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ASTRONAUTICS

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$\theta_2 - \theta_1 = 360° = 2\pi$. For this reason, the artifice of evaluating Eq. (1-23) for $\theta_2 - \theta_1 = \pi$ and multiplying the result by 2 is used. Thus

$$T = 2\frac{H^3}{\mu^2}\left[\frac{e \sin\theta}{(1-e^2)(1-e\cos\theta)} + \frac{1}{(1-e^2)^{3/2}}\sin^{-1}\frac{e-\cos\theta}{1-e\cos\theta}\right]_0^{\pi}$$

$$= 2\frac{H^3}{\mu^2}\left\{0 - 0 + \frac{1}{(1-e^2)^{3/2}}\left[\sin^{-1}\frac{e+1}{1+e} - \sin^{-1}\frac{e-1}{1-e}\right]\right\}$$

$$= \frac{2H^3}{\mu^2(1-e^2)^{3/2}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \frac{2\pi H^3}{\mu^2(1-e^2)^{3/2}}$$

From the development of Eq. (1-22),

$$\frac{H^2}{\mu} = a(1-e^2)$$

Raising each side of this equation to the $3/2$ power and then multiplying both sides by $\sqrt{1/\mu}$ yields

$$\frac{H^3}{\mu^2} = \left(\frac{1}{\mu}\right)^{1/2} a^{3/2}(1-e^2)^{3/2}$$

Substituting this into the time-of-flight expression yields finally,

$$T = \frac{2\pi a^{3/2}}{\mu^{1/2}} \tag{1-24}$$

This equation indicates that the period of an elliptical orbit is completely determined by the major axis of that orbit. If both sides of the equation are squared, the result is

$$T^2 = \frac{(2\pi)^2}{\mu}a^3$$

This is Kepler's third law of planetary motion which states that the squares of the periods of the planetary orbits are proportional to the cubes of their respective major axes. Equation (1-24) also shows that the period of an elliptical orbit of major axis $2a$ is identical to that of a circular orbit of diameter $2a$. It may also be observed that since a is related to the energy of the orbit, the periods of all orbits of the same energy are the same regardless of their angular momentum.

It is informative to apply Eq. (1-24) for the period of a satellite to some interesting examples. Consider first an earth satellite which has been launched so as to have a major axis equal to the distance to the moon. This would represent a sort of minimum energy trip around the moon. The average distance to the moon is about 239,000 miles; therefore

$$2a = 0.239 \times 10^6 \times 5{,}280 \text{ ft}$$
$$a = 0.631 \times 10^9 \text{ ft}$$

Therefore,

$$T = \frac{2\pi a^{3/2}}{\mu_e^{1/2}} = \frac{2\pi(0.631\times 10^9)^{3/2}}{(14.05\times 10^{15})^{1/2}} = 84.0 \times 10^4 \text{ sec} = 9.7 \text{ days}$$

Consider next a satellite having a 24-hr period.

$$a^3 = \frac{T^2\mu_e}{2\pi}$$

$$a^3 = \frac{(24\times 60\times 60)^2 \times 14.05\times 10^{15}}{2\pi} = 16.7 \times 10^{24}$$

$$a = 2.56 \times 10^8 \text{ ft}$$

The major axis is $2a$ or 5.12×10^8 ft.

If a circular orbit having a 24-hr period were established, the diameter of this orbit would be $\frac{a}{r_e} = (2.56 \times 10^8)/(20.9 \times 10^6)$ or $12\frac{1}{4}$ times the diameter of the earth.

1.9 MINIMUM-ENERGY INTERPLANETARY TRAJECTORIES

The study of satellite trajectories leads naturally to the idea of the minimum-energy interplanetary trajectories, since these trajectories are actually sun satellite trajectories. These minimum-energy trajectories which are tangent to the orbit of one planet at perihelion and tangent to the orbit of another planet at aphelion are known as *Hohmann transfer ellipses*. The approximate lower bound of their energy levels may be obtained from a simplification of the solar system. Thus, all the orbits of all planets in the solar system are presumed to lie in the same plane and to be circular with the sun at the center. Under these assumptions the minimum energy interplanetary trajectory between the earth and another planet is an elliptical satellite trajectory tangent at one extreme to the earth's orbit and at the other extreme to the orbit of the other planet. Figure 1-24 gives a sketch of this basic idea for the trajectory between the earth and Mars.

