tan \phi \tag{11.74}$$

where $E = Q/P$ is the eccentricity of the terminal ellipses and

$$y = \sqrt{\frac{p}{P}} \tag{11.75}$$

Division of Eq. (11.74) by Eq. (11.73) yields

$$\tan (\theta + \alpha) = \frac{y(x+y+1)}{y^2 - x} \tan \phi \tag{11.76}$$

Given E and α, Eqs. (11.69)–(11.74) determine x, y, e, θ, ϕ. Equation (11.19) then gives the characteristic velocity for the maneuver to be $\gamma^{1/2}X$, where

$$P^{-1/2}X = \frac{2}{x}(1-y)y \sec \phi \tag{11.77}$$

The form taken by Eqs. (11.69)–(11.74) implies that if, when

$$E = E_0, \qquad \alpha = \alpha_0 \tag{11.78}$$

these equations are satisfied by

$$x = x_0, \quad y = y_0, \quad e = e_0, \quad \theta = \theta_0, \quad \phi = \phi_0 \tag{11.79}$$

then, when

$$E = E_0, \qquad \alpha = \pi - \alpha_0 \tag{11.80}$$

the equations are satisfied by

$$x = x_0, \quad y = y_0, \quad e = -e_0, \quad \theta = \pi - \theta_0, \quad \phi = 2\pi - \phi_0 \tag{11.81}$$

Also, when

$$E = E_0, \qquad \alpha = \pi + \alpha_0 \tag{11.82}$$

the equations are satisfied by

$$x = x_0, \quad y = y_0, \quad e = -e_0, \quad \theta = \pi + \theta_0, \quad \phi = \phi_0 \tag{11.83}$$

and, when

$$E = E_0, \qquad \alpha = 2\pi - \alpha_0 \tag{11.84}$$

the equations are satisfied by

$$x = x_0, \quad y = y_0, \quad e = e_0, \quad \theta = 2\pi - \theta_0, \quad \phi = 2\pi - \phi_0 \tag{11.85}$$

Also, if the solution (11.79) leads to a positive value for X from Eq. (11.77) (as is necessary), each of the solutions (11.81), (11.83), and (11.85) will also yield a positive value of the same magnitude. It follows that any solution to a problem of the type we are considering immediately provides solutions to three related problems. The four cases (11.78), (11.80), (11.82), and (11.84), together with the appropriate transfer orbits, are shown in Figs. 4(i)–(iv), respectively. It will be observed that cases (i) and (iii) differ only in respect to orientation and are essentially identical, as also are cases (ii) and (iv). Further, cases (i) and (iv) differ only in that the orbit of departure in the first case is the orbit of arrival in the second and complementary major and minor arcs of a single ellipse are employed to effect transfer; cases (ii) and (iii) are similarly related. It follows from these considerations that it is only necessary for us to consider the possibility that α is a positive acute angle, for if α lies outside this range the solution can be derived from the corresponding acute angle configuration.

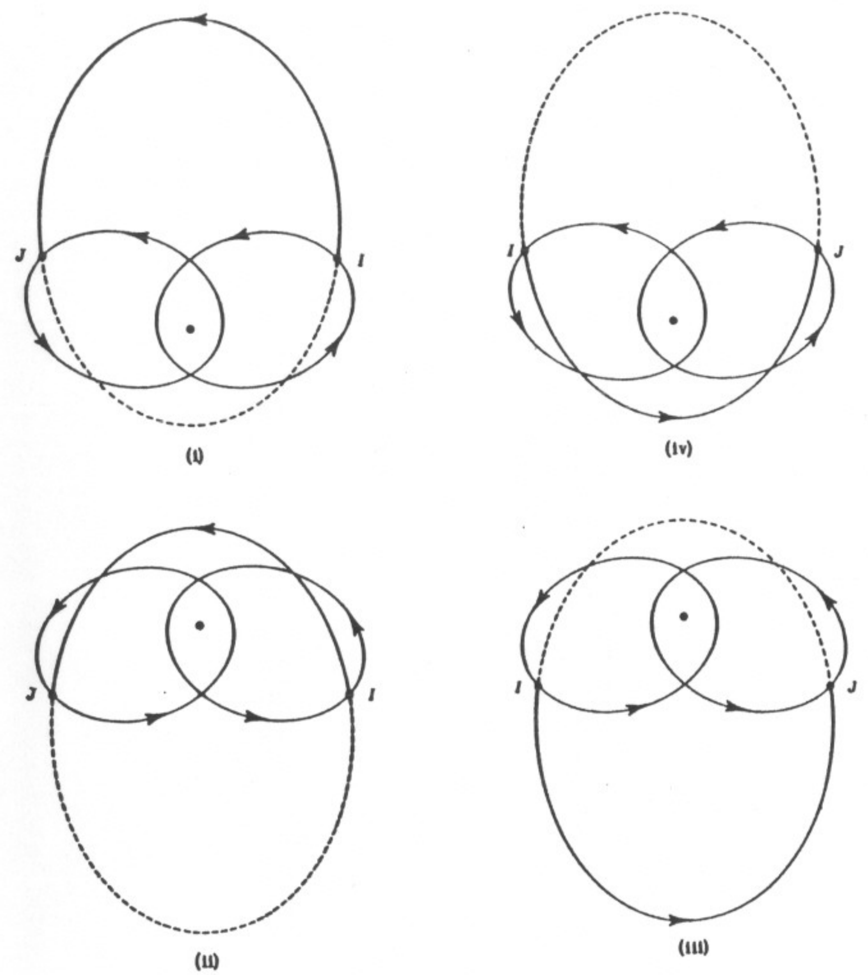

FIG. 4. Similar transfers between equal ellipses.

