AUTHOR: M. Minovich August 28, 1961

TITLE: The Determination of Miss Distances for Conic Trajectories due to Velocity Errors

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In this paper we consider the effect of varying the velocity vector by a small amount $d\vec{V}$ of a free-fall vehicle moving on a conic trajectory under the influence of one body. In particular if $\vec{R}_P$, $\vec{V}_P$ and $\vec{R}_Q$ , $\vec{V}_Q$ are the position and velocity vectors at two points P and Q on the vehicles trajectory we shall determine the distance of closest approach between the vehicle and the point Q if the velocity vector of the vehicle is changed by $d\vec{V}_P$ at the point P. The conic may be either elliptic or hyperbolic.

If the position and velocity of a free-fall vehicle moving in a known gravitational field are known at any time its trajectory is completely determined. Consequently if one desires a free-fall vehicle at any point P at time $t_P$ to pass through a prescribed point Q at the time $t_Q$, one and only one velocity $\vec{V}_P$ exists for the vehicle at P which will enable it to be at Q at the time $t_Q$. Now no matter how accurate a guidance system may be for the powered portion of flight there will always be some initial error when the vehicle begins its free-fall flight. Thus, if P is the initial point of free-fall flight to the target point Q its velocity will not be $\vec{V}_P$ but $\vec{V}_P + d\vec{V}_P$ where $d\vec{V}_P$ is the amount the actual velocity at P differs from a velocity $\vec{V}_P$ which would have taken the vehicle to the point Q. We shall assume that $\sum$ is some orthogonal inertial frame with origin at the center of the gravitating body such that $d\vec{V}_P$ can be expressed by three components $dV^1$ $dV^2$ $dV^3$ where

(1) $d\vec{V}_P = (dV_P^1 \; dV_P^2 \; dV_P^3)$

It has been found very convenient to employ two important vectors $\vec{\epsilon}$ and $\vec{h}$ when conic trajectories are being studied. The vector $\vec{\epsilon}$ is directed toward the vehicle's point of perihelion and has a magnitude equal to the conic's eccentricity. The vector $\vec{h}$ is the vehicle's angular momentum about the center of the gravitating body. Taking the mass of the vehicle as unity the angular momentum vector is

(2) $\vec{h} = \vec{R} \times \vec{V}$

where $\vec{R}$ and $\vec{V}$ are position and velocity vectors at any point on the vehicle's trajectory. By formulas (4) and (5) of T. M. No. 312-130 these vectors are related by

(3) $\vec{\epsilon} = \frac{1}{\mu} \vec{V} \times \vec{h} - \hat{R}$

(4) $\vec{V} = \frac{1}{\ell} \vec{h} \times (\hat{R} + \vec{\epsilon}\,)$

where $\hat{R} = \frac{\vec{R}}{R}$ and $\mu$ = GM, M being the mass of the body and G the gravitational constant. The semi-latus rectum $\ell$ of the conic trajectory is related to $\vec{h}$ and $\mu$ by

(5) $\ell = \frac{h^2}{\mu}$

The magnitude of $\vec{R}$ corresponding to $\hat{R}$ can be calculated by

$$(6) \quad R = \frac{\ell}{1 + \hat{R} \cdot \vec{\epsilon}}$$

Now when $\vec{V}_P$ is changed by an amount $d\vec{V}_P$ the vectors $\vec{\epsilon}$ and $\vec{h}$ will change by $d\vec{\epsilon}$ and $d\vec{h}$ where, by employing (2) and (3)

$$(7) \quad d\vec{h} = \vec{R}_P \times d\vec{V}_P$$

$$(8) \quad d\vec{\epsilon} = \frac{1}{\mu} (d\vec{V}_P \times \vec{h} + \vec{V}_P \times d\vec{h})$$

Let Q' be the point on the new trajectory which is closest to Q and $\vec{R}_{Q'}$, $\vec{V}_{Q'}$ its position vector and the vehicles velocity vector at Q'. If we denote $\vec{d} = \overrightarrow{Q'Q}$ = vehicles miss vector, it follows that

$$(9) \quad \vec{V}_{Q'} \cdot \vec{d} = 0$$

This can be easily seen since at closest approach the vehicle is neither approaching nor reseding from Q. Consequently employing (4) and noting that $\overrightarrow{Q'Q} = \vec{R}_Q - \vec{R}_{Q'}$, (9) may be written as

$$(10) \quad \frac{1}{\ell + d\ell} (\vec{h} + d\vec{h}) \times (\hat{R}_{Q'} + \vec{\epsilon} + d\vec{\epsilon}) \cdot (\vec{R}_Q - \vec{R}_{Q'}) = 0$$