It may be seen that the desired trajectory is created in the same way as the earlier earth satellite trajectories. When being launched from the earth for a planet farther away from the sun, a spaceship is already

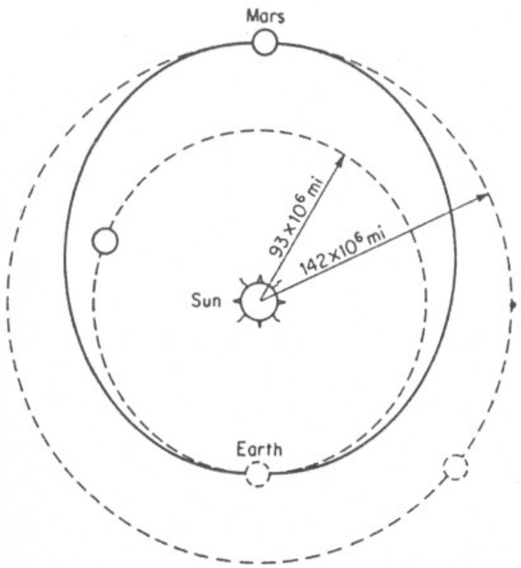

Fig. 1-24 Minimum energy: Earth to Mars. The small dotted circles show the relative positions of the earth and Mars at the time of launch from the earth; the small solid circles show the relative positions of the earth and Mars at the time of rendezvous with Mars.

at the perigee of the desired orbit. Presuming that the spaceship is fired to take advantage of the earth's orbital velocity, the rocket must then supply the velocity required for the elliptical trajectory plus sufficient velocity to escape the earth.

For a trip to the inner planets, Mercury and Venus, the spaceship, before launching from the earth, is at the apogee of the desired elliptical trajectory. In this instance, the rocket would be fired opposite to the direction of the earth's orbital travel with sufficient velocity to escape from the earth and to reduce its velocity so as to create an elliptical trajectory inside the earth's orbit. The idea of this trajectory is shown in Fig. 1-25.

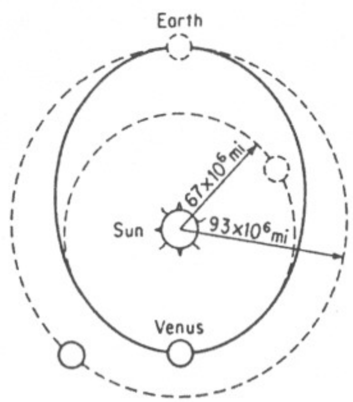

Fig. 1-25 Minimum energy: Earth to Venus. The small dotted circles show the relative positions of the earth and Venus at the time of launch from the earth; the small solid circles show the relative positions of the earth and Venus at the time of rendezvous with Venus.

Turning first to the Earth-Mars trajectory, the major axis of the required ellipse is seen to be the sum of the orbital radii of the earth and Mars. Therefore

$$2a = (93 + 142) \times 10^6 \text{ miles}$$
$$a = 117.5 \times 10^6 \text{ miles}$$

From the data of Table 1-1 the value of μ for the sun is

$$\mu_s = g_s r_s^2 = 900.0 \times \frac{0.864^2}{2} \times 10^{12} \times 5.28^2 \times 10^6$$
$$= 4.69 \times 10^{21} \text{ ft}^3/\text{sec}^2$$

From the earlier derivations,

$$a = -\frac{\mu}{2E}$$
$$2E = \frac{-4.69 \times 10^{21}}{117.5 \times 10^6 \times 5{,}280} = -7.59 \times 10^9$$

Therefore,

$$v^2 = 2E + 2\frac{\mu}{r} = -7.59 \times 10^9 + \frac{2 \times 4.69 \times 10^{21}}{93 \times 10^6 \times 5{,}280}$$

or

$$v = 107{,}300 \text{ ft/sec}$$

This is the velocity required at the perigee of the orbit of the space vehicle. Presuming the launching to be precisely controlled so as to take full advantage of the earth's orbital motion the velocity increment of the rocket is reduced accordingly. The velocity increment needed at a great distance from the earth is the difference between this velocity and earth's orbital velocity, 107,300 − 97,800 = 9,500 ft/sec. The velocity increment needed at the surface of the earth may be calculated from the conservation of energy requirement as follows:

$$E = \frac{1}{2}v^2 - \frac{\mu_e}{r}$$
$$= \frac{1}{2}(9{,}500)^2 - \frac{\mu_e}{\infty} = \frac{1}{2}v^2 - \frac{\mu_e}{r_e}$$

from which the velocity required at the surface of the earth v is computed to be

$$v = 37{,}900 \text{ ft/sec}$$

It is to be noted that by far the largest part of required velocity increment is that needed to overcome the earth's gravitational attraction.