Assuming, then, that $0 \leq \alpha \leq \frac{1}{2}\pi$, Eqs. (11.69)–(11.74) are solved most easily by an indirect procedure as follows: Assume a pair of values for the quantities x, y and derive the corresponding value of ϕ from Eq. (11.69). This must be chosen to be such that the value of X given by Eq. (11.77) is positive; this condition will restrict ϕ to lie in one of two quadrants but will leave the sign of $\tan\phi$ undetermined. For each sign of $\tan\phi$, Eq. (11.72) now yields a value of $\tan\theta$ and hence two distinct values of θ differing by π. Of the four alternative values of θ, only two can yield a positive acute value of α when substituted in Eq. (11.76), and of these

two only one will yield a positive value for E from either of the equations (11.73) or (11.74). At this stage all ambiguities can be resolved and a solution of the type being sought can be written down.

This procedure has been followed through for a large number of pairs of values of x and y and from these data the families of curves x = constant, y = constant have been drawn in the αE-plane (Fig. 5). It was found that, in most cases, to each pair of values of α, E, there corresponded two pairs of values of x, y; in one pair y was less than unity and in the other greater.

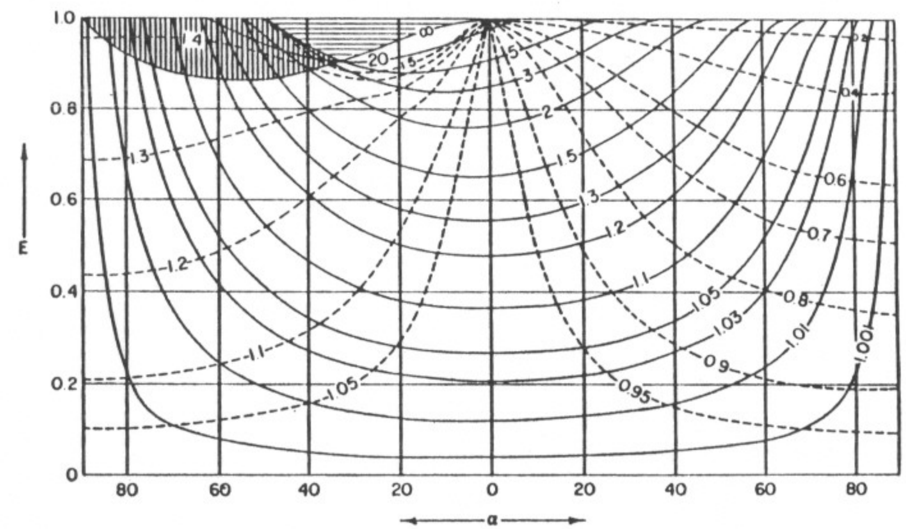

FIG. 5. ———, curves x = constant; - - - - -, curves y = constant.

These two types of solution have been separated in Fig. 5 by employing an α-axis proceeding to the right in the case of solutions for which $y < 1$ and an α-axis proceeding to the left in the case of solutions for which $y > 1$. This arrangement is made very convenient by the facts that the E-axis is also the curve $y = 1$ and that the curves x = constant are continuous across this axis. The curve $x = 1$ comprises the two vertical lines $\alpha = 90°$ and the horizontal line $E = 0$. The curve $y = 0$ comprises that portion of the line $E = 1$ which lies in the right-hand quadrant. The constant values of x and y appropriate to the remaining curves have been marked on each. It will be observed that, in the left-hand quadrant, the curves $x = x_0$ with $x_0 > 3$, intersect some of the curves $x = x_0$ with $x_0 < 3$. This implies that, in the region bounded by the curves $x = 3$ (approximately), $x = \infty$ and $E = 1$, and indicated by vertical shading, each point (α, E) lies on two distinct curves x = constant and hence that, in such