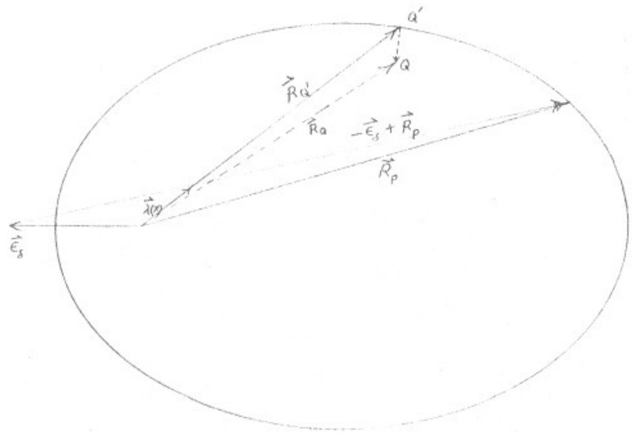
Since $\vec{R}_{Q'}$ is a point on the new trajectory

(11) $(\vec{h} + d\vec{h}) \cdot \vec{R}_{Q'} = 0$

There will be two solutions of (10) satisfying (11) corresponding to the point of closest approach $\vec{R}_{Q'}$ and another vector corresponding to the farthest point from Q since at this point the vehicle also neither approaches nor recedes from Q. These two solutions can be distinguished from each other by noting the correct solution will yield a positive scalor when dotted into $\vec{R}_Q$ and the second solution will yield a negative scalor when dotted into $\vec{R}_Q$. Hence we obtain a second condition which the solution of (10) must satisfy:

$$\vec{R}_{Q'} \cdot \vec{R}_Q > 0$$

The condition expressed by (11) can be eliminated by finding a solution of (10) from a class of vectors already in the plane of the trajectory. This can be accomplished by considering the following special example:




The vectors $\vec{R}_Q$ and $\overrightarrow{Q'Q}$ are not in general in the plane of the new trajectory drawn above; $\vec{\epsilon}_s = \vec{\epsilon} + d\vec{\epsilon}$

From this special case we observe that $\hat{\lambda}(x_o) = \hat{R}_Q$, where

$$\vec{\lambda}(x_o) = \vec{\epsilon} + x_o(-\vec{\epsilon} + \vec{R}_P) \qquad 0 \leqq x_o \leqq 1$$

for $\vec{R}_P$ is in the plane of the old and new trajectories. Consequently the solution can be obtained by varying x in the interval $0 \leqq x \leqq 1$ such that

$$\vec{h}_s \times (\hat{\lambda}(x) + \vec{\epsilon}_s)\cdot(\vec{R}_Q - \frac{\ell_s}{1 + \hat{\lambda}(x)\cdot\vec{\epsilon}_s}\hat{\lambda}(x)) = 0$$

where $\vec{h}_s = \vec{h} + d\vec{h}$ and $\ell_s = \ell + d\ell$. From (5) $d\ell = \frac{2}{\mu} h \mid \vec{R}_P x\, d\vec{V}_P \mid$
Hence instead of varying 3 scalors representing the components of possible solutions one has to vary only one scalor x which is restricted to the interval $0 \leqq x \leqq 1$.

Now for all possible situations there exists some scalor x in the interval $0 \leqq x \leqq 1$ such that $\hat{\lambda}(x) = \hat{R}_Q$, where $\vec{\lambda}(x)$ is given by one of the following four formulas:

$$(12)\begin{cases} \vec{\lambda}_1(x) = \vec{\epsilon}_s - x(\vec{\epsilon}_s + \vec{R}_P) & \vec{\lambda}_2(x) = -\vec{\epsilon}_s + x(\vec{\epsilon}_s + \vec{R}_P) \\ \vec{\lambda}_3(x) = \vec{\epsilon}_s + x(-\vec{\epsilon}_s + \vec{R}_P) & \vec{\lambda}_4(x) = -\vec{\epsilon}_s + x(\vec{\epsilon}_s - \vec{R}_P) \end{cases}$$

Thus the problem is solved by finding x and j ($0 \leq x \leqq 1$, $j = 1$ or 2 or 3 or 4) such that

$$(13) \quad \vec{h}_\delta \; x(\hat{\lambda}_j(x) + \vec{\epsilon}_\delta \;)\cdot(\vec{R}_Q - \frac{\ell_\delta}{1 + \lambda_j(x)\cdot\vec{\epsilon}_\delta}\,\hat{\lambda}_j(x) = 0$$

and

$$(14) \quad \hat{\lambda}_j(x)\cdot \vec{R}_Q > 0$$

This should not be too difficult for a high speed computer for the process of finding x and j may proceed by setting j - 1, 2, 3, or 4 and for each value of j consider the possible values of x = 0, .02, .04, ..., .88, 1.00 until the solution satisfying (13) and (14) has been found. After finding x and j the miss vector $\vec{d}$ and the miss distance d can be readily calculated since

$$\vec{R}_{Q'} = \frac{\ell_\delta}{1 + \lambda_j(x)\cdot\vec{\epsilon}}\,\hat{\lambda}_j(x)$$

and

$$\vec{d} = \vec{R}_Q - \vec{R}_{Q'}$$

These vectors can be easily expressed in any other orthogonal inertial frame of reference with the same origin as $\sum$ by multiplying the vectors in $\sum$ expressed as (3 x 1) column matrices by the orthogonal matrix

$$A = \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}$$

where $\ell_{ij}$ is the cosine of the angle between the i'th new axis and the j'th old axis.

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