The period of this trajectory is computed from Eq. (1-24) as follows:

$$T = \frac{2\pi a^{3/2}}{\mu_s^{1/2}} = \frac{2\pi(117 \times 10^6 \times 5{,}280)^{3/2}}{(4.69 \times 10^{21})^{1/2}} = 0.446 \times 10^8 \text{ sec}$$
$$= \frac{0.446 \times 10^8}{24 \times 60 \times 60} = 516 \text{ days} = 1.4 \text{ years}$$

This is the period of the round trip; the one-way journey would be 258 days.

Turning now to the Earth-Venus trajectory, the major axis is again the sum of the radii of the two orbits. Thus

$$2a = (67 + 93) \times 10^6 \text{ miles}$$
$$a = 80 \times 10^6 \times 5{,}280 = 0.422 \times 10^{12} \text{ ft}$$
$$2E = -\frac{\mu_s}{a} = -\frac{4.69 \times 10^{21}}{0.422 \times 10^{12}} = -11.1 \times 10^9$$
$$v^2 = 2E + 2\frac{\mu_s}{r} = -11.1 \times 10^9 + \frac{2 \times 4.69 \times 10^{21}}{93 \times 10^6 \times 5{,}280} = 80 \times 10^8$$
$$v = 89{,}450 \text{ ft/sec}$$
$$E = \frac{1}{2}(89{,}450 - v_{\text{earth}})^2 = \frac{1}{2}v^2 - \frac{\mu_e}{r_e}$$
$$v = \text{velocity required at the surface of the earth} = 37{,}600 \text{ ft/sec}$$

It is to be noted that it requires almost the same velocity increment to create the trajectory to Venus as it takes to create the trajectory to Mars, even though the former is a much lower energy trajectory. This occurs because the Venus trajectory represents about the same change in energy level as does the Mars trajectory. The period of the Earth-Venus trajectory is determined as follows:

$$T = \frac{2\pi(0.422 \times 10^{12})^{3/2}}{(4.69 \times 10^6)^{1/2}} = 0.252 \times 10^8 \text{ sec} = 290 \text{ days}$$

The one-way trip would be about 145 days.

Actual interplanetary trajectories will require somewhat more velocity than the idealized trajectories studied here. The fact that the orbits are not coplanar requires that sufficient velocity be available to transfer the trajectory from one plane to another. This situation is entirely analogous to that discussed in connection with earth satellites in which it was seen that the latitude of the launch site largely determined the minimum tilt of the orbital plane with respect to the earth's axis. In the same way the plane of the interplanetary vehicle will be essentially that of the earth's orbital plane unless a considerable amount of energy is used to move the trajectory out of the earth's orbital plane.

1.10 HISTORICAL NOTE

The full import of this chapter cannot be appreciated without recognizing the relation of the trajectories of man-made space vehicles to the early development of the science of celestial mechanics. A very brief review of this history is given in the next few paragraphs.

The scholars of ancient Greece (ca. 600 B.C.) presumed that the earth was the center of the universe and that the stars were fixed in some gigantic sphere which revolved around the earth from east to west once each day. They recognized that certain objects followed the same general course through the sky but had motions somewhat different from those of the stars. They recognized these as the planets of the solar system and, of course, gave special recognition to the moon and the sun. The motion of the moon and the sun presented little difficulty since a simple presumption that they traveled at a slightly different speed from the stars accounted for their special behavior. However, the planets presented a difficulty since they sometimes went faster than the stars and sometimes slower. The most widely accepted solution was that recorded in Ptolemy's Almagest and known as the Ptolemaic System. In this system, each planet, in addition to partaking of the circular motion of the other stars, moved in a circle whose center moved with the stars. These combinations of large and small circles were known as *epicycles*. As observations were refined, the motions of the planets were seen to be more complex and were accounted for by adding epicycles on top of epicycles.

The Copernican system was proposed by Nicolas Copernicus in the first half of the sixteenth century. He showed that a much simpler and more accurate model of the planetary system resulted if the planets, including the earth, were assumed to revolve in circular paths about a stationary sun. For many reasons, including even some of the most careful observations by the Danish astronomer Tycho Brahe, this heliocentric system was rejected. It seems that Brahe, toward the end of the sixteenth century, reasoned that if the earth moved about the sun, his observation of the stars would demonstrate parallax which would show an annual variation. In spite of very careful measurement, however, he could detect no parallax and hence his observations were offered as scientific proof that the heliocentric system could not be true. It was many years until the science of photography was sufficiently developed to detect the parallax of the nearer stars and used, in fact, to measure their distance. Nonetheless, Brahe made a great contribution through his careful determinations of the positions of the planets which he continued over an extended period.

Johannes Kepler, a German astronomer, joined Brahe around the