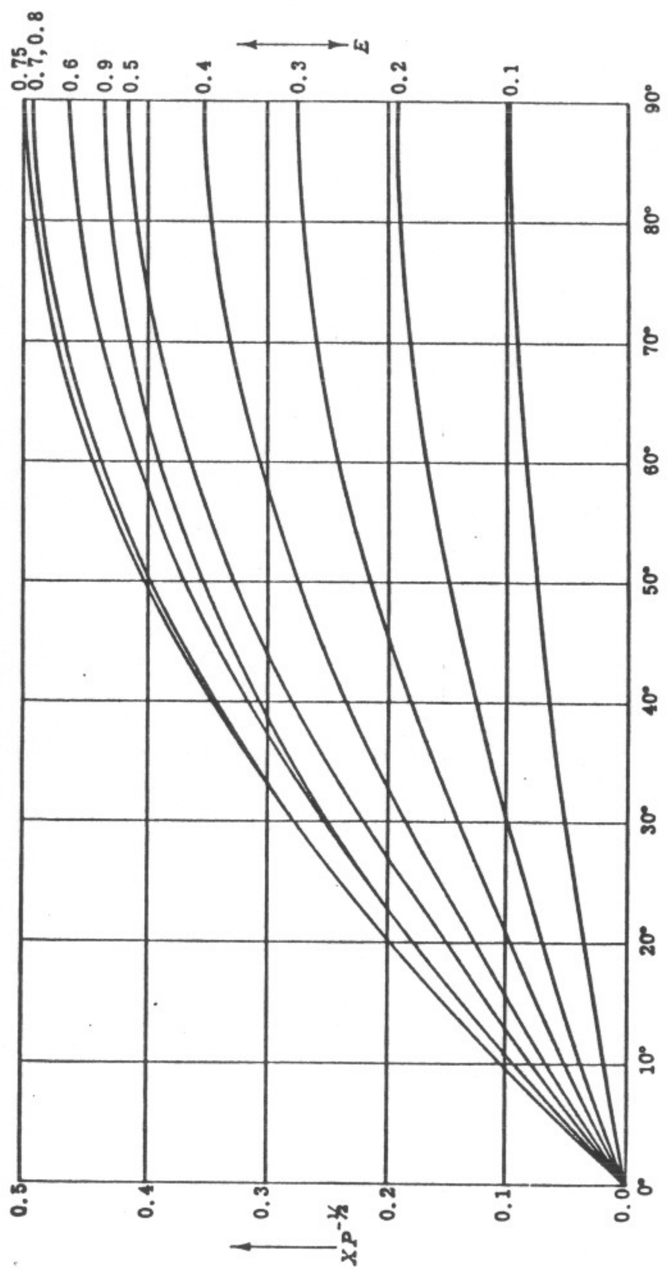

FIG. 6. Characteristic velocity graphs.

cases, two solutions are available from the left-hand quadrant, making a total of three possible solutions altogether. By contrast, the region bounded by the same curves and indicated by horizontal shading is traversed by no curves x = constant and it follows that for values of α, E corresponding to points in this region, only one solution is possible; this is provided by the curves in the right-hand quadrant.

To calculate an optimal transfer with the aid of these curves, it is necessary to identify the two points having coordinates (α, E) and then to estimate which curves of the two families pass through these points, thus yielding appropriate values for x and y. These can be refined in accuracy by substitution into Eqs. (11.69)–(11.74) and subsequent adjustment and the values of e, θ, ϕ, X determined. The solution providing the least value for X is, of course, the absolute optimal solution. Calculation suggests that this absolute optimal solution will invariably be found in the right-hand quadrant of Fig. 5.

For each value of E in the range $0{\cdot}1(0{\cdot}1)0{\cdot}9$, values of $XP^{-1/2}$ were calculated for a number of values of α and from the results the family of curves shown in Fig. 6 were constructed. These indicate that, for a given elliptical orbit, the propellant expenditure necessary to bring about a rotation of its axis increases with the angle of rotation 2α, reaching a maximum when $\alpha = 90°$. For a given value of α, the propellant expenditure is greatest when $E = \frac{3}{4}$. The curves $E = 0{\cdot}7$ and $0{\cdot}8$ are indistinguishable at the scale to which Fig. 6 has been drawn. It was found that the formula

$$XP^{-1/2} = 2{\cdot}469(1 - E)^{1/2}\{1 - (1 - E)^{1/2}\}\alpha(180 - \alpha) \times 10^{-4} \quad (11.86)$$

where α is to be measured in degrees, is in agreement with data obtained from Fig. 6 to an accuracy of within 10% and often to an accuracy of better than 2%.

Further numerical data relating to the problem of this section will be found in the paper by Plimmer.[10]

11.7 Transfer between Orbits Whose Axes Are Aligned

Another special case in which a solution to Eqs. (11.50)–(11.60) can be found without difficulty is that where the axes of the terminal orbits $(p_1, q_1, \tilde{\omega}_1)$ and $(p_2, q_2, \tilde{\omega}_2)$ are aligned. This case has been studied by Smith[9] and Plimmer.[10] As has already been remarked, the orbits $(p, q, \tilde{\omega} + \pi)$ and $(p, -q, \tilde{\omega})$ are identical and hence, provided negative q-values are permitted, if the axes of the terminal orbits are aligned, we may take $\tilde{\omega}_1 = \tilde{\omega}_2$. Then, if q_1 and q_2 are of like sign, the orbital axes will be in the same sense, whereas, if the signs of q_1 and q_2 are opposite, the senses of the axes will also be opposite. It will then be convenient to choose the direction of the reference line $\theta = 0$ so that $\bar{\omega}_1 = \bar{\omega}_2 = 0$.

It has been shown by Plimmer[10] that if one of the terminal orbits is a circle, so that one of q_1 or q_2 vanishes, Eqs. (11.50)–(11.60) are easily

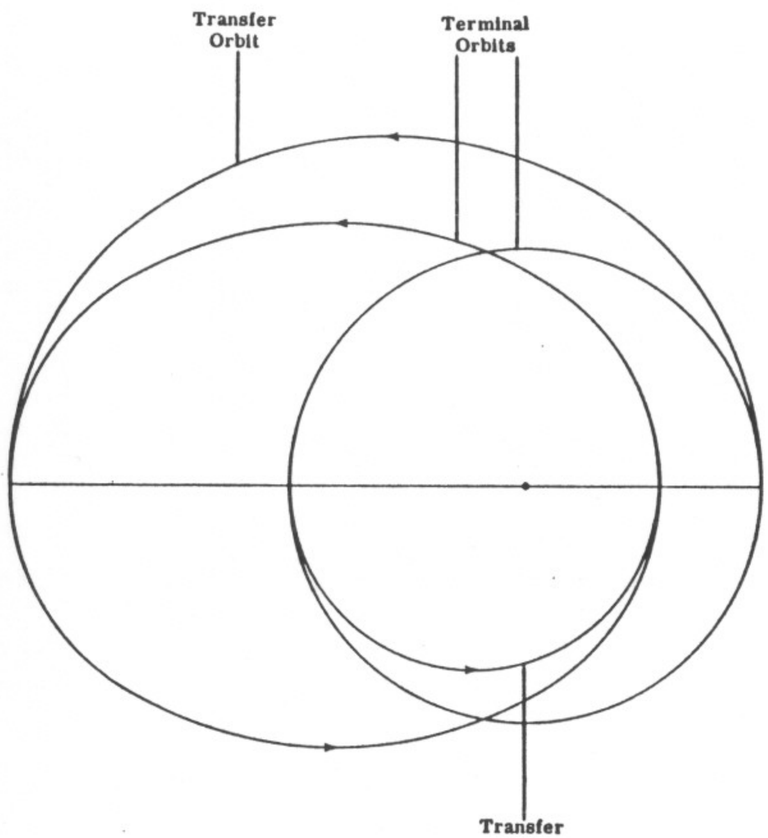

FIG. 7. Transfer between circle and ellipse.

solved. The optimal transfer ellipse is then tangential at its two apses to the terminal orbits and its axis is aligned with that of the elliptical terminal. There are always two possible such transfer orbits as indicated in Fig. 7, but, in both cases, the impulsive thrusts are applied at its apses and are in a direction perpendicular to the radius from the center of attraction,

i.e., ϕ_1, ϕ_2 take the values 0 or π. In case both terminals are ellipses, Eqs. (11.50)–(11.60) have not yet been solved directly. However, it is clear that, in these circumstances also, it is possible to find two transfer ellipses which are tangential at both their apses to the terminals. The previous result then renders it probable that optimal transfer will be effected along one of these ellipses by the application of two impulses in a direction perpendicular to the radius from the center of attraction. We shall accordingly seek solutions of Eqs. (11.50)–(11.60) for which $\phi_1, \phi_2 = 0$ or π. It is conjectured that, in the conditions being considered in this section, any other type of solution (if such exists) will lead to a higher characteristic velocity.

With $\bar{\omega}_1 = \bar{\omega}_2 = 0$ and $\phi_1, \phi_2 = 0$ or π, Eqs. (11.50)–(11.60) reduce to the set

$$\begin{aligned} q_1 \cos\theta_1 &= s_1 - p_1, & q_1 \sin\theta_1 &= 0 \\ q\cos(\theta_1 - \bar{\omega}) &= s_1 - p, & q\sin(\theta_1 - \bar{\omega}) &= 0 \\ q\cos(\theta_2 - \bar{\omega}) &= s_2 - p, & q\sin(\theta_2 - \bar{\omega}) &= 0 \\ q_2\cos\theta_2 &= s_2 - p_2, & q_2\sin\theta_2 &= 0 \end{aligned} \tag{11.87}$$

$$\frac{s_1 + p}{Z_1 p^{1/2}} + 1 = \pm\left(\frac{s_2 + p}{Z_2 p^{1/2}} + 1\right) \tag{11.88}$$

$$\left(1 + \frac{p^{1/2}}{Z_1}\right)(s_1 - p) = \pm\left(1 + \frac{p^{1/2}}{Z_2}\right)(s_2 - p) \tag{11.89}$$

where the positive sign is to be taken in Eqs. (11.88) and (11.89) if $\phi_1 = \phi_2$ and the negative sign otherwise. It follows from the right-hand set of equations (11.87) that θ_1, θ_2, $\bar{\omega}$ are all integral multiples of π. If negative values of q are permitted, there is no loss of generality in taking $\bar{\omega} = 0$. If $\theta_1 = \theta_2 = 0$ (or π), the transfer ellipse has degenerated into a straight line which is coincident with the axes of the terminals. To effect transfer into this orbit by an impulse for which $\phi_1 = 0$ or π, it would be necessary to bring the rocket to rest; the rocket would then fall freely towards the center of attraction and, provided its path intersected the second terminal, a second impulse could be employed to establish it in the required orbit. However, this impulse would not be in a direction perpendicular to the radius from the center of attraction. It follows that $\theta_1 \neq \theta_2$ and there are only two cases to consider, viz., *case I*, $\theta_1 = 0$, $\theta_2 = \pi$ and *case II*, $\theta_1 = \pi$,

$\theta_2 = 0$. Substituting for $\theta_1, \theta_2, \bar{\omega}$, in the left-hand set of equations (11.87), it will be found that:

$$\text{Case I.}\quad s_1 = p_1 + q_1, \quad s_2 = p_2 - q_2$$

$$p = \tfrac{1}{2}(p_1 + p_2 + q_1 - q_2)$$

$$q = \tfrac{1}{2}(p_1 - p_2 + q_1 + q_2)$$

$$\text{Case II.}\quad s_1 = p_1 - q_1, \quad s_2 = p_2 + q_2$$

$$p = \tfrac{1}{2}(p_1 + p_2 - q_1 + q_2)$$

$$q = \tfrac{1}{2}(-p_1 + p_2 + q_1 + q_2)$$

Having satisfied Eqs. (11.87), the remaining conditions (11.88) and (11.89) can be satisfied by appropriate choice of Z_1, Z_2 and serve, in fact, to determine these quantities; their values are of no interest.

Equation (11.19) now specifies the characteristic velocity for the transfer maneuver. Denoting this by $\gamma^{1/2}X$ and writing

$$a = p_1 - q_1, \qquad b = p_2 - q_2$$
$$a' = p_1 + q_1, \qquad b' = p_2 + q_2 \tag{11.90}$$

it will be found that in case I

$$X = X_1 = \sqrt{2}a'\left|\frac{1}{(a+a')^{1/2}} - \frac{1}{(a'+b)^{1/2}}\right| + \sqrt{2}b\left|\frac{1}{(b+b')^{1/2}} - \frac{1}{(a'+b)^{1/2}}\right| \tag{11.91}$$

and in case II

$$X = X_2 = \sqrt{2}a\left|\frac{1}{(a+a')^{1/2}} - \frac{1}{(a+b')^{1/2}}\right| + \sqrt{2}b'\left|\frac{1}{(b+b')^{1/2}} - \frac{1}{(a+b')^{1/2}}\right| \tag{11.92}$$

a, b, a', b' are the reciprocals of the distances from the center of attraction of the four apses on the terminals.

Suppose, first, that the axes of the terminals are directed in the same sense. Then there are two possibilities: (a) the orbits intersect or (b) the orbits do not intersect. These alternative possibilities are illustrated in Fig. 8. The initial orbit will be taken to be that marked "A" and the final

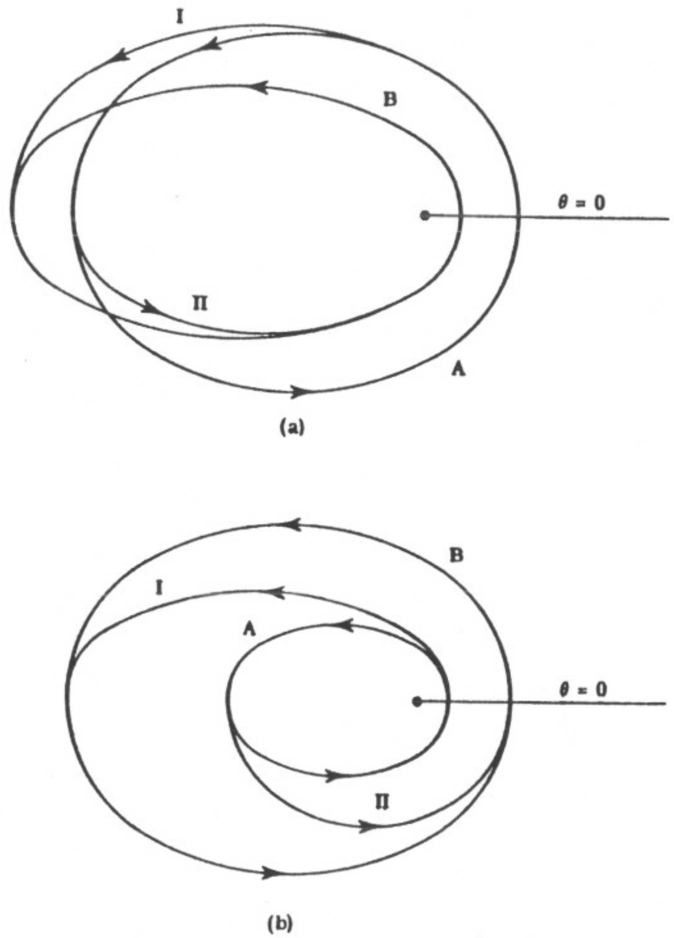

FIG. 8. Ellipses with axes in identical senses.

orbit to be that marked "B"; reversal of the roles of the two orbits or of the senses of both their axes are trivial modifications which do not lead to any essentially distinct possibilities. The transfer orbits corresponding to the cases I and II solutions are also indicated in the figure.

In Fig. 8a, it is clear that $a > b$, $a' < b'$, $a < a'$, $b < b'$. Employing these inequalities to resolve the ambiguous senses of the differences in Eqs. (11.91) and (11.92), it will be found upon subtraction that

$$\frac{1}{\sqrt{2}}(X_1 - X_2) = (a' + b)^{1/2} - (a' + a)^{1/2} - (b' + b)^{1/2} + (b' + a)^{1/2}$$

$$= f(a', a, b) - f(b', a, b) \tag{11.93}$$

where

$$f(x, a, b) = (x + b)^{1/2} - (x + a)^{1/2} \tag{11.94}$$

It is shown in the Appendix at the end of this chapter that, when $a > b$, f is an increasing function of x. It accordingly follows from Eq. (11.93) that, since $a' < b'$, then $X_1 - X_2 < 0$. Thus $X_1 < X_2$ and in Fig. 8a the transfer orbit I is the more economical.

In Fig. 8b the following inequalities are valid: $a > b$, $a' > b'$, $a < a'$, $b < b'$. From Eqs. (11.91) and (11.92), it now follows that

$$\frac{1}{\sqrt{2}}(X_1 - X_2) = \frac{a' - b}{(a' + b)^{1/2}} - \frac{a' - a}{(a' + a)^{1/2}} - \frac{b' - b}{(b' + b)^{1/2}} + \frac{b' - a}{(b' + a)^{1/2}}$$

$$= g(a', a, b) - g(b', a, b) \tag{11.95}$$

where

$$g(x, a, b) = \frac{x - b}{(x + b)^{1/2}} - \frac{x - a}{(x + a)^{1/2}} \tag{11.96}$$

It is also shown in the Appendix that, when $a > b$, the graph of $g(x, a, b)$ for $x \geq b$ has the form indicated in Fig. 9, i.e., g first increases monotonically as x increases to the value ξ and then decreases monotonically so that $g \to 0$ as $x \to \infty$. It is also clear that $g(a, a, b) = g(b, a, b) = G$. It now follows from Fig. 9 that, since $a' > a$ and $a' > b'$, therefore

$$g(a', a, b) < g(b', a, b) \tag{11.97}$$

Thus, from Eq. (11.95), we deduce that $X_1 - X_2 < 0$ and transfer orbit I is again the more economical.

Now suppose that the axes of the terminal orbits are directed in opposite senses. We shall consider the intersecting and nonintersecting cases shown in Figs. 10a and 10b, respectively. No other possibilities are essentially distinct from these two.

In Fig. 10a we have $a > b$, $a' < b'$, $a > a'$, $b < b'$ and Eq. (11.93) follows as before. The inequality $X_1 < X_2$ may now be deduced as already

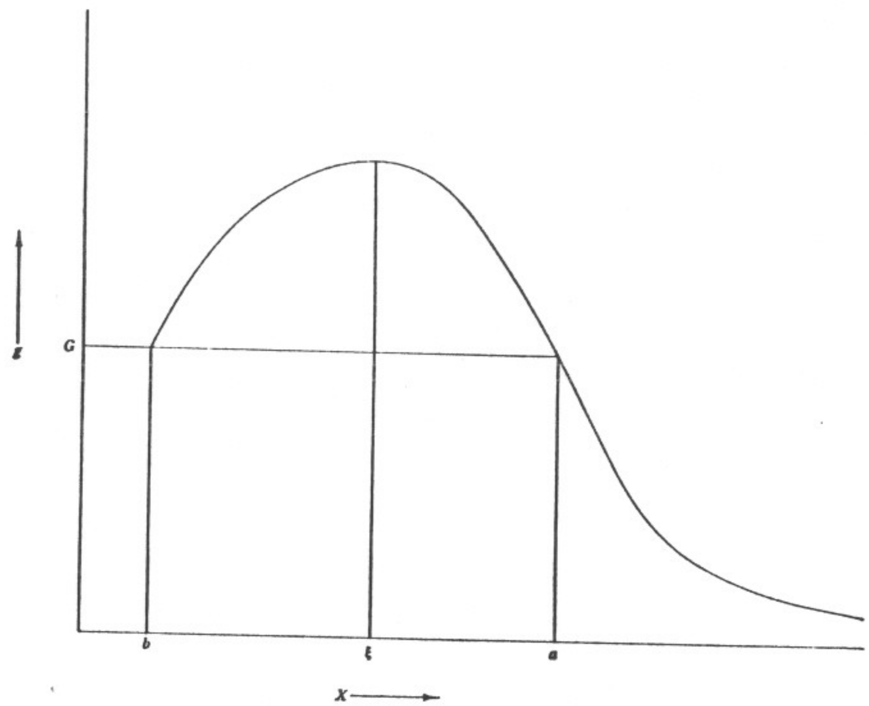

FIG. 9. Graph of $g(x, a, b)$.

explained. Again, therefore, the transfer orbit I has been proved to be the over-all optimal.

In Fig. 10b we have $a > b$, $a' > b'$, $a > a'$, $b < b'$ and Eq. (11.95) is accordingly valid. However, in this instance, a' and b' both lie in the interval (a, b) and it follows from Fig. 9 that either of the inequalities $X_1 < X_2$, $X_1 > X_2$ may be valid depending upon the values of the quantities a' and b'. In this case there appears to be no simple criterion by which the smaller of X_1, X_2 may be selected and each individual case must accordingly be decided by substitution of numerical values in Eq. (11.95).

The findings of this section may be summarized thus: *If two orbits have their axes aligned and the orbits either intersect or have their axes directed in the same sense, then the over-all optimal transfer orbit is that which is tangential to both terminals at an apse on each and which passes through the apse most distant from the center of attraction. In cases where the terminals have their axes directed in opposite senses and are nonintersecting, the optimal transfer orbit may be either of the two ellipses tangential to the terminals at apses depending upon the relative dimensions of the terminals.*

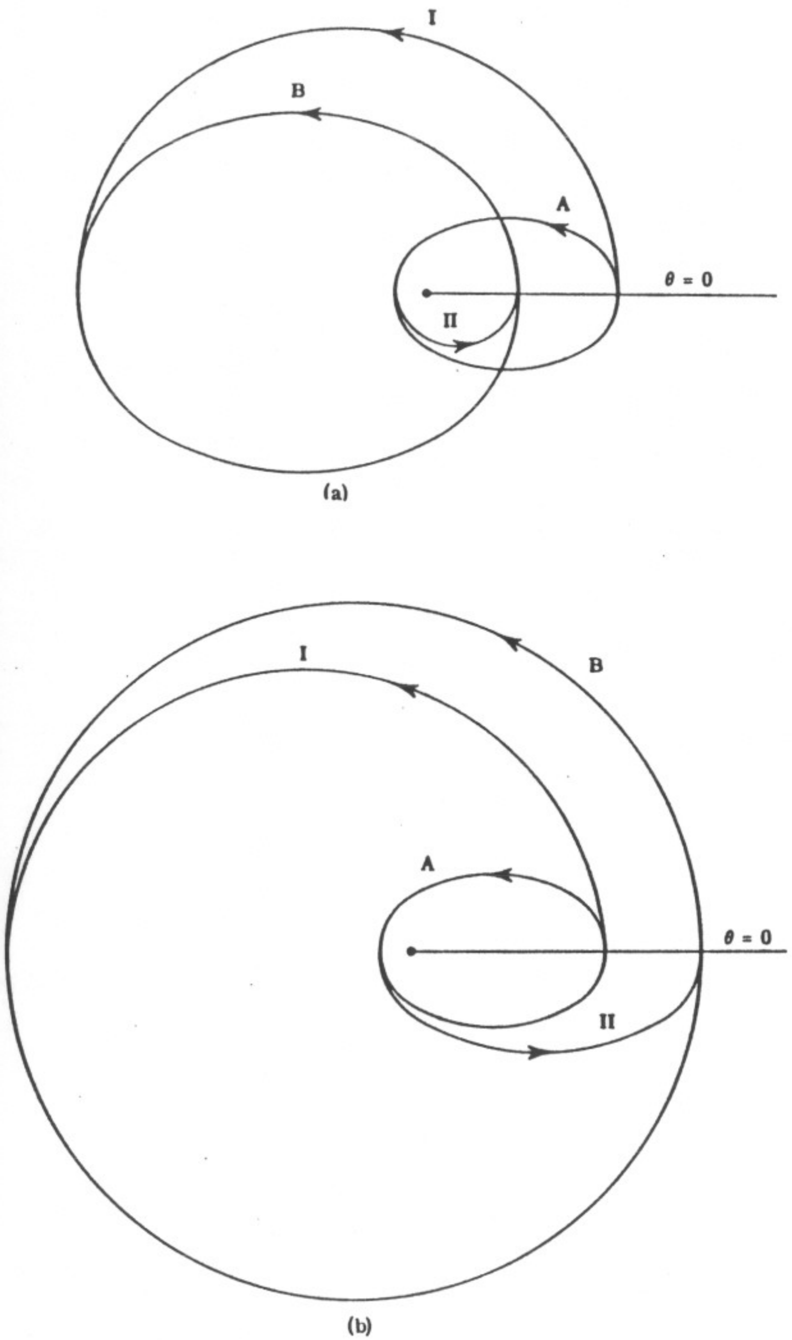

FIG. 10. Ellipses with axes in opposite senses.

11.8 Appendix

11.81 The Function $f(x, a, b)$

This function is defined by the equation

$$f(x, a, b) = (x + b)^{1/2} - (x + a)^{1/2} \tag{11.98}$$

Differentiating with respect to x, it will be found that

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x + b)^{-1/2} - \tfrac{1}{2}(x + a)^{-1/2} \tag{11.99}$$

Thus, if $a > b$ and x is positive, then $\partial f/\partial x > 0$ and hence f is an increasing function of x.

11.82 The Function $g(x, a, b)$

This function is defined by the equation

$$g(x, a, b) = \frac{x - b}{(x + b)^{1/2}} - \frac{x - a}{(x + a)^{1/2}} \tag{11.100}$$

We shall suppose that a, b are given fixed quantities such that $a > b > 0$ and shall abbreviate $g(x, a, b)$ to $g(x)$.

Then $g(a) = g(b)$ and it therefore follows from Rolle's theorem that a number ξ can be found such that $b < \xi < a$ and $g'(\xi) = 0$.

Now

$$g'(x) = \frac{x + 3b}{2(x + b)^{3/2}} - \frac{x + 3a}{2(x + a)^{3/2}} \tag{11.101}$$

Writing $h(x) = x^{1/2}g'(x)$ and differentiating then yields the following result:

$$h'(x) = \frac{3b(b - x)}{4x^{1/2}(x + b)^{5/2}} + \frac{3a(x - a)}{4x^{1/2}(x + a)^{5/2}} \tag{11.102}$$

Hence, provided $b \le x \le a$, then $h'(x) < 0$, i.e., $h(x)$ is decreasing. But $h(\xi) = 0$ and consequently $h(x) > 0$, provided $b \le x < \xi$, and $h(x) < 0$, when $\xi < x \le a$. It now follows immediately that

$$\begin{aligned} g'(x) &> 0, \qquad b \le x < \xi \\ &< 0, \qquad \xi < x \le a \end{aligned} \tag{11.103}$$

Now suppose that x is fixed and a is variable and put

$$k(a) = \frac{x + 3a}{(x + a)^{3/2}} \tag{11.104}$$

Then, differentiating with respect to a, we obtain

$$k'(a) = \frac{3(x - a)}{2(x + a)^{5/2}} \tag{11.105}$$

and hence $k'(a) > 0$, provided $a < x$. This implies that $k(a)$ is an increasing function for such values of a and thus, if $b < a < x$, we have

$$k(b) < k(a) \tag{11.106}$$

It now follows from Eq. (11.101) that

$$g'(x) = \tfrac{1}{2}\{k(b) - k(a)\} < 0 \tag{11.107}$$

provided $x > a$.

Finally, it is easy to prove that $g(x) \to 0$ as $x \to \infty$. Taking this result in conjunction with Ineqs. (11.103) and (11.107), it will be immediately obvious that the graph of the function $g(x, a, b)$ takes the form shown in Fig. 9.

References

1. D. F. Lawden, Interplanetary ocket Trajectories, *Advances in Space Sci.* **1**, 1, Academic Press, New York (1959).
2. W. Hohmann, 'Die Erreichbarkeit der Himmelskörper,' Oldenbourg, Munich (1925).
3. D. F. Lawden, Optimal transfer via tangential ellipses, *J. Brit. Interplanet. Soc.* **11**, 278 (1952).
4. D. F. Lawden, Inter-orbital transfer of a rocket, *J. Brit. Interplanet. Soc.* **11**, 321 (1952).
5. D. F. Lawden, The determination of minimal orbits, *J. Brit. Interplanet. Soc.* **11**, 216 (1952).
6. D. F. Lawden, Minimal rocket trajectories, *J. Am. Rocket Soc.* **23**, 360 (1953).
7. D. F. Lawden, Fundamentals of space navigation, *J. Brit. Interplanet. Soc.* **13**, 87 (1954).
8. R. S. Long, Transfer between non-coplanar elliptical orbits, *Astronaut. Acta* **6**, 167 (1960).
9. G. C. Smith, The calculation of minimal orbits, *Astronaut. Acta* **5**, 253 (1959).
10. R. N. A. Plimmer, Fuel requirements for inter-orbital transfer of a rocket, *Proc. 10th Intern. Astronaut. Congr., London*, 1959 (1960).
11. L. G. Vargo, Optimal transfer between two coplanar terminals in a gravitational field, *Am. Astronaut. Soc. Preprint* No. 58-20 (1958).
12. H. Munick, R. McGill and G. E. Taylor, Minimization of characteristic velocity for two-impulse orbital transfer, *ARS Journal* **30**, 638 (1960).

13. J. M. Horner, Optimum orbital transfer, Personal communication from ABMA, Huntsville, Alabama (1960).
14. T. N. Edelbaum, Some extensions of the Hohmann transfer maneuver, *ARS Journal* **29**, 864 (1959).
15. R. F. Hoelker and R. Silber, The bi-elliptical transfer between circular coplanar orbits, *DA Tech. Memo.* No. 2-59, Army Ballistic Missile Agency (1959).
16. L. Rider, Ascent from inner circular to outer coplanar elliptic orbits, *ARS Journal* **30**, 254 (1960).